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Optimal boundary control problems of retarded parabolic systems

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Optimal boundary control problems of retarded parabolic systems are presented. Necessary and sufficient conditions of optimality are derived for the Neumann problem. A simple example of application is also presented.

Key words: boundary control, parabolic systems, time delays, linear quadratic problem, quadratic programming method

1. Introduction

Various optimal control problems of infinite dimensional systems with point and distributed delays were considered in [1], [2], [3], [5], [6], [7], [8], [9], [10] and [15].

In [15], optimal control problems for parabolic systems with Neumann boundary conditions involving constant time delays were considered. Such systems constitute in a linear approximation, a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced at the boundary of a system's spatial domain.For example, in the area of plasma control, it is of interest to confine the plasma in a given bounded spatial domain Ω by introducing a finite electric potential barrier or a "magnetic mirror" surrounding Ω. For a collision-dominated plasma, its particle density is describable by a parabolic equation. Due to the particle inertia and finiteness of the electric potential barrier or the magnetic mirror field strength, the particle reflection at the domain boundary is not instantaneous. Consequently, the particle flux at the boundary of Ω at any time depends on the flux of particles which escaped earlier and reflected back into Ω at a later time. This leads to Neumann boundary conditions involving time delays. Necessary and sufficient conditions which the optimal controls must satisfy

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were derived. Estimates and a sufficient condition for the boundedness of solutions were obtained for parabolic systems with specified forms of feedback controls.

Subsequently, in [5], the time-optimal control problems of linear parabolic systems with the Neumann boundary conditions involving constant time delays were considered. Using the results of [15], the existence of a unique solution of such parabolic systems were discussed. A characterization of the optimal control in terms of the adjoint system is given. This characterization was used to derive specific properties of the optimal control (bang-bangness, uniqueness, etc.). These results were also extended to certain cases of nonlinear control without convexity and to certain fixed time problems.

In particular, in [8] optimization problems of parabolic systems with time delays given by the integral form with $h \in (a,b)$ and $a > 0$ were considered. Such optimal control problems can be extended onto the case where $h \in (0,b)$ with $a = 0$.

Consequently, in the paper [9] an optimal distributed control problem for a linear parabolic system in which time delays appear in the integral form with $h \in (0,b)$ in the state equation and with $k \in (0,c)$ in the Neumann boundary condition is considered. Using the isomorphism between two Hilbert spaces and a constructive method, sufficient conditions for the existence of a unique solution of such retarded parabolic equations with the Neumann boundary conditions are proved.

Making use of Lion's framework ([11]) necessary and sufficient conditions of optimality are derived for a linear quadratic problem.

Finally, in the paper [10] optimal boundary control problems for distributed parabolic systems in which retarded arguments with $h \in (0,b)$ appear in the integral form in the state equations are presented. Necessary and sufficient conditions of optimality are derived for the non-homogeneous Dirichlet problem.

In this paper, we consider an optimal boundary control problem for a linear parabolic system in which time delays appear in the integral form with $h \in (0,b)$ in the state equation and with $k \in (0,c)$ in the Neumann boundary condition.

Using the transposition method and some interpolation theorems sufficient conditions for the existence of a unique solution for such retarded parabolic systems are proved.

The performance functional has a quadratic form. The time horizon is fixed. Finally, we impose some constraints on the boundary control. Making use of Lion's framework ([11]) necessary and sufficient conditions of optimality are derived for a linear quadratic problem. A simple mathematical example of application is also provided.

2. Existence of solutions in the space $H^{2,1}(Q)$

Consider now the distributed-parameter system described by the following parabolic delay equation:

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$$\frac{\partial y}{\partial t}+A(t)y+\int_0^b y(x,t-h)dh=v \qquad x\in\Omega,\ t\in(0,T),\ h\in(0,b) \tag{1}$$

$$y(x,t')=\Phi_o(x,t') \qquad x\in\Omega,\ t'\in[-b,0) \tag{2}$$

$$\frac{\partial y}{\partial\eta_A}=\int_0^c y(x,t-k)dk+u \qquad x\in\Gamma,\ t\in(0,T),\ k\in(0,c) \tag{3}$$

$$y(x,t')=\Psi_0(x,t') \qquad x\in\Gamma,\ t'\in[-c,0) \tag{4}$$

$$y(x,0)=y_0(x) \qquad x\in\Omega \tag{5}$$

where: $\Omega\subset R^n$ is a bounded, open set with boundary Γ, which is a C^∞ manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ.

$$y\equiv y(x,t;u), \qquad u\equiv u(x,t), \qquad v\equiv v(x,t),$$

$$Q=\Omega\times(0,T), \quad \bar{Q}=\bar{\Omega}\times[0,T], \quad Q_0=\Omega\times[-b,0), \quad \Sigma=\Gamma\times(0,T),$$
$$\Sigma_0=\Gamma\times[-c,0)$$

where: T is a specified positive number representing a time horizon, h and k are time delays such that $h\in(0,b)$ and $k\in(0,c)$. Φ_0,Ψ_0 are initial functions defined on Q_0 and Σ_0, respectively.

