sciendo
**Control and Cybernetics**
vol. **52** (2023) No. 2
pages: 129-180
DOI: 10.2478/candc-2023-0035

# Network optimality conditions*

by

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**Abstract:** Optimality conditions for optimal control problems arising in network modeling are discussed. We confine ourselves to the steady state network models. Therefore, we consider only control systems described by ordinary differential equations. First, we derive optimality conditions for the nonlinear problem for a single beam. These conditions are formulated in terms of the *local* Pontryagin maximum principle and the matrix Riccati equation. Then, the optimality conditions for the control problem for networks posed on an arbitrary planar graph are discussed. This problem has a set of independent variables $x_i$ varying within their intervals $[0, l_i]$, associated with the corresponding beams at network edges. The lengths $l_i$ of intervals are not specified and must be determined. So, the optimization problem is non-standard, it is a combination of control and design of networks. However, using a linear change of the independent variables, it can be reduced to a standard one, and we show this. Two simple numerical examples for the single-beam problem are considered.

**Keywords:** network, optimal control problem, weak local minimum, Pontryagin's maximum principle, critical cone, quadratic form, second order optimality conditions, Riccati equation

*Submitted: October 2023; Accepted: December 2023.

# 1. Introduction

## 1.1. Motivation

We are interested in the optimum design of optimal control systems for networks. We restrict ourselves in this study to the steady state nonlinear network models. First, a single element is considered. Then, a network with the star graph is studied. The geometric domain for the network is a star graph for the sake of simplicity.

Optimization problems for steady state models are important for networks that enjoy some specific features as regards the control problems. Roughly speaking, the control strategy with long time horizons includes two parts. The first part is constituted by an exact controllability problem for the fixed time interval with the aim of attaining some steady state solution, which is then followed by the stabilization of the steady state solution. The cost functional is chosen of tracking type with some regularization components for the state and the control, if necessary to assure *the turnpike property* for the control problems under consideration. The steady state solution could be selected by the optimization of the steady state network model. In other words, it turns out that for some control problems with nonlinear state equations, the so-called turnpike property occurs. This means that the optimal control and optimum design for a steady state system can be used for the evolution system in the specific case of the cost. Therefore, our analysis of the optimality conditions is performed for the nonlinear steady state models. Such an analysis can be useful for the real systems that are governed by the networks of Nonlinear Partial Differential Equations.

The practical examples for our framework include, e.g., the Gas and Hydrogen Distribution (GHD) Networks, see Gugat and Herty (2011, 2022), and the Geometrically Exact Beam (GEB) Networks, which lead to the Intrinsic Geometrically Exact Beam (IGEB) network models, see Rodriguez and Leugering (2020) and Leugering et al. (2021). The GHD Networks are modeled by quasilinear hyperbolic systems. The IGEB Networks are governed by semilinear hyperbolic systems under some assumptions on the transformation of GEB models. The steady state equations for two types of networks are given by ODEs.

The quadratic tracking type cost, depending on the specific solution to the steady state equation, is considered for the optimal control problem. The optimal control cost is augmented by an auxiliary term depending on design, usually in a finite-dimensional space, which models the cost of manufacturing the networks. We present an example of the elastic networks governed by the static GEB state equations.

### 1.2. Model for a single beam

The networks of elastic beams are of primal importance for applications that we have in mind. Thus, we describe in detail the nonlinear models of beams which lead to semilinear state equations for static and evolution problems. The optimal steady state can be determined by solving the control problem for static model. We are looking for optimal control and optimum design in the framework of the systems which enjoy the turnpike property.

