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Imperfect maintenance in correlated bivariate Wiener model

Keywords

Wiener process, Pearson correlation coefficient, bivariate degradation, imperfect maintenance

Abstract

A two-unit system subject to imperfect maintenance is analyzed in this chapter. The deterioration of the system follows a bivariate Wiener degradation process. This process is built from the trivariate reduction method by sharing a common noise that describes the dependence between both units. This dependence is measured through the Pearson's correlation coefficient between both degradation trajectories of the bivariate Wiener process at time t. A maintenance strategy consisting of periodic inspections in which the accumulated deterioration of the system is reduced by a certain quantity is implemented. Some results on the monotonicity of the Pearson's correlation coefficient in different scenarios are obtained.

1. Introduction

Research on degradation modelling in industrial systems has been rising in recent years. It does not only focus on univariate models but also considers multivariate models (Palayangoda et al., 2021). These multivariate models are essential to assess the maintenance and safety of complex degrading systems in a more realistic way (Kolowrocki & Magryta, 2020).

In most of multi-component systems it is assumed that their components degrade independently. Although such as assumption allows tractable mathematical models, it remains unrealistic for system where stochastic dependence is indeed present. Examples of multi-component with dependent units can be found in lithium batteries, rubidium discharge lamps or LED lighting systems. It is necessary to study the dependencies to evaluate the overall behaviour and reliability of such systems, since, for example, availability or reliability measures are highly dependent on the number of components, even more on its dependencies.

Several mathematical models have been proposed in the literature to analyze multiple dependent degradation processes (Liu et al., 2014; Assaf et al., 2018), such as those based on copulas, multivariate joint distributions, or degradation rate interaction methods. However, these methods have

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drawbacks. Copula functions link marginal distribution of different performance characteristics (PCs) of a system, that is, the dependence structure of random variables can be characterized separately from their marginal distribution functions. They have more flexibility and have gained great attention in recent years. For example, Palayangoda and Ng (Palayangoda & Ng, 2021) provided a non-parametric modelling structure based on the Lévy process via Archimedean copula. Although copula functions address the multivariate problem, the choice of the copula function is difficult, and an inappropriate copula function will lead to incorrect assessment of reliability level (Fang et al., 2020; Mercier & Verdier, 2022). Degradation rate interaction models (DRI) assume that the degradation process of one component can affect the deterioration of other components. The failure occurs when one component, called the *influencing component*, transitions to a more severe degradation state, which increases the degradation rates of all the other components, which are the *affected components*. For example, in a wind turbine system. Bian and Gebraeel (Bian & Gebraeel, 2014) developed a DRI model for the stochastic modelling of multi-components, separating it into the self-induced degradation of the component and the degradation inducted by other components. Other authors have analyzed a two-component system degradation based on gamma processes. Rasmekomen and Parlikad (Rasmekomen & Parlikad, 2016) also applied DRI models to study the stochastic dependency between two PCs, which is characterized by a linear regression model.

Multivariate base models are usually employed because multivariate distributions easily extend results and properties from univariate ones. In these models, components of the system are dependent with a known joint distribution. The multivariate Gaussian distribution model is the most known multivariate model. With this, each degradation path is described by a marginal Gaussian distribution from the joint multivariate Gaussian distribution.

2. Related literature in bivariate models

There are recent works dealing with bivariate degradation processes, and they use the assumption of dependency (Zhang et al., 2017, 2018; Pan et al., 2011). Lawless and Crowder (Lawless & Crowder, 2004) suggested that *independent assumption should only be used if we are sure that a failure has absolutely no direct or indirect link to the likelihood of other failure mechanism, which is not usual, due to the existence of the same environmental, operational or stresses and wear factors.*

It is very common to find a two-units system (series or parallel) with two very distinguishable parts whose deterioration can be described by using a bivariate degradation model. Since it is a special case of the multivariate distribution models, the degradation paths of each unit are normally correlated. For example, fatigue cracks of two terminals of electronic devices, or machines composed of positioning accuracy and output power. In multivariate distribution models, in particular, for the bivariate case, environmental conditions are represented through a common factor influencing the degradation paths of the systems' components (PCs). Both processes share a common diffusion parameter or a common noise, for example in lithium-ion battery systems sharing a common environmental stress.