The parabolic operator $\frac{\partial}{\partial t}+A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 of ([12], Vol. 2, p. 2) and $A(t)$ is given by

$$A(t)y=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{6}$$

and the functions $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$(closure of Q) satisfying the ellipticity condition

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\varphi_i\varphi_j\geqslant\alpha\sum_{i=1}^{n}\varphi_i^2,\alpha>0, \qquad \forall(x,t)\in\bar{Q},\forall\varphi_i\in R. \tag{7}$$

The equations (1) - (5) constitute a Neumann problem. The left-hand side of (4) is written in the following form

$$\frac{\partial y}{\partial\eta_{A(t)}}=\sum_{i,j=1}^{n}a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j}=q(x,t) \qquad x\in\Gamma,\ t\in(0,T) \tag{8}$$

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where: $\dfrac{\partial y}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_i)$ is an i-th direction cosine of n, with n being the normal at Γ exterior to Ω and

$$q(x,t) = \int_0^c y(x,t-k)dk + u(x,t) \qquad x \in \Gamma,\ t \in (0,T),\ k \in (0,c). \tag{9}$$

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (1) - (5) for the case where the boundary control $u \in L^2(\Sigma)$. For this purpose, for any pair of real numbers $r,s \geqslant 0$, we introduce the Sobolev space $H^{r,s}(Q)$ ([12], Vol. 2, p. 6) defined by

$$H^{r,s}(Q) = H^0(0,T;H^r(\Omega)) \cap H^s(0,T;H^0(\Omega)) \tag{10}$$

which is a Hilbert space normed by

$$\left(\int_0^T \| y(t) \|^2_{H^r(\Omega)} dt + \| y \|^2_{H^s(0,T;H^0(\Omega))} \right)^{1/2} \tag{11}$$

where: the spaces $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined in ([12], Vol.1, Chapter 1) respectively.

The case of time delays given in the integral form with $h \in (0,b)$ and $k \in (0,c)$ is very sophisticated. We cannot use in this case a classical constructive method in the proof about the existence of a unique solution of the parabolic problem (1)-(5), since the values of lower limits of integration are equal to zero. Consequently, using the transposition method and some interpolation theorems we can ommit such restriction.

Theorem 1 *Let $y_0, \Phi_0, \Psi_0, u, v$ be given with $y_0 \in H^1(\Omega)$, $\Phi_0 \in H^{2,1}(Q_0)$, $\Psi_0 \in H^{1/2,1/4}(\Sigma_0)$ $u \in H^{1/2,1/4}(\sum)$ and $v \in L^2(Q)$. Then, there exists a unique solution $y \in H^{2,1}(Q)$ for the mixed initial-boundary value problem* (1)-(5).

Proof The parabolic delay equation (1) with initial and boundary conditions (2)-(5) may be rewritten as

$$\frac{\partial y}{\partial t} + A(t)y = Ny + f \tag{12}$$

$$\frac{\partial y}{\partial \eta_A} = My + q \tag{13}$$

$$f(x,t) := v(x,t) + \int_{\min(0,t-b)}^{0} \Phi_0(x,\tau)d\tau \tag{14}$$

$$Ny(x,t) := \int_{\max(0,t-b)}^{0} y(x,\tau)d\tau \tag{15}$$

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$$q(x,t) = \int_{\min(0,t-c)}^{0} \Psi_0(x,\tau)d\tau + u(x,t) \tag{16}$$

$$My = \int_{\max(0,t-c)}^{0} y(x,\tau)d\tau. \tag{17}$$

Let

$$\left.\begin{array}{lcr} G_0 & : & H^1(\Omega) \to H^{2,1}(Q) \\ G_1 & : & H^{1/2,1/4}(\Sigma) \to H^{2,1}(Q) \\ S & : & L^2(Q) \to H^{2,1}(Q) \end{array}\right\} \tag{18}$$

denote the continuous solution operators provided by Theorem 6.1 and Remark 6.3 of [12] (Vol. 2, pp.33 and 37).
Then the parabolic problem (1)-(5) is equivalent to the fixed point of equation

$$y = G_0 y_0 + G_1 q + G_1 My + Sf + SNy. \tag{19}$$

We need to find an estimate for $\| SN \|_{\mathcal{L}(H^{2,1}(Q),H^{2,1}(Q))}$ and $\| G_1 M \|_{\mathcal{L}(H^{1/2,1/4}(\Sigma),H^{2,1}(Q))}$ respectively. We have

$$\| SNy \|_{H^{2,1}(Q)} \leqslant \| S \|_{\mathcal{L}(L^2(Q),H^{2,1}(Q))} \| Ny \|_{L^2(Q)} \leqslant$$

$$\leqslant c \int_{\max(0,t-c)}^{t} \| y(x,\tau) \|_{H^{2,1}(Q)} d\tau \leqslant cT \| y \|_{H^{2,1}(Q)} \tag{20}$$

$$\| G_1 My \|_{H^{2,1}(Q)} \leqslant \| G_1 \|_{\mathcal{L}(H^{1/2,1/4}(\Sigma),H^{2,1}(Q))} \| My \|_{H^{1/2,1/4}(\Sigma)} \leqslant$$

$$\leqslant c \int_0^t \| y(x,\tau) \|_{H^{1/2,1/4}(\Sigma)} d\tau \leqslant cT \| y \|_{H^{1/2,1/4}(\Sigma)} . \tag{21}$$