The mathematical framework describing geometrically exact beams (GEB) focuses on the position of the beam's centerline and the orientation of its cross sections with a fixed coordinate system, denoted as $\{e_j\}_{j=1}^3$ (representing the standard basis of $\mathbb{R}^3$). In the GEB context, the system state is denoted as $(\mathbf{p}, \mathbf{R})$, expressed in the basis $\{e_j\}_{j=1}^3$. This state comprises the position of the centerline, denoted as $\mathbf{p}(x,t) \in \mathbb{R}^3$, and the orientation of the cross sections, represented by the columns $\left\{\mathbf{b}^j\right\}_{j=1}^3$ of the rotation matrix $\mathbf{R}(x,t) \in \mathrm{SO}(3)$. Here, SO(3) denotes the special orthogonal group, which comprises unitary real matrices of size 3 with a determinant equal to 1. For visual reference, we could refer to Fig. 1. The figure illustrates three pivotal states of a deformable beam: the unchanged reference beam; the initial beam characterized by a curvature described as $\Upsilon_c = \operatorname{vec}\left(R^\top \frac{\mathrm{d}}{\mathrm{d}x} R\right)$, where $R = \left[\begin{array}{ccc} b^1 & b^2 & b^3 \end{array}\right]$; and the beam at time $t$, represented by the state variables $\mathbf{p}$ and $\mathbf{R} = \left[\begin{array}{ccc} \mathbf{b}^1 & \mathbf{b}^2 & \mathbf{b}^3 \end{array}\right]$.

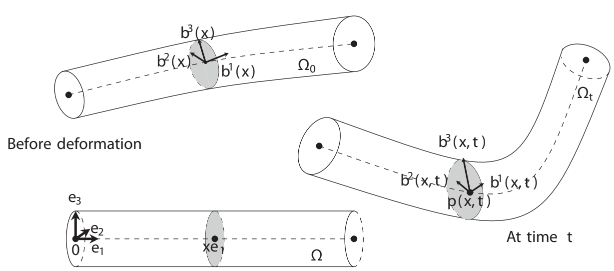


Figure 1. The straight reference beam (bottom), the beam before deformation (upper left), and the beam at time $t$ (upper right)

For a beam with a length $l > 0$, positioned within the domain $(0,l) \times (0,T)$, the governing system is defined as follows:

$$\left[\begin{array}{cc} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{array}\right] \left[\left[\begin{array}{cc} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{array}\right] \mathbf{M} v\right] = \left[\begin{array}{cc} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{array}\right] \left[\left[\begin{array}{cc} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{array}\right] z\right] + \left[\begin{array}{c} \bar{\phi} \\ \bar{\psi} \end{array}\right], \quad (1)$$

given external forces and moments $\bar{\phi}(x,t), \bar{\psi}(x,t) \in \mathbb{R}^3$, the mass matrix $\mathbf{M}(x) \in \mathbb{S}^6_{++}$(the set of positive definite symmetric matrices), the flexibility (or compliance) matrix $\mathbf{C}(x) \in \mathbb{S}^6_{++}$, and the curvature before deformation $\Upsilon_c(x)$, where $v, z$ depend on $(\mathbf{p}, \mathbf{R})$:

$$v = \begin{bmatrix} \mathbf{R}^\top \partial_t \mathbf{p} \\ \operatorname{vec}\left(\mathbf{R}^\top \partial_t \mathbf{R}\right) \end{bmatrix}, \quad s = \begin{bmatrix} \mathbf{R}^\top \partial_x \mathbf{p} - e_1 \\ \operatorname{vec}\left(\mathbf{R}^\top \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix}, \quad z = \mathbf{C}^{-1} s. \tag{2}$$

Here, for any $u \in \mathbb{R}^3$, the skew-symmetric matrix $\widehat{u}$ is defined as follows:

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

Consider the Intrinsic Geometrically Exact Beam (IGEB) model for a single beam. The governing semilinear system consists of twelve equations. The state variable is denoted as

$$y = \begin{bmatrix} v \\ z \end{bmatrix}$$

expressed on a moving basis. Here, $v(x,t) \in \mathbb{R}^6$ represents linear and angular velocities, and $z(x,t) \in \mathbb{R}^6$ represents internal forces and moments. We use $v_f$, $z_f$, $v_l$, and $z_l$ to denote the first and last three components of $v$ and $z$, respectively. The notation $\bar{\Phi}(x,t)$ and $\bar{\Psi}(x,t) \in \mathbb{R}^3$ is employed for external forces and moments expressed in the moving basis. Within the domain $(0,l) \times (0,T)$, the governing system of IGEB reads:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \partial_t y - \begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix} \partial_x y - \mathcal{A} y = -\mathcal{B}(v,z) \begin{bmatrix} \mathbf{M} v \\ \mathbf{C} z \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ \bar{\Psi} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{3}$$

where

$$\mathcal{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \widehat{\Upsilon}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{e}_1 & \widehat{\Upsilon}_c \\ \widehat{\Upsilon}_c & \widehat{e}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widehat{\Upsilon}_c & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathcal{B}(v,z) = \begin{bmatrix} \widehat{v}_l & \mathbf{0} & \mathbf{0} & \widehat{z}_f \\ \widehat{v}_f & \widehat{v}_l & \widehat{z}_f & \widehat{z}_l \\ \mathbf{0} & \mathbf{0} & \widehat{v}_l & \widehat{v}_f \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{v}_l \end{bmatrix}, \tag{4}$$

and $\mathbb{I}_6$ is the identity matrix having the size $6 \times 6$. The system (3) is semilinear, because of the presence on the right-hand side of the quadratic terms

$$(v,z) \mapsto \mathcal{B}(v,z) \begin{bmatrix} \mathbf{M} v \\ \mathbf{C} z \end{bmatrix}.$$

We introduce the matrix $\mathbf{E}(x) \in \mathbb{R}^{6 \times 6}$, which contains information about curvature and twist at rest, and the matrix $Q^{\mathcal{P}}(x) \in \mathbb{S}^{12}_{++}$, defined by

$$\mathbf{E} = \begin{bmatrix} \widehat{\Upsilon}_c & \mathbf{0} \\ \widehat{e}_1 & \widehat{\Upsilon}_c \end{bmatrix}, \quad Q^{\mathcal{P}} = \operatorname{diag}(\mathbf{M}, \mathbf{C}).$$

We present a simple example, that of a single beam clamped at $x = 0$ and with the zero velocities at $x = l$. The IGEB system with boundary conditions reads

$$\begin{cases} \partial_t y + \bar{A}(x)\partial_x y + \bar{B}(x)y = \bar{g}(x,y) & \text{in } (0,l)\times(0,T) \\ v(0,t) = 0 & \text{for } t \in (0,T) \\ z(l,t) = 0 & \text{for } t \in (0,T) \\ y(x,0) = y^0(x) & \text{for } x \in (0,l) \end{cases} \tag{5}$$

where the coefficients $\bar{A}, \bar{B}$ and the source $\bar{g}$ depend on $\mathbf{M}, \mathbf{C}$ and $\mathbf{R}$, and $y^0(x)$ is the initial velocity. The governing system is derived by left-multiplying Eq. (3) by the inverse of $Q^{\mathcal{P}}$. Specifically, the functions $\bar{A}(x)$ and $\bar{B}(x)$ are defined over the interval $[0,l]$ and map to $\mathbb{R}^{12\times 12}$,

$$\bar{A} = -\left(Q^{\mathcal{P}}\right)^{-1}\begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix}, \quad \bar{B} = \left(Q^{\mathcal{P}}\right)^{-1}\begin{bmatrix} \mathbf{0} & -\mathbf{E} \\ \mathbf{E}^\top & \mathbf{0} \end{bmatrix}. \tag{6}$$