Wang et al. (Wang et al., 2021) construct a multivariate Wiener process as a baseline model, proposing two types of degradation models to characterize the time-variant covariates and imperfect maintenance effects. The difference between the two models lies in the way of capturing the influence of covariates and maintenance. Mercier et al. (Mercier et al., 2018) considered two components subject to common external shocks, which are independent of the intrinsic wear of the components, but one shock can induce the immediate failure of both components. A mathematical formulation was provided for the joint distribution of the bivariate lifetime, studying the influence of the shock parameter in it.

Sari et al. (Sari et al., 2009) studied a bivariate constant stress degradation assuming dependency between the PCs. The degradation data is modelled with a General Linear Model (GLM) and the dependency is described by copula functions. This model is applied to Light-emitting diode (LED) real data.

In (Song & Cui, 2021) a bivariate degradation model based on gamma processes is proposed, which could capture the dependency between the two degradation processes naturally and model the linear and non-linear degradation paths flexibly. The dependency between the two degradation processes is capture by a common random effect

naturally.
Xu et al. (Xu et al., 2018) proposed a bivariate degradation model based on the Wiener process with time-scale transformation. The two PCs are dependent on each other, and this dependency is described by introducing a common random effect, unlike the copula method.
Another interesting case could be a system with two PCs in which one of them works as a marker for the system. Sari et al. (Sari et al., 2009) defined a bivariate model with an underlying basic process and a second associated process built by applying a functional to the basic process. The latter process could also be a covariate variable or an imperfect observation of the basic process with a relevant role in decision making. This model is known as marker-latent model and has important applications in a wide range of fields, such as epidemiology or risk analysis. Marker processes have been usually employed in the study of HIV infected patients. Based on the works of Whitmore (Whitmore, 1998) the PhD dissertation of Conroy (Conroy, 2016) proposed a bivariate Wiener model, where marker processes are used in the study of the effect of treatments. The monotone process can represent the level of a drug in blood, temperature, or blood pressure monitoring. Regarding to risk management, fatigue crack growth in pressure vessels and in aircraft structures can be also considered. Here, the associated process incorporates the process history of some other important aspects of the main process. Similarly, Whitmore (Whitmore, 1998,) proposed a bivariate Wiener process for two components: one for represents the marker or covariate and the second determines the failure time. Failure occurs when the latent component crosses a fixed threshold level. They also consider that failure is not related deterministically to the observable marker. An extension of the model permits the construction of a composite marker from several candidates.

3. Framework of model

This work focuses on the study of a system subject to a bivariate degradation model. Two dependent degradation processes are created from three independent degradation processes using the so-called *trivariate reduction method* as follows: two dependent degradation processes are created from three independent degradation processes. These correlated processes obtained represent a system consisting of two dependent components that share a common noise. Specifically, the Wiener process is used to model this bivariate degradation.
Periodically imperfect preventive maintenance actions are performed, and the overall system deterioration is reduced by reducing the accumulated degradation level of each component from its installation in a fixed percentage given by the repair efficiency parameter. This maintenance model is known as Arithmetic Reduction of Degradation (ARD).
Under these assumptions, the Pearson correlation coefficient at time t is obtained at each inspection for two random paths of the bivariate Wiener process considering linear and non-linear drifts. The correlation as a function of the repair efficiency parameter is also studied.
Furthermore, the analytical cost model is obtained, and the expected cost rate is minimized considering the repair efficiency and the time between repairs as optimization parameters.

4. Trivariate reduction method

It is a multivariate-based method commonly used to model the correlation between several components (Lai, 1995). For the bivariate case, given three independent random variables X_1, X_2 and X_3, the dependent variables X and Y are formed as:

$$X = X_1 + X_3,$$

$$Y = X_2 + X_3,$$

where X_3 stands for the common noise between both components.
The use of this method is not new in the stochastic processes' literature, and it has extended to reliability. For example, it has been employed by Mercier et al. (Mercier et al., 2018) for modelling correlated gamma processes.

5. Wiener degradation process

The Wiener degradation process has gained attention due to its interesting mathematical properties for modelling non-monotonic degradation. This model is considered, for example, when there is a dependence between the drift parameter, which represents the degradation rate, and the diffusion parameter. In general, a component with a higher degradation rate also presents a higher dispersion, so this correlation is positive. Xu et al. (Xu et al., 2018) obtained the correlation coefficient between two Wiener processes depending on the variance and the mean degradation parameter at a certain time t, which is a non-decreasing function of t due to the fact that the remaining useful life (RUL) changes over time.

The Wiener process with linear drift (or stationary Wiener process) is considered for describing the system degradation since it is a simple model for non-monotonic degradation.