From (20) and (21) we deduce

$$\| SN \|_{\mathcal{L}(H^{2,1}(Q),H^{2,1}(Q))} < 1 \quad \text{if } T < \frac{1}{C} \tag{22}$$

$$\| G_1 My \|_{\mathcal{L}(H^{1/2,1/4}(\Sigma),H^{2,1}(Q))} < 1 \quad \text{if } T < \frac{1}{C}. \tag{23}$$

Evidently, we can extend our result to any $T < +\infty$.

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3. The adjoint problem

The adjoint problem is

$$A^*(t)p - p' + \int_0^b p(x,t+h)dh = \varphi \quad \text{in } Q \tag{24}$$

$$p(x,t') = 0 \quad x \in \Omega,\ t' \in (T,c] \tag{25}$$

$$\frac{\partial p}{\partial \eta_{A^*}}(x,t) = \int_0^c p(x,t+k)dk \quad x \in \Gamma,\ t \in (0,T),\ k \in (0,c) \tag{26}$$

$$p(x,T) = 0 \quad \text{on } \Omega \tag{27}$$

where

$$A^*(t)p = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right)$$

and p' denotes the derivative with respect to t.
The problem (24)-(27) can be solved backwards in time. For this purpose, we may apply Theorem 1 (with an obvious change of variables).
The following result can be proved.

Lemma 1 *Let φ be given in $L^2(Q)$. Then, there exists a unique solution $p \in H^{2,1}(Q)$ for the problem* (24)-(27).

Let us denote by $X(Q)$ the space described by the solutions of the adjoint problem (24)-(27) as φ describes $L^2(Q)$.
We have

$$X(Q) \in H^{2,1}(Q). \tag{28}$$

We can equivalently define

$$\begin{aligned} X(Q) \quad = \quad & \Bigg\{ p \mid p \in H^{2,1}(Q) : \frac{\partial p}{\partial \eta_{A^*}} = \int_0^c p(x,t+k)dk \text{ on } \Sigma, \\ & p(x,T) = 0, \\ & A^*p - p' + \int_0^b p(x,t+h)dh \in L^2(Q) \Bigg\}. \end{aligned} \tag{29}$$

Providing $X(Q)$ with the norm of the graph we get

$$P^*\left(= A^*p - p' + \int_0^c p(x,t+h)dh\right) \tag{30}$$

which is an isomorphism of $X(Q)$ onto $L^2(Q)$.

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4. Transposition of the adjoint isomorphism

By transposition we deduce from (30):

Lemma 2 *Let $p \to L(p)$ be a continuous linear form on $X(Q)$. Then, there exists a unique solution $y \in L^2(Q)$ such that*

$$\left\langle y, A^* p - p' + \int_0^b p(x,t+h)dh \right\rangle = L(p) \quad \forall p \in X(Q). \tag{31}$$

We choose L in the following form

$$L(p) = \int_\Omega \int_0^b \int_{-h}^0 \Phi_0(x,t)p(x,t+h)dtdhdx + \left\langle v, p \right\rangle +$$

$$+ \int_\Gamma \int_0^c \int_{-h}^0 \Psi_0(x,t)p(x,t+k)dtdkd\Gamma + \left\langle u, \frac{\partial p}{\partial \eta_{A^*}} \right\rangle + \left\langle y_0, p(x,0) \right\rangle. \tag{32}$$

We take

$$\Phi_0 \in (L^2(Q_0)) \tag{33}$$

$$v \in (H^{2,1}(Q))' \tag{34}$$

$$\Psi_0 \in (H^{3/2,3/4}(\Sigma_0))'. \tag{35}$$

Since $p \to \dfrac{\partial p}{\partial \eta_{A^*}}$ is a continuous linear mapping of $X(Q) \to H^{3/2,3/4}(\Sigma)$, we may take

$$u \in (H^{3/2,3/4}(\Sigma))'. \tag{36}$$

Similarly, since $p \to p\ (x,0)$ is a continuous linear mapping of $X(Q) \to H^1(\Omega)$, we may take

$$y_0 \in (H^1(\Omega))'. \tag{37}$$

According to Lemma 2 we have

Theorem 2 *Let $\Phi_0, v, \Psi_0, u, y_0$ be given with* (33), (34), (35), (36) *and* (37). *Then there exists a unique solution $y \in L^2(Q)$ such that* (31) *holds with* (32).