The function $\bar{g} : [0,l]\times\mathbb{R}^{12} \to \mathbb{R}^{12}$ is defined by

$$\bar{g}(x,u) = Q^{\mathcal{P}}(x)^{-1}\mathcal{G}(u)Q^{\mathcal{P}}(x)u$$

for all $x \in [0,l]$ and $u = \left(u_1^\top, u_2^\top, u_3^\top, u_4^\top\right)^\top \in \mathbb{R}^{12}$ with each $u_j \in \mathbb{R}^3$, where the map $\overline{\mathcal{G}}$ is defined by

$$\mathcal{G}(u) = -\begin{bmatrix} \widehat{u}_2 & \mathbf{0} & \mathbf{0} & \widehat{u}_3 \\ \widehat{u}_1 & \widehat{u}_2 & \widehat{u}_3 & \widehat{u}_4 \\ \mathbf{0} & \mathbf{0} & \widehat{u}_2 & \widehat{u}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{u}_2 \end{bmatrix}.$$

For the static problem, the nonlinear transformation results in $v = 0$. Denote $\mathbf{L}(z) := \begin{bmatrix} 0 & \widehat{z}_f \\ \widehat{z}_f & \widehat{z}_l \end{bmatrix}$ and we have a steady state system:

$$\begin{cases} -\partial_x z = \mathbf{E}(x)z - \mathbf{L}(z)\mathbf{C}z + \begin{bmatrix} f(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \text{in } (0,l) \\ z(l) = \mathbf{0} \end{cases} \tag{7}$$

where $f(x)$ is control.

Remark 1 *We are going to present numerical examples for the evolution state equation in a separate paper. We are interested in the steady state models as* *well as dynamic models. The framework and the analysis of optimization problems for the steady state models are presented in this paper. The steady state problems for one edge of the network are considered in Section 3. In the simplest case, the model problem for the semilinear state equation for the steady state of a single edge of the network can be considered in the form of the semilinear ordinary differential equation* (14). *We refer the reader to Section 4 for elementary numerical examples.*

The paper is organized as follows. In Section 2, we recall the formulations of known first-order necessary conditions and second-order sufficient conditions for a weak local minimum for problems of optimal control of ordinary differential equations.

In Section 3, we discuss a problem of optimal control of a single beam that arises in network modeling and obtain optimality conditions for a weak local minimum in this problem. An elementary numerical example of a single beam problem is considered in Section 4.

Section 5 studies a general optimal control problem with $m$ beams that arises in network modeling, which is not a standard optimal control problem. The characteristic of our setting is the optimum design part of the cost, which allows to include the variable geometry of network in our analysis of the optimal control and at the same time of shape optimum design. With the help of a change of independent variables, we transform such a complex problem to the standard one in the reference geometry, and in the latter, we use the known optimality conditions. We then rewrite these conditions in terms of the original problem. In shape optimization this is a standard approach, which is called the material derivative method, see Sokołowski and Zolésio (1992) in the reference domain setting, in contrast to the shape derivative method in the variable domain setting. Note that a similar technique was used by A.V. Dmitruk and A.M. Kaganovich (2008, 2011) with slightly different goals. An example ends Section 5.

## 2. Preliminaries

### 2.1. Formulation of the first-order necessary optimality conditions for an autonomous problem on the interval $[0,1]$

Consider the following autonomous problem of optimal control:

$$\left.\begin{aligned}&J(x,u)=\textstyle\int_0^1 F(x(t),u(t))\,\mathrm{d}t\to\min,\\&\dot x(t)=f(x(t),u(t))\ \forall t\in[0,1],\ \kappa(x(0),x(1))\le 0,\ K(x(0),x(1))=0.\end{aligned}\right\}\tag{8}$$

Here $x:[0,1]\to\mathbb{R}^n$ is a continuously differentiable function, $u:[0,1]\to\mathbb{R}^m$ is a continuous function, and $\dot{x} = \mathrm{d}x/\mathrm{d}t$. Hence, the problem is considered in the space

$$\mathcal{W} := C^1([0,1],\mathbb{R}^n) \times C([0,1],\mathbb{R}^m).$$

A local minimum in this space is called a *weak local minimum*. We call $x$ the *state variable* and $u$ the *control*. All data $F : \mathbb{R}^{n+m} \to \mathbb{R}$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^k$, $K : \mathbb{R}^{2n} \to \mathbb{R}^s$ are assumed to be continuously differentiable.