The stationary Wiener process can be generally described by:

$$X(t) = \mu t + \sigma B(t),$$

where $B(t)$ is the standard Brownian motion with mean 0 and variance t, μ is called the drift parameter and σ is the diffusion parameter.

By using the approach described in *the trivariate reduction method*, the bivariate Wiener degradation processes is given by:

$$X_1(t) = \mu_1 t + \sigma_1 B_1(t) + \sigma_0 B_0(t), \quad (1)$$

$$X_2(t) = \mu_2 t + \sigma_2 B_2(t) + \sigma_0 B_0(t), \quad (2)$$

with μ_i being the drift parameters, σ_i the diffusion parameters and $B_i(t)$ independent Brownian processes, for $i = 0,1,2$. The common noise is given by the term $\sigma_0 B_0(t)$.

The expectation and the variance of processes (1) and (2) are given by:

$$E[X_i(t)] = \mu_i t,$$

$$\text{Var}[X_i(t)] = (\sigma_0^2 + \sigma_i^2)t.$$

Now, considering a transformation in the Wiener process formulation given by (1) and (2) by including general time scales function that represent the nonlinearity of the degradation paths, we have:

$$X_1(t) = \mu_1 \Lambda(t) + \sigma_1 B_1\big(\Lambda(t)\big) + \sigma_0 B_0\big(\Lambda_0(t)\big),$$

$$X_2(t) = \mu_2 \Lambda(t) + \sigma_2 B_2\big(\Lambda(t)\big) + \sigma_0 B_0\big(\Lambda_0(t)\big).$$

Wiener process from (1) and (2) is a particular case of this model, when $\Lambda(t) = \Lambda_0(t) = t$. The expectation and variance are calculated in a similar way.

A bivariate Wiener degradation model considering imperfect maintenance is described in this chapter. The dependence between the two processes is characterized by a common noise.

Since the system is subject to periodic inspections in which imperfect maintenance is performed, the Pearson correlation coefficient is computed for the bivariate process (Mercier & Pham, 2017).

There are few works that combined imperfect maintenance with a bivariate degradation and they do not focus on the correlation coefficient between the Wiener processes.

The Arithmetic reduction of degradation model for the imperfect maintenance is next presented.

6. Arithmetic reduction of degradation maintenance model

Gaudoin and Doyen (Gaudoin & Doyen, 2006) defined the Arithmetic Reduction of Degradation (ARD) and Arithmetic Reduction of Age (ARA) models.

However, we think that the ARA models are more difficult to implement or, sometimes, unrealistic when dealing with maintenance, so that we will focus mainly on ARD models for imperfect maintenance (Nguyen et al., 2017). Their main difference with ARA models is that ARD models reduce the system deterioration without rejuvenating it, unlike the ARA model, which do rejuvenate the system age.

ARD and ARA models can be of different *order* depending on the way of reducing degradation; in particular, the ARD of infinite order removes a certain percentage of the deterioration accumulated by the system since the beginning. On the other hand, the ARD of order one also removes a percentage of the accumulated deterioration, but, in this case, since the last repair or maintenance instead of the beginning of the degradation process (Moschopoulos, 1985).

Each T time units, an imperfect preventive maintenance action consisting of Arithmetic Reduction of Degradation is performed. The degradation of each component is reduced in a ρ%, with $0 \leq \rho \leq 1$.

Figures 1 and 2 show the Arithmetic Reduction of Degradation model of order infinite (ARD(∞)) with repair efficiency parameter $\rho = 0.1$ and $\rho = 0.9$, respectively.

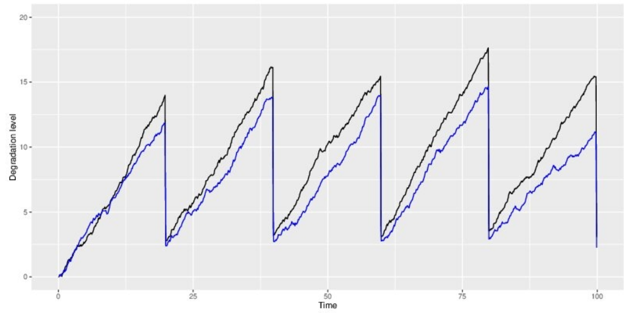

Figure 1. ARD(∞) with repair efficiency parameter $\rho = 0.9$.