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5. Existence of solutions in the space $H^{3/2,3/4}(Q)$

We consider the mapping G

$$G\ :\ \{\Phi_0, v, \Psi_0, u, y_0\} \to y = G(\Phi_0, v, \Psi_0, u, y_0). \tag{38}$$

Then from Theorem 1 and Theorem 2 it follows that it is a continuous mapping of

$$H^{2,1}(Q_0) \times H^0(Q) \times H^{1/2,1/4}(\Sigma_0) \times H^{1/2,1/4}(\Sigma) \times H^1(\Omega) \to H^{2,1}(Q) \tag{39}$$

$$H^0(Q_0) \times (H^{2,1}(Q))' \times (H^{3/2,3/4}(\Sigma_0))' \times (H^{3/2,3/4}(\Sigma))' \times (H^1(\Omega))' \to H^0(Q). \tag{40}$$

We shall now interpolate between (39) and (40).

Theorem 3 ([12], Vol. 2, p.68) *We set*

$$A_0 = \left\{ \prod_{j=0}^{m-1} H^{2m-(m_j+\frac{1}{2}),1-(m_j+\frac{1}{2})/2m}(\Sigma) \times H^m(\Omega) \right\} \tag{41}$$

$$A_1 = \prod_{j=0}^{m-1} (H^{m_j+\frac{1}{2},(m_j+\frac{1}{2})/2m}(\Sigma))' \times (H^m(\Omega))'. \tag{42}$$

We also set

$$B_0 = \prod_{j=0}^{m-1} H^{2m-(m_j+\frac{1}{2}),1-(m_j+\frac{1}{2})/2m}(\Sigma_0) \tag{43}$$

$$B_1 = \prod_{j=0}^{m-1} (H^{m_j+\frac{1}{2},(m_j+\frac{1}{2})/2m}(\Sigma_0))'. \tag{44}$$

Let m_j be the order of boundary operator $\left\{ C_j\left(x, t_0, \frac{\partial}{\partial x}\right)\right\}_{j=0}^{m-1}$ such that $0 \leqslant m_j \leqslant 2m-1$.
If

$$\Theta > \frac{1}{2} - \frac{1}{4m} - \frac{\min m_j}{2m} = \eta \leqslant 0 \tag{45}$$

then

$$[A_0, A_1]_\Theta = \prod_{j=0}^{m-1} H^{2m(1-\Theta)-(m_j+\frac{1}{2}),1-\Theta-(m_j+\frac{1}{2})/2m}(\Sigma) \times H^{(1-2\Theta)m}(\Omega) \tag{46}$$

and

$$[B_0, B_1]_\Theta = \prod_{j=0}^{m-1} H^{2m(1-\Theta)-(m_j+\frac{1}{2}),1-\Theta-(m_j+\frac{1}{2})/2m}(\Sigma_0). \tag{47}$$

Remark 1 In (45) we always take $0 < \Theta < 1$.

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Remark 2 The condition (45) is therefore always satisfied if $\eta \leqslant 0$, i.e. if $\min_j m_j \geqslant m - \frac{1}{2}$, therefore if $m_j \geqslant m \ \ \forall\ j$. Then (46),(47) hold $\forall\ \Theta \in (0,1)$.

Consequently, we set (with $m = 1, m_0 = 1$)

$$A_0 = H^{1/2,1/4}(\Sigma) \times H^1(\Omega) \tag{48}$$

$$A_1 = (H^{3/2,3/4}(\Sigma))' \times (H^1(\Omega))' \tag{49}$$

and

$$B_0 = H^{1/2,1/4}(\Sigma_0) \tag{50}$$

$$B_1 = (H^{3/2,3/4}(\Sigma_0))'. \tag{51}$$

Using the Theorem 3 we have

$$[A_0, A_1]_{\Theta=1/4} = H^{0,0}(\Sigma) \times H^{1/2}(\Omega) \tag{52}$$

and

$$[B_0, B_1]_{\Theta=1/4} = H^{0,0}(\Sigma_0). \tag{53}$$

According to the results of [12] (Vol. 2, Chapter 4, Sections 15.1 and 2.1) we have

$$\left[H^0(Q), (H^{2,1}(Q))'\right]_{\Theta=1/4} = (H^{1/2,1/4}(Q))' \tag{54}$$

$$\left[H^{2,1}(Q_0), H^0(Q_0)\right]_{\Theta=1/4} = H^{3/2,3/4}(Q_0) \tag{55}$$

$$\left[H^{2,1}(Q), H^0(Q)\right]_{\Theta=1/4} = H^{3/2,3/4}(Q). \tag{56}$$

Then G defined by (38), is a continuous linear mapping of

$$H^{3/2,3/4}(Q_0) \times (H^{1/2,1/4}(Q))' \times H^{0,0}(\Sigma_0) \times H^{0,0}(\Sigma) \times H^{1/2}(\Omega) \to H^{3/2,3/4}(Q). \tag{57}$$

Theorem 4 *Let Φ_0, Ψ_0, v, u and y_0 be given with $\Phi_0 \in H^{3/2,3/4}(Q_0), \Psi_0 \in L^2(\Sigma_0)$, $v \in (H^{1/2.1/4}(Q))', u \in L^2(\Sigma)$ and $y_0 \in H^{1/2}(\Omega)$. Then there exists a unique solution $y \in H^{3/2,3/4}(Q)$ for the mixed initial-boundary value problem* (1)-(5) *(in the sense of Theorem* 2*).*

6. Optimal boundary control

We shall now formulate the optimal control problem for the Neumann problem (1) - (5).

Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

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The performance functional is given by

$$I(u) = \lambda_1 \int\limits_Q |y(x,t;u) - z_d|^2 dxdt + \lambda_2 \int\limits_0^T \int\limits_\Gamma (Nu)u \, d\Gamma dt \tag{58}$$

where $\lambda_i \geqslant 0$ and $\lambda_1 + \lambda_1 > 0$; z_d is a given element in $L^2(Q)$; N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

Finally, we assume the following constraint on controls $u \in U_{ad}$, where

$$U_{ad} \text{ is a closed, convex subset of } U. \tag{59}$$

The starting point for our considerations will be the following theorem, which can be found in ([11], p. 10):

Theorem 5 *Assume that the function $u \to I(u)$ is strictly convex, differentiable such that $I(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in U_{ad}$ (the last hypothesis may be omitted if U_{ad} is bounded). Then the unique element u_0 in U_{ad} satisfying $I(u_0) = inf_{u \in U_{ad}} I(u)$ is characterized by*

$$I'(u_0)(u - u_0) \geqslant 0 \quad \forall\, u \in U_{ad}. \tag{60}$$

The solving of the formulated optimal control problem is equivalent to seeking a $u_0 \in U_{ad}$ such that $I(u_0) \leqslant I(u) \quad \forall u \in U_{ad}$.

From Theorem 5 it follows that, for $\lambda_2 > 0$ a unique optimal control u_0 exists; moreover, u_0 is characterized by the condition (60).

Using the form of the performance functional (58) we can express (60) in the following form

$$\lambda_1 \int\limits_Q (y(u_0) - z_d)(y(u) - y(u_0))dxdt + \lambda_2 \int\limits_\Sigma Nu_0(u - u_0)dxdt \geqslant 0 \quad \forall u \in U_{ad}. \tag{61}$$

To simplify (61) we introduce the adjoint equation and for every $u \in U_{ad}$ we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the equation

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) + \int\limits_0^b p(x,t+h;u)dh = \lambda_1(y(u) - z_d)$$

$$(x,t) \in \Omega \times (0,T),\ h \in (0,b) \tag{62}$$

$$p(x,t;u) = 0 \qquad x \in \Omega,\ t' \in (T, T+d),\ d = max\{b,c\} \tag{63}$$

$$p(x,T;u) = 0 \qquad x \in \Omega \tag{64}$$

$$\frac{\partial p(u)}{\partial \eta_{A^*}} = \int\limits_0^c p(x,t+k;u)dk \quad (x,t) \in \Gamma \times (0,T),\ k \in (0,c) \tag{65}$$

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where

$$\frac{\partial p(u)}{\partial \eta_{A^*}}(x,t) = \sum_{i,j=1}^{n} a_{ji}(x,t)cos(n,x_i)\frac{\partial p(u)}{\partial x_j}(x,t)$$

$$A^*(t)p = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right).$$

Then it is easy to notice that for given z_d and u, problem (62)-(65) can be solved backwards in time starting from t=T, i.e. first solving (62)-(65) on the subcylinder Q_k and in turn on Q_{k-1}, etc. , until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 1 (with an obvious change of variables).

Hence, using Theorem 4, the following result can be proved:

Lemma 3 *Let the hypothesis of Theorem* 4 *be satisfied. Then, for given $z_d \in L^2(Q)$, there exists a unique solution $p(u) \in H^{3/2,3/4}(Q)$ for the problem* (62)-(65).

We simplify (61), using the adjoint equation (62)-(65). For this purpose, setting $u = u_0$ in (62)-(65), multiplying both sides of (62) by $(y(u) - y(u_0))$, then integrating over Q we get

$$\lambda_1 \int_Q (y(u_0)-z_d)(y(u)-y(u_0))dxdt = \int_Q \left(-\frac{\partial p(u_0)}{\partial t} + A^*(t)p(u_0)\right)(y(u)-y(u_0))dxdt +$$

$$+\int_Q \left(\int_0^b p(x,t+h;u_0)dh\right)(y(u)-y(u_0))dxdt =$$

$$= \int_Q p(u_0)\frac{\partial}{\partial t}(y(u)-y(u_0))dxdt + \int_Q A^*(t)p(u_0)(y(u)-y(u_0))dxdt +$$

$$\int_0^b \int_\Omega \int_0^T p(x,t+h;u_0)(y(t,x;u)-y(t,x,u_0))dtdxdh. \tag{66}$$