We say that $w = (x,u) \in \mathcal{W}$ is an *admissible point* if it satisfies all the constraints of the problem. For brevity, we set $\xi = (x(0), x(1))$.

Let us formulate the first-order necessary optimality conditions for this problem. We introduce the Hamiltonian (Pontryagin) function and the endpoint Lagrange function:

$$H(x,u,p,\alpha_0) = pf(x,u) + \alpha_0 F(x,u), \quad L = \alpha\kappa(\xi) + \beta K(\xi),$$

where $p$, $\alpha$, $\beta$ are row vectors of the same dimensions as the column vectors $f$, $\kappa$, $K$, respectively, $\alpha_0$ is a number. By definition, $pf = \sum\limits_{i=1}^{n} p_i f_i$, where $p_i$ and $f_i$ are the components of the vectors $p$ and $f$, respectively.

Denote by $\mathbb{R}^{n\top}$ the space of row vectors of dimension $n$.

By $F_x$ and $F_u$ we denote the partial derivatives $\partial F/\partial x$ and $\partial F/\partial u$, respectively, considered as row vectors, i.e. $F_x \in \mathbb{R}^{n\top}$, $F_u \in \mathbb{R}^{m\top}$. Similarly, $f_x := \partial f/\partial x$ and $f_u := \partial f/\partial u$, which are matrices of order $n \times n$ and $n \times m$, respectively. Note that $H_x \in \mathbb{R}^{n\top}$, $H_u \in \mathbb{R}^{m\top}$ are row vectors, and $H_p = f \in \mathbb{R}^n$ is a column vector.

We say that at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$ the *local minimum principle* (LMP) is satisfied if there exists a continuously differentiable function $p : [0,1] \to \mathbb{R}^{n\top}$, a number $\alpha_0$, and row vectors $\alpha \in \mathbb{R}^{k\top}$, $\beta \in \mathbb{R}^{s\top}$ such that the following system of optimality conditions holds:

(a) the nonnegativity conditions: $\alpha_0 \geq 0$, $\alpha \geq 0$,
(b) the nontriviality condition: $\alpha_0 + |\alpha| + |\beta| > 0$,
(c) the complementary slackness condition: $\alpha\kappa(\xi^0) = 0$, where $\xi^0 = (x^0(0), x^0(1))$,
(d) the adjoint equation: $-\dot{p}(t) = H_x(w^0(t), p(t), \alpha_0) \ \ \forall\, t \in [0,1]$,
(e) the transversality conditions: $(-p(0), p(1)) = L_\xi(\xi^0, \alpha, \beta)$,
(f) the stationarity of the Hamiltonian with respect to the control: $H_u(w^0(t), p(t), \alpha_0) = 0 \ \ \forall t \in [0,1]$.

From the equation $\dot{x}^0 = f(w^0)$ and conditions (d) and (f) there follows

(g) the *condition for the Hamiltonian to be constant*: there exists a constant $c_H$ such that $H(w^0(t), p(t), \alpha_0) = c_H \ \ \forall\, t \in [0,1]$.

Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} H(w^0(t), p(t), \alpha_0) = H_x(w^0(t), p(t), \alpha_0)\dot{x}^0(t) + H_u(w^0(t), p(t), \alpha_0)$$

$$\dot{u}^0(t) + \dot{p}(t) H_p(w^0(t), p(t), \alpha_0) = -\dot{p}(t)\dot{x}^0(t) + \dot{p}(t)\dot{x}^0(t) = 0. \qquad \square$$

The following theorem is well known, see, for example, Alekseyev et al.(1979), Dubovitskii and Milyutin (1965), Milyutin and Osmolovskii (1998), Milyutin et al. (2004), Pontryagin et al. (1961).