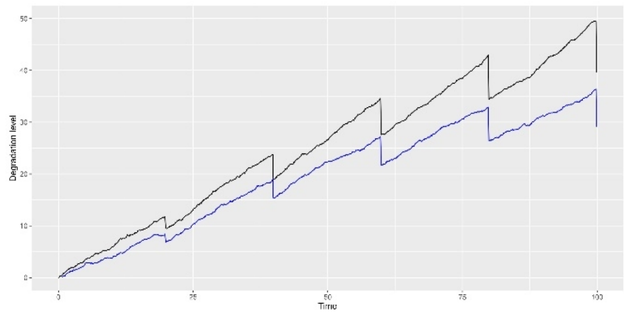

Figure 2. ARD(∞) with repair efficiency parameter $\rho = 0.1$.

For that, let $Y_1(t)$ and $Y_2(t)$ be the processes that describe the degradation of the system after a preventive maintenance action. Since the Wiener processes have independent increments, the piece-wise evolution of the maintained system $Y(t) = \{Y_i(t),\ t \geq 0\}$ is developed in the following paragraphs.

7. Evolution of maintained system

At time t, with $0 \leq t < T$, the degradation level of the i-th component is given by:

$$Y_i(t) = X_i(t).$$

The first imperfect preventive maintenance action is performed at time T and the degradation of the two components is reduced in a 100ρ%. Let us denote by T^+ the instant of time just after the first preventive maintenance action. With that, we get that

$$Y_i(T^+) = (1-\rho)X_i(T).$$

Then when $T < t < 2T$, the evolution of the degradation of the i-th component is given by:

$$Y_i(t) = Y_i(T^+) + X_i(t) - X_i(T).$$

At time 2T^-, just before the second imperfect maintenance action and at time 2T^+, just after it, the degradation of the ***i***-th component is given by:

$$Y_i(2T^-) = Y_i(T^+) + X_i(2T) - X_i(T)$$

$$Y_i(2T^+) = (1-\rho)\big(Y_i(T^+) + X_i(2T) - X_i(T)\big).$$

In general, we can say that just before the ***n***-th preventive maintenance action, the degradation of the ***i***-th component is represented by:

$$Y_i(nT^-)$$

$$= Y_i\big((n-1)T^+\big) + X_i(nT) - X_i\big((n-1)T\big),$$

$$Y_i(nT^+) = (1-\rho)\cdot$$

$$\cdot\Big(Y_i\big((n-1)T^+\big) + X_i(nT) - X_i\big((n-1)T\big)\Big).$$

After some calculations, we get that

$$Y_i(nT^+)$$

$$= \sum_{j=1}^{n}(1-\rho)^{n-j+1}\Big(X_i(jT) - X_i\big((j-1)T\big)\Big).$$

Finally, for $nT \leq t < (n+1)T$, the degradation of the i-th component at time t is

$$Y_i(t) = Y_i(nT^+) + \big(X_i(t) - X_i(nT)\big)$$

$$= \sum_{j=1}^{n}(1-\rho)^{n-j+1}\Big(X_i(jT) - X_i\big((j-1)T\big)\Big)$$

$$+\big(X_i(t) - X_i(nT)\big).$$

Now, using the additivity property of the normal distribution, the degradation after the ***n***-th preventive maintenance action is normally distribution with expectation

$$E[Y_i(nT^+)] = \mu_i \sum_{j=1}^{n}(1-\rho)^{(n-j+1)}\,\Delta\Lambda(jT)$$

and variance

$$\mathrm{Var}[Y_i(nT^+)]$$

$$= (\sigma_0^2 + \sigma_i^2) \sum_{j=1}^{n} (1-\rho)^{2(n-j+1)} \, \Delta\Lambda(jT),$$

where $\Delta\Lambda(jT)$ denotes the increments of the function Λ, that is,

$$\Delta\Lambda(jT) - \Lambda((j-1)T).$$

Figures 3 and 4 represent the degradation processes of components 1 and 2 (in blue and red colours, respectively) and the baseline degradation ARD(∞) processes of each one without repairs (in black). The periodic time to perform ARD equals T = 30 time units and it is represented in the above line. The bivariate Wiener process is obtained with 10000 divisions over 100 time units, represented in the X-axis. The dashed lines show the times at which the ARD is performed. Figure 5 shows a realization of the bivariate degradation process applying the ARD model.

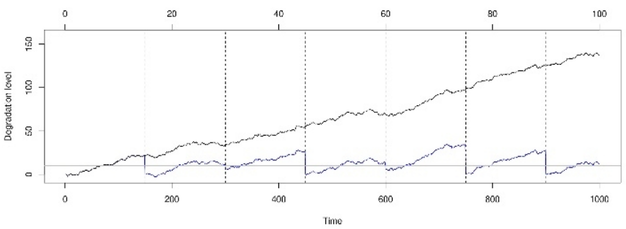

Figure 3. ARD(∞) of the first component with ρ = 0.5.