Using the equation (1), the first integral on the right-hand side of (66) can be rewritten as

$$\int_Q p(u_0)\frac{\partial}{\partial t}(y(u)-y(u_0))dxdt =$$

$$= -\int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt -$$

$$\int_Q p(u_0)\left(\int_0^b (y(x,t-h;u)-y(x,t-h,u_0))dh\right)dxdt$$

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$$= -\int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt-$$

$$\int_0^b \int_\Omega \int_{-h}^{T-h} p(x,t'+h;u_0)(y(x,t';u)-y(x,t',u_0))dt'dxdh =$$

$$= -\int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt-$$

$$\int_0^b \int_\Omega \int_0^{T-h} p(x,t'+h;u_0)(y(x,t';u)-y(x,t',u_0))dt'dxdh. \tag{67}$$

The second integral on the right-hand side of (66), in view of Green's formula, can be expressed as:

$$\int_Q A^*(t)p(u_0)(y(u)-y(u_0))dxdt = \int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt+$$

$$\int_0^T \int_\Gamma p(u_0)(\frac{\partial y(u)}{\partial \eta_A} - \frac{\partial y(u_0)}{\partial \eta_A})d\Gamma dt - \int_0^T \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_{A^*}}(y(u)-y(u_0))d\Gamma dt. \tag{68}$$

Using the boundary condition (3) the second component on the right-hand side of (68) can be written as

$$\int_0^T \int_\Gamma p(u_0)(\frac{\partial y(u)}{\partial \eta_A} - \frac{\partial y(u_0)}{\partial \eta_A})d\Gamma dt =$$

$$= \int_0^T \int_\Gamma p(x,t;u_0)(\int_0^c (y(x,t-k;u)-y(x,t-k;u_0))dk)d\Gamma dt+$$

$$+\int_0^T \int_\Gamma p(u_0)(u-u_0)d\Gamma dt =$$

$$\int_0^c \int_\Gamma \int_{-k}^{T-k} p(x,t'+k;u_0)(y(x,t';u)-y(x,t';u_0))dt'd\Gamma dk+$$

$$+\int_0^T \int_\Gamma p(u_0)(u-u_0)d\Gamma dt =$$

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$$\int_0^c \int_\Gamma \int_0^{T-k} p(x,t'+k;u_0)(y(x,t';u)-y(x,t';u_0))dt'd\Gamma dk+$$

$$+\int_0^T \int_\Gamma p(u_0)(u-u_0)d\Gamma dt. \tag{69}$$

The last component in (68) can in view of (65) be written as:

$$\int_0^T \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_{A^*}}(y(u)-y(u_0))d\Gamma dt$$

$$=\int_0^c \int_0^T \int_\Gamma p(x,t+k;u_0)(y(u)-y(u_0))d\Gamma dtdk. \tag{70}$$

Substituting (69) and (70) into (68) yields

$$\int_Q A^*(t)p(u_0)(y(u)-y(u_0))dxdt = \int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt-$$

$$-\int_0^c \int_\Gamma \int_{T-k}^{T} p(x,t+k;u_0)(y(x,t;u)-y(x,t;u_0))dtd\Gamma dk+$$

$$+\int_0^T \int_\Gamma p(u_0)(u-u_0)d\Gamma dt = \int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt-$$

$$-\int_0^c \int_\Gamma \int_T^{T+k} p(x,t';u_0)(y(x,t'-k;u)-y(x,t'-k;u_0))dt'd\Gamma dk+$$

$$+\int_0^T \int_\Gamma p(u_0)(u-u_0)d\Gamma dt. \tag{71}$$

Substituting (67) and (71) into (66), yields

$$\lambda_1 \int_Q (y(u_0)-z_d)(y(u)-y(u_0))dxdt = -\int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt+$$

$$-\int_0^b \int_\Omega \int_0^{T-h} p(x,t+h;u_0)(y(x,t;u)-y(x,t;u_0))dtdxdh+$$

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$$+\int_0^b\int_\Omega\int_0^T p(x,t+h;u_0)(y(x,t;u)-y(x,t;u_0))dtdxdh+$$

$$+\int_Q p(u_0)A(t)(y(u)-y(u_0))dxdt$$

$$-\int_0^c\int_\Gamma\int_{T-k}^T p(x,t+k;u_0)(y(x,t;u)-y(x,t;u_0))dtd\Gamma dk+$$

$$+\int_0^T\int_\Gamma p(u_0)(u-u_0)d\Gamma dt=$$

$$\int_0^b\int_\Omega\int_T^{T+h} p(x,t;u_0)(y(x,t-h;u)-y(x,t-h;u_0))dtdxdh$$

$$-\int_0^c\int_\Gamma\int_T^{T+k} p(x,t;u_0)(y(x,t-k;u)-y(x,t-k;u_0))dtd\Gamma dk+$$

$$+\int_0^T\int_\Gamma p(u_0)(u-u_0)d\Gamma dt=\int_0^T\int_\Gamma p(u_0)(u-u_0)d\Gamma dt. \tag{72}$$

Substituting (72) into (61) gives

$$\int_\Sigma (p(u_0)+\lambda_2 Nu_0)(u-u_0)d\Gamma dt \geqslant 0 \quad \forall u\in U_{ad}. \tag{73}$$

We now summarize the foregoing result.