THEOREM 2.1 *If $w^0$ is a weak local minimum in problem (8), then it satisfies the LMP.*

The case, when the cost Lagrange multiplier $\alpha_0$ is not equal to zero (for any quadruple $(\alpha_0, \alpha, \beta, p(\cdot))$ satisfying the LMP conditions), is called *normal*. Let us formulate a condition that guarantees the normal case for the point $w^0$. Introduce a *set of active indices*

$$I = \{i \in \{1, \dots, k\} : \kappa_i(\xi^0) = 0\}.$$

We say that the *Mangasarian-Fromovitz constraint qualification* (MFCQ) is satisfied for the point $w^0 = (x^0, u^0) \in \mathcal{W}$ if there exists a pair $(x, u) \in \mathcal{W}$ such that

$$\kappa_i'(\xi^0)\xi < 0 \quad \forall\, i \in I, \quad K'(\xi^0)\xi = 0, \quad \xi = (x(0), x(1)), \quad \dot{x} = f'(w^0)w,$$

where, for example, $f'(w^0)w = f_x(w^0)x + f_u(w^0)u$. In this case, in the LMP conditions, we can set $\alpha_0 = 1$.

## 2.2. Formulation of the second-order sufficient optimality conditions for an autonomous problem on the interval $[0,1]$

Consider again the autonomous problem (8). Now we suppose that all data $F$, $f$, $\kappa$, $K$ are twice continuously differentiable.

Let us formulate sufficient second-order conditions for a weak local minimum at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$, satisfying necessary first-order conditions with the adjoint variable $p$ and Lagrange multipliers $\alpha_0$, $\alpha$, $\beta$. Define the *critical cone* at the point $w^0$:

$$\begin{aligned} \mathcal{C} := \big\{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad & \delta\dot{x}(t) = f'(w^0(t))\delta w(t),\ K'(\xi^0)\delta\xi = 0, \\ & \kappa_i'(\xi^0)\delta\xi \le 0,\ i \in I,\ \textstyle\int_0^1 F'(w^0(t))\delta w(t)\,\mathrm{d}t \le 0 \big\}, \end{aligned}$$

where $\delta\xi = (\delta x(0), \delta x(1))$. The equation $\delta\dot{x} = f'(w^0)\delta w$ is called the *equation in variations*.

In the normal case, where $\alpha_0 = 1$, the inequality $\int_0^1 F'(w^0(t))\delta w(t)\,\mathrm{d}t \le 0$ can be excluded from the definition of the critical cone, but then we must add the equalities $\alpha_i\kappa_i'(\xi^0)\delta\xi = 0$, $i \in I$. Thus, in the normal case, we have

$$\begin{aligned}\mathcal{C} := \{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad & \delta\dot{x}(t) = f'(w^0(t))\delta w(t),\ K'(\xi^0)\delta\xi = 0,\\ & \kappa_i'(\xi^0)\delta\xi \le 0,\ i \in I,\ \alpha_i\kappa_i'(\xi^0)\delta\xi = 0,\ i \in I\}.\end{aligned}$$

This is easy to prove using the LMP conditions. Later, in Section 3, where we consider the normal case, we will use this critical cone representation.

Define the *strengthened Legendre condition*: there exists $c_L > 0$ such that for all $t \in [0,1]$ we have $\langle H_{uu}(w^0(t), p(t), \alpha_0)u, u\rangle \ge c_L|u|^2 \quad \forall\, u \in \mathbb{R}^m$. Here $H_{uu} = \partial^2 H/\partial u^2$ stands for the second partial derivative of $H$ with respect to the control.

Next, define a *quadratic form*:

$$2\Omega(\delta w) = \langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi\rangle + \int_0^1 \langle H_{ww}(w^0(t), p(t), \alpha_0)\delta w(t), \delta w(t)\rangle\,\mathrm{d}t.$$

Note that if $\kappa(\xi)$ and $K(\xi)$ are affine functions, then $L = \alpha\kappa + \beta K$ is also an affine function of $\xi$, and therefore, $L_{\xi\xi} = 0$. In this case, the endpoint term $\langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi\rangle$ vanishes, and $\Omega$ reduces to the integral only.