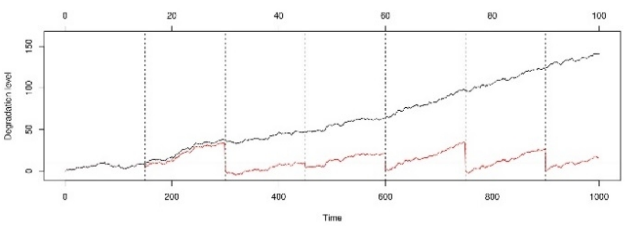

Figure 4. ARD(∞) of the second component with ρ = 0.5.

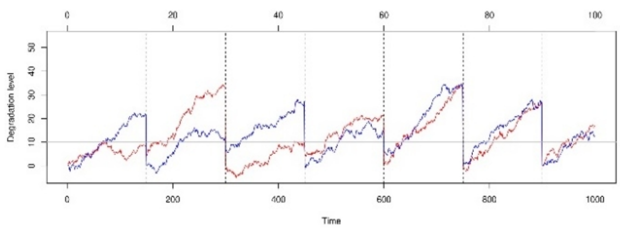

Figure 5. ARD(∞) realization of both components with ρ = 0.5.

Since it is often difficult to make a choice with the order and repair efficiency in ARD models, stochastic comparisons (Muller, 2001) are useful to be made between the different models and were proposed by Mercier and Castro (Mercier & Castro, 2019) for this purpose.

8. Correlation study

The Pearson's correlation coefficient is the most common method for measuring a linear correlation between two sets of data or two processes (Chen & Zhao, 2020; Kong et al., 2022). It is a normalized measure of the covariance. In the case of stochastic processes depending on time, given a pair processes X(t) and Y(t), this coefficient has the following expression:

$$\theta = \frac{\mathrm{Cov}(X(t),Y(t))}{\sqrt{\mathrm{Var}(X(t))Var(Y(t))}},$$

where

$$\mathrm{Cov}(X(t),Y(t))$$

$$= \mathrm{E}[X(t) - E[X(t)]][Y - E[Y(t)]].$$

With that, the Pearson correlation coefficient between $X_1(t)$ and $X_2(t)$ is calculated as:

$$\theta = \frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma_1^2}\sqrt{\sigma_0^2+\sigma_2^2}}. \quad (3)$$

Notice that this coefficient θ given by (3) does not depend on time. It is equal to 1 if $\sigma_1^2 = \sigma_2^2 = 0$, and the correlation is stronger when the common noise is dominant, that is, when $\sigma_0^2 \gg \sigma_i^2$. On the other hand, if the common noise is negligible, the two degradation paths of the bivariate Wiener process evolve almost independently.
In the case of non-linear drift function $\Lambda(t)$ and $\Lambda_0(t)$, the Pearson correlation coefficient between $X_1(t)$ and $X_2(t)$ is given by:

$$\theta(t) = \frac{\mathrm{Cov}(X_1(t),X_2(t))}{\sqrt{\mathrm{Var}(X_1(t))}\sqrt{\mathrm{Var}(X_2(t))}}$$

$$= \frac{\sigma_0^2\Lambda_0(t)}{\sqrt{\sigma_0^2\Lambda_0(t)+\sigma_1^2\Lambda(t)}\sqrt{\sigma_0^2\Lambda_0(t)+\sigma_2^2\Lambda(t)}}$$

$$= \frac{1}{\sqrt{1 + \sigma_1^2/\sigma_0^2 h(t)}\sqrt{1 + \sigma_1^2/\sigma_0^2 h(t)}},$$

where $h(t) = \Lambda(t)/\Lambda_0(t)$ is the ratio of the two general time scales. We can deduce from this formula that:

- the Pearson correlation coefficient decreases in t when $h(t)$ increases in t. This is due to the fact that if $h(t)$ increases in t, then $\Lambda(t)$ is dominant with respect to $\Lambda_0(t)$ and the Pearson correlation decreases,
- the Pearson correlation coefficient increases when σ_0^2 increases, since σ_0^2 is linked to the common part of both degrading components,
- the Pearson correlation coefficient decreases when σ_i^2 increases. It is clear since σ_i is linked to the independent part of the two degrading processes,
- if $\Lambda(t) = \Lambda_0$, then the correlation coefficient $\theta(t)$ is constant and equal to the one obtained in (3).