Theorem 6 *For the problem* (1)-(5) *with the performance functional* (58)*, with* $z_d\in L^2(Q)$ *and* $\lambda_2>0$ *and with constraints on controls* (59)*, there exists a unique optimal control* u_0 *which satisfies the maximum condition* (73)*.*

We can also consider an analogous optimal control problem where the performance functional is given by

$$\hat{I}(v)=\lambda_1\int_\Sigma \left|y(u)\big|_\Sigma - z_{\Sigma d}\right|^2 d\Gamma dt+\lambda_2\int_\Sigma (Nu)u\,d\Gamma dt. \tag{74}$$

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From Theorem 4 and the trace theorem (Theorem 2.1 in [12], Vol. 2, p.9), for each $u \in L^2(\Sigma)$ there exists a unique solution $y \in H^{3/2,3/4}(Q)$ with $y|_\Sigma \in H^{1,1/2}(\Sigma) \subset L^2(\Sigma)$. Thus $\hat{I}(u)$ is well-defined. Then the optimal control v_0 is characterized by

$$\lambda_1 \int\limits_\Sigma \Big(y(u_0)\big|_\Sigma - z_{\Sigma d}\Big)\Big(y(u_0)\big|_\Sigma - y(u_0)\big|_\Sigma\Big) d\Gamma dt + \lambda_2 \int\limits_\Sigma (Nu)(u - u_0)\, d\Gamma dt \geqslant 0 \qquad \forall u \in U_{ad}. \tag{75}$$

We introduce the following equation

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) + \int\limits_0^b p(x,t+h;u)dh = 0 \qquad (x,t) \in \Omega \times (0,T),\ h \in (0,b) \tag{76}$$

$$p(x,t;u) = 0 \qquad x \in \Omega,\ t' \in (T, T+d),\ d = \max\{b,c\} \tag{77}$$

$$p(x,T;u) = 0 \qquad x \in \Omega \tag{78}$$

$$\frac{\partial p(u_0)}{\partial \eta_{A^*}} = \int\limits_0^c p(x,t+k;u)dk + \lambda_1\Big(y(u)\big|_\Sigma - z_{\Sigma d}\Big) \qquad \forall (x,t) \in \Gamma \times (0,T),\ k \in (0,c). \tag{79}$$

Using Theorem 4 the following lemma can be proved:

Lemma 4 *Let the hypothesis of Theorem* 4 *be satisfied. Then, for given $z_{\Sigma d} \in L^2(\Sigma)$ and any $u_0 \in L^2(\Sigma)$, there exists a unique solution $p(u_0) \in H^{3/2,3/4}(Q)$ for the problem* (76)-(79).

In this case the condition (75) can be also rewritten in the form (73). The following theorem is now fulfilled.

Theorem 7 *For the problem* (1)-(5) *with the performance functional* (74)*, with $z_{\Sigma d} \in L^2(\Sigma)$ and $\lambda_2 > 0$ and with constraints on controls* (59)*, there exists a unique optimal control u_0 which satisfies the maximum condition* (73).

Consider now the particular case where $U_{\Sigma_{ad}} = L^2(\Sigma)$. Thus the maximum condition (73) is satisfied when

$$u_0 = -\lambda_2^{-1} N^{-1} p(u_0)\big|_\Sigma. \tag{80}$$

If N is the identity operator on $L^2(\Sigma)$, then from Lemmas 3 and 4 it follows that $u_0 \in L^2(\Sigma)$.

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7. Example

In the case of performance functional (58) with $\lambda_1 > 0$ and $\lambda_2 = 0$ the optimization problem is equivalent to a quadratic programming one ([7]) which can be solved by the use of algorithms, e.g. Gilbert's ([4]). We shall formulate the following control problem as an example: equation of the system control (1)-(5), performance functional (58) with $\lambda_1 = 1$ and $\lambda_2 = 0$, i.e.

$$I(u) = \|y - z_d\|^2_{L^2(Q)} \tag{81}$$

constraints on controls

$$U_{ad} = \{u \in L^2(\Sigma) : \|u(x,t)\|_{L^2(\Sigma)} \leqslant 1\}. \tag{82}$$

We shall define the attainable set Y_{ad}

$$\begin{aligned} Y_{ad} = \Bigg\{ y(u) : \frac{\partial y(u)}{\partial t} + A(t)y + \int_0^b y(x,t-h)dh = f \\ y(x,t') = \Phi_0(x,t') \\ y(x,0;u) = y_0(x) \\ \frac{\partial y}{\partial \eta_A} = \int_0^c y(x,t-k)dk + u, \\ y(x,t') = \Psi_0(x,t'), \\ u \in U_{ad} \Bigg\}. \end{aligned} \tag{83}$$

Theorem 8 *The set Y_{ad} is closed, convex and bounded in the space $Y = L^2(Q)$.*

The proof of this theorem is similar to that in the case of the hyperbolic equation which is given in Kowalewski's doctoral dissertation [6].