The following theorem holds, see, for example, Maurer and Osmolovskii (2012).

Theorem 2.2 *Assume that for the point $w^0$*

*(a) the strengthened Legendre condition is satisfied,*
*(b) there exists a constant $c_\Omega > 0$ such that $\Omega(\delta w) \ge c_\Omega(|\delta x(0)|^2 + \|\delta u\|_2^2)$ $\forall\, \delta w \in \mathcal{C}$.*

*Then, there are $c > 0$ and $\varepsilon > 0$ such that $J(w) - J(w^0) \ge c\Big(\|x - x^0\|_\infty^2 + \int_0^1 |u(t) - u^0(t)|^2\,\mathrm{d}t\Big)$ for all admissible $w = (x,u)$ such that $\|w - w^0\|_\infty < \varepsilon$, and hence $w^0$ is a weak local minimum in the problem.*

Remark 2 *Since $\Omega(-\delta w) = \Omega(\delta w)$ for all $\delta w \in \mathcal{W}$, condition (b) in this theorem is equivalent to the condition $\Omega(\delta w) \ge c_\Omega(|\delta x(0)|^2 + \|\delta u\|_2^2)$ $\forall\, \delta w \in \Sigma$, where $\Sigma = \mathcal{C} \cup (-\mathcal{C})$. In particular, let $\mathcal{C} = \{\delta w \in \Gamma,\ l(\delta w) \le 0\}$, where $\Gamma$ is a subspace, and $l$ is a linear functional. Then, obviously, $\Sigma = \Gamma$.*

### 2.3. Matrix Riccati equation

Now we consider a sufficient condition for positive definiteness of the quadratic form $\Omega$ on the subspace $\Gamma$. Assume that $\Gamma$ has the form:

$$\Gamma = \Big\{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad \delta\dot{x} = f_x(w^0)\delta x + f_u(w^0)\delta u, \quad \mathcal{E}\delta\xi = 0\Big\},$$

where $\mathcal{E}$ is a constant matrix, $\delta\xi = (\delta x(0), \delta x(1))$. Let us show that the quadratic form $\Omega$ could be transformed into a perfect square if the corresponding Riccati equation has a solution $Q(t)$, defined on $[0,1]$. Assume that the strengthened Legendre condition is satisfied. Define the Riccati matrix equation along $(x^0(t), u^0(t), p(t))$ by

$$\dot{Q} + Qf_x + f_x^\top Q + H_{xx} - (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q) = 0, \quad t \in [0,1], \tag{9}$$

where $Q = Q(t)$ is a symmetric matrix of order $n$, whose elements belong to $C^1$, $f_x = f_x(w^0)$, $H_{xx} = H_{xx}(w^0, p, \alpha_0)$, etc., $f_x^\top$ means the transposed matrix $f_x$.

THEOREM 2.3 *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution $Q$ (with the entries belonging to $C^1$) of the matrix Riccati equation on $[0,1]$. Then, the quadratic form $\Omega$ has the following transformation into a perfect square on the subspace $\Gamma$:*

$$2\Omega(\delta w) = \int_0^1 \langle H_{uu}^{-1}\delta v, \delta v\rangle \, \mathrm{d}t + \langle M\delta\xi, \delta\xi\rangle \quad \forall\, \delta w \in \Gamma, \tag{10}$$

*where $\delta v := (H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u$, $H_{uu}^{-1}$ is the inverse matrix of matrix $H_{uu}$, and*

$$M := \begin{pmatrix} L_{x_0x_0} + Q(0) & L_{x_0x_1} \\ L_{x_1x_0} & L_{x_1x_1} - Q(1) \end{pmatrix}.$$

For the reader's convenience, we give a proof of this theorem. We follow Maurer and Osmolovskii (2012) (see also Maurer and Pickenhein, 1995).