Some numerical examples related to the mean, variance and the Pearson correlation coefficient of the above processes are next presented.

9. Numerical examples

Figures 6 and 7 show the mean and variance of the maintained process subject to ARD for the power law case $\Lambda(t) = \Lambda_0(t) = t^b$ with different values of the parameter b. The mean is increasing when $b > 1$ and decreasing when $b < 1$. Notice that the variance is obviously always increasing.

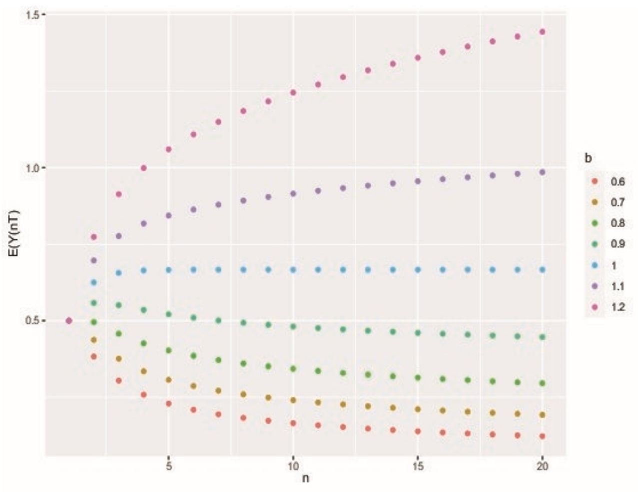

Figure 6. Mean of ARD(∞) with different values for b and $\rho = 0.5$.

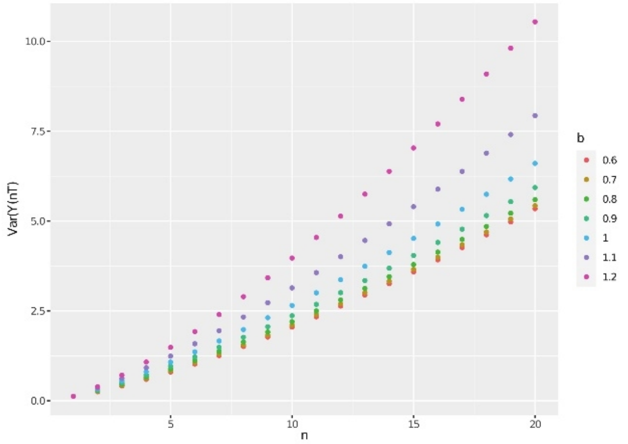

Figure 7. Variance of the ARD(∞) with different values for b and $\rho = 0.5$.

The Pearson correlation coefficient is calculated in Figures 8 and 9 at each maintenance time with different values of the repair efficiency for the power law case $\Lambda(t) = t^{b_1}$ and $\Lambda_0(t) = t^{b_2}$. Notice that it is increasing when $b_1 > b_2$ and decreasing when $b_1 < b_2$.

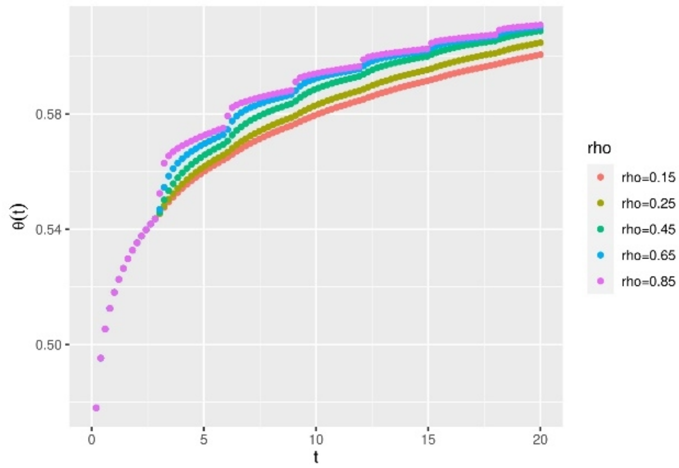

Figure 8. Pearson correlation coefficient for power law case with $b_1 > b_2$.