We shall now describe the iteration procedure for solving our quadratic programming problem.

Let $\{Y^i_{ad}\}$ be a system of closed and convex subsets of the set Y_{ad} . We denote by $y^i \in Y^i_{ad}$ an element whose distance from element z_d is minimal, i.e. the following condition is fulfilled

$$\|y^i - z_d\| = \min_{y \in Y^i_{ad}} \|y - z_d\|. \tag{84}$$

By $\overline{y}^{i+1}$ we denote the element such that

$$\left\langle y^i - z_d, y - \overline{y}^{i+1} \right\rangle_{L^2(Q)} \geqslant 0 \quad \forall y \in Y_{ad}. \tag{85}$$

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The point $\overline{y}^{i+1}$ is a support of the set Y^i_{ad} determined by the hyperplane M^i orthogonal to the vector $(z_d - y^i)$.

In [13] it is shown that if the system of sets $\{Y^i_{ad}\}$ has the structure

$$Y^{i+1}_{ad} \supset \{y^i\} \cup \left\{\overline{y}^{i+1}\right\} \tag{86}$$

then the sequence $\{y^i\}$ is strongly convergent to y_0 in the space Y. Element y_0 which corresponds to a given control $u_0 \in U_{ad}$ is the solution of the formulated optimal control problem.

The step- by - step algorithms for finding the sequence y_i convergent to y_0 differ from each other by the construction of the sets Y^i_{ad}, only. The simplest one of them has been proposed by Gilbert ([4]) and applied in [7], [8], [9] and [10].

The shortcoming of Gilbert's algorithm mentioned above is a very slow rate of convergence. In this respect, the algorithm due to the Nahi and Wheeler [14] is better.

We now describe the method of determining the element $\overline{y}^{i+1}$ for the optimal control problem (1)-(5), (81) and (82). We introduce the following notation.

$$y^i = y(u^i), \quad \overline{y}^{i+1} = y(\overline{u}^{i+1}), \quad p^i = p(u^i). \tag{87}$$

Here, we introduce the adjoint equation

$$-\frac{\partial p^i}{\partial t} + A^*(t)p^i + \int_0^b p^i(x,t+h;u^i)dh = y^i - z_d,\ (x,t) \in \Omega \times (0,T),\ h \in (0,b) \tag{88}$$

$$p^i(x,t;u^i) = 0, \quad x \in \Omega,\ t \in (T, T+d),\ d = max\{b,c\} \tag{89}$$

$$p^i(x,T) = 0, \quad x \in \Omega \tag{90}$$

$$\frac{\partial p^i(u^i)}{\partial \eta_{A^*}}(x,t) = \int_0^c p^i(x,t+k;u^i)dk, \quad (x,t) \in \Gamma \times (0,T),\ k \in (0,c). \tag{91}$$

Proceeding in a similar way as in deriving the formula (73), the condition (85) is written as

$$\int_0^T \int_\Gamma p^i(u - \overline{u}^{i+1})dxdt = \langle p^i, u - \overline{u}^{i+1} \rangle_{L^2(Q)} \geqslant 0 \quad \forall u \in U_{ad}. \tag{92}$$

Taking into consideration the form of the set U_{ad} from (92), we get

$$\overline{u}^{i+1} = -\frac{p^i}{\|p^i\|_{L^2(\Sigma)}}. \tag{93}$$

From formula (93) we find out $\overline{u}^{i+1}$ for p^i which we determine from (88)-(91), knowing y^i from the previous iteration. Then, having $\overline{u}^{i+1}$, we compute $\overline{y}^{i+1}$ from (1)-(5).

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8. Conclusions

The results presented in this paper can be treated as a generalization of the results obtained by Kowalewski ([8]) and Krakowiak ([10]) pertaining to the case of retarded arguments appearing in the integral form with $h \in (0,b)$ in the state equations and with $k \in (0,c)$ in the Neumann boundary conditions, respectively.

The existence and uniqueness of solutions for such parabolic systems were proved - Theorems 1 and 4.

The optimal control was characterized by using the adjoint equation - Lemmas 3 and 4. Necessary and sufficient conditions of optimality with the quadratic performance functional and constrained controls are derived for the Neumann problem - Theorems 6 and 7.

As an example, a quadratic programming method in a Hilbert space, which can be used in solving certain optimization problems for retarded parabolic systems is also presented.

In this paper we have considered optimal boundary control problems for retarded parabolic systems where deviating arguments appear in the integral form both in the state equations and in the Neumann boundary conditions.

We can also consider optimal boundary control problems for retarded parabolic systems with non-homogeneous Dirichlet boundary conditions involving retarded arguments given in the integral form such that $k \in (0,c)$.

Finally, we can consider similar optimization problems for retarded hyperbolic systems.

Moreover, with regard to the controllability condition (i.e. there exists a $T > 0$ and $u \in U$ with $y(T;u) \in Y$), we can also investigate the exact controllability problem for time-delay parabolic system (1)-(5).

The ideas mentioned above will be developed in forthcoming papers.

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