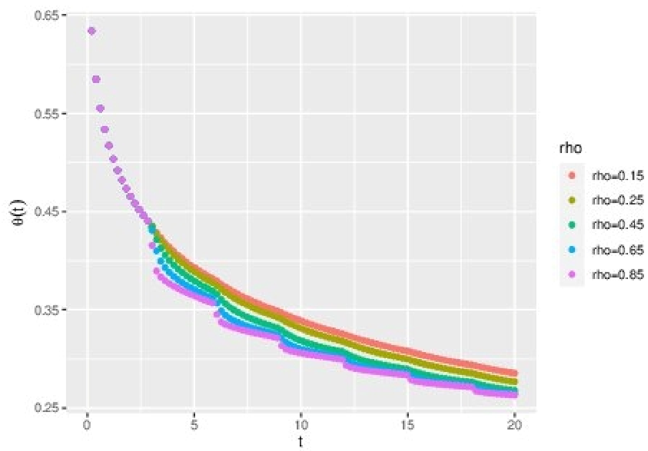

Figure 9. Pearson correlation coefficient for power law case with $b_1 < b_2$.

10. Maintenance and cost model

Barlow and Proschan (Barlow & Proschan, 1975) proposed different classical maintenance models in the 1960s, first for single-component systems. These approaches were far from reflecting the real behaviour and complexity of industrial systems. In (Cho & Parlar, 1991), the existing models were improved and research on maintenance optimization progressed by developing strategies for multi-component systems. Nowadays, research on maintenance optimization takes into account both the characteristics of the system and the nature of the implemented maintenance operations.
Simulations are generally used to assess the performance of a maintenance policy in multi-component systems. Maintenance policies no longer consist only of complete replacement strategies, but also take into account the condition of the system and system repairs are carried out.
The deterioration level of the system is usually not known, unless it is a monitored system, which involves high costs. It may be that the system can only be observed by stopping it, or may not even be possible to observe the state of deterioration, but only whether or not the system is still functioning. To overcome these drawbacks, the state of the system is evaluated periodically at pre-established times, called inspection times or repair times. Imperfect maintenance actions consisting on ARD repairs are performed at those times.
Each maintenance task implies a certain cost, so a good maintenance policy is the one that balances the cost of performing frequent preventive maintenance with that of performing infrequent, but more costly, corrective maintenance, which happens when the system deterioration reaches a certain threshold. After a preventive imperfect repair, the system begins to operate again.
A realization of the maintenance and repair process is shown in Figure 10. The threshold value is represented by the red line and the deterioration processes are coloured in blue and black.

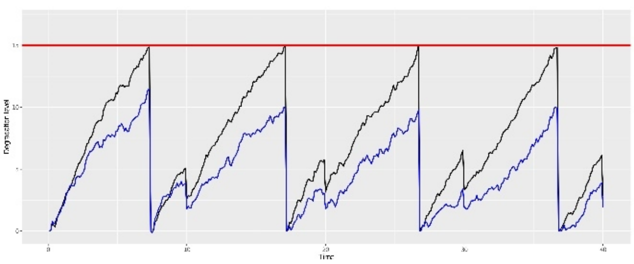

Figure 10. Realization of the ARD maintenance model.

In this case, the following assumptions regarding the maintenance strategy are assumed:

- an imperfect preventive maintenance policy consisting on periodic repairs with ARD(∞) is performed while the system is not failed. With that, the accumulated degradation of each component is reduced in a 100ρ%, with $0 < \rho < 1$. This preventive maintenance cost is denoted by $C_I(\rho)$ monetary units, which depends on the repair efficiency parameter,
- a corrective maintenance is performed when the system is failed in an inspection time. This corrective action implies the system replacement by a new one (both components are replaced even) with a cost of C_c monetary units,
- the system is considered to be failed when one of the degrading components exceeds a certain failure threshold, which has the same value for both components and it is denoted by L,
- failure are only detected at periodic inspection times, so a system downtime cost of C_d monetary units per time unit is implemented,
- the maintenance duration is assumed to be negligible.

Let

$$C(T, \rho) = \frac{E[C(T_{max})]}{T_{max}}$$

be the maintenance cost function in the finite time horizon, where T_{max} is the maximum time that the system is considered to operate.
Then, the decisions variables considered for the optimization of the maintenance policy are the time between imperfect repairs T and the repair efficiency parameter \rho. The optimal maintenance policy is given by

$$C\left(T_{opt}, \rho_{opt}\right) = \inf\{ C_{T_{max}}(T, \rho),$$

$$0 < T < T_{max}, 0 \leq \rho \leq 1.$$

The expected cost rate in asymptotic approaches, where the system is supposed to operate with an infinite life cycle, has been widely employed in the literature. However, this circumstance does not occur in practice, therefore some authors recommend studying maintenance policies based on finite life cycles. These policies are more realistic than the ones based on infinite life cycles, but they have some added handicaps since their analytical and computational treatment is more challenging.

The goal behind the modelling of the system state is to find the optimal maintenance policy for such a system. That is, identify the maintenance policy that minimizes the expected cost rate previously developed. The main difficulty lies in obtaining this long-run average cost per time unit with analytical methods based on the statistical properties of the system behaviour. Sometimes it is not possible to obtain it analytically, so it is necessary to resort to simulations.

Let R be the replacement time of the system, being $R = (\lfloor Tf/T \rfloor + 1)T$ and where T_f is the time to the system failure. The expected cost in a replacement cycle, that is, in the time between successive replacement of the system is given by

$$C_I(\rho)\mathrm{E}(\lfloor Tf/T \rfloor) + C_d\mathrm{E}\left(R - T_f\right) + C_c$$

where $\lfloor T\mathrm{f}/T \rfloor$ is the floor function of T_f/T.

Some numerical examples are next given to illustrate the proposed model. The expected cost rate is calculated using EQ. and Monte Carlo simulation. The maintenance policy is optimized by obtaining the optimal values for T and ρ.

For the bivariate Wiener degradation process, following Eqs. (1) and (2), the following parameters are considered:

$$\sigma_0 = 0.5, \quad \sigma_1 = 1, \qquad \sigma_2 = 1.5,$$

$$\mu_1 = \mu_2 = 2, \; \Lambda(t) = \Lambda_0(t) = t.$$

The threshold value is $L = 20$. The following sequence of costs is also assumed:

$$C_c = 120 m.u., \quad C_d = 20 m.u.,$$

$$C_I(\rho) = 50\rho \quad m.u.$$

For the optimization, a grid of size 11 is considered for ρ between (0,1) and a grid of 10 points is considered in (0,10) for T.

The results obtained are shown in Figures 11 and 12. The optimal values are $T_{\text{opt}} = 6$ and $\rho_{\text{opt}} = 0.8$, with an expected cost rate of 45.87 m.u. per t.u.

11. Conclusion

A system with two dependent components subject to bivariate Wiener degradation is analyzed in this chapter. The bivariate process is obtained with the

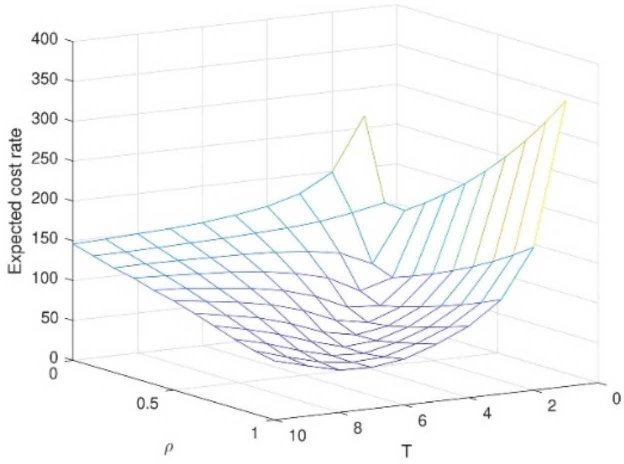

Figure 11. Cost model for bivariate ARD process.

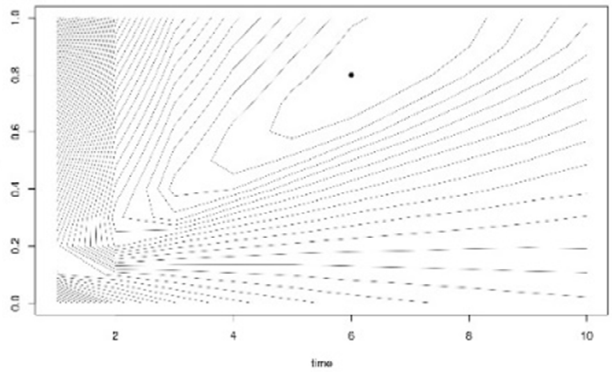

Figure 12. Contour plot for bivariate ARD process.

trivariate reduction method and an imperfect maintenance policy is implemented, in particular, an Arithmetic Reduction of Degradation of order infinite model. The Pearson correlation coefficient is analyzed for the two degradation processes with the implementation of the ARD imperfect maintenance. The cost model for the proposed maintenance is also developed and optimized considering the repair efficiency and the time between repairs as optimization parameters.

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