PROOF Let $(\delta x, \delta u) \in \Gamma$. Then

$$\begin{aligned} 2\langle Q\delta\dot{x}, \delta x\rangle &= 2\langle Q(f_x\delta x + f_u\delta u), \delta x\rangle \\ &= \langle (Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle Q\delta x, \delta x\rangle &= \langle \dot{Q}\delta x, \delta x\rangle + 2\langle Q\delta\dot{x}, \delta x\rangle \\ &= \langle \dot{Q}\delta x, \delta x\rangle + \langle (Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle \\ &= \langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle. \end{aligned}$$

Integrating over $[0,1]$, we get

$$\begin{aligned} &\langle Q(1)\delta x(1), \delta x(1)\rangle - \langle Q(0)\delta x(0), \delta x(0)\rangle \\ &= \textstyle\int_0^1 \Big(\langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle\Big)\, \mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned} &\textstyle\int_0^1 \Big(\langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle\Big)\, \mathrm{d}t \\ &+\langle Q(0)\delta x(0), \delta x(0)\rangle - \langle Q(1)\delta x(1), \delta x(1)\rangle = 0. \end{aligned}$$

Adding this zero form to the form $2\Omega(\delta w)$, we obtain

$$\begin{aligned}2\Omega(\delta w) \quad = \quad & \textstyle\int_0^1 \Big(\langle(\dot{Q}+Qf_x+f_x^\top Q+H_{xx})\delta x,\delta x\rangle \\ & +\langle(Qf_u+H_{xu})\delta u,\delta x\rangle+\langle(f_u^\top Q+H_{ux}\delta x,\delta u\rangle+\langle H_{uu}\delta u,\delta u\rangle\Big)\,\mathrm{d}t \\ & +\langle Q(0)\delta x(0),\delta x(0)\rangle-\langle Q(1)\delta x(1),\delta x(1)\rangle+\langle L_{\xi\xi}\delta\xi,\delta\xi\rangle.\end{aligned}$$

Now let $Q$ satisfy the Riccati equation (9). Then

$$\begin{aligned}2\Omega(\delta w) \quad = \quad & \textstyle\int_0^1 \Big(\langle(H_{xu}+Qf_u)H_{uu}^{-1}(H_{ux}+f_u^\top Q)\delta x,\delta x\rangle \\ & +\langle(Qf_u+H_{xu})\delta u,\delta x\rangle+\langle(f_u^\top Q+H_{ux}\delta x,\delta u\rangle+\langle H_{uu}\delta u,\delta u\rangle\Big)\,\mathrm{d}t \\ & +\langle Q(0)\delta x(0),\delta x(0)\rangle-\langle Q(1)\delta x(1),\delta x(1)\rangle+\langle L_{\xi\xi}\delta\xi,\delta\xi\rangle.\end{aligned}$$

Since

$$\langle H_{uu}\delta u,\delta u\rangle=\langle(H_{uu})^{-1}H_{uu}\delta u,H_{uu}\delta u\rangle \text{ and } \langle Q(0)\delta x(0),\delta x(0)\rangle-\langle Q(1)\delta x(1),\delta x(1)\rangle+\langle L_{\xi\xi}\delta\xi,\delta\xi\rangle=\langle M\delta\xi,\delta\xi\rangle,$$

we obtain

$$\begin{aligned}2\Omega(\delta w)=\int_0^1 & \Big(\langle(H_{xu}+Qf_u)H_{uu}^{-1}(H_{ux}+f_u^\top Q)\delta x,\delta x\rangle \\ & +\langle(Qf_u+H_{xu})\delta u,\delta x\rangle \\ & +\langle(f_u^\top Q+H_{ux}\delta x,\delta u\rangle+\langle(H_{uu})^{-1}H_{uu}\delta u,H_{uu}\delta u\rangle\Big)\,\mathrm{d}t \\ & +\langle M\delta\xi,\delta\xi\rangle.\end{aligned}$$

Further,

$$\begin{aligned}&\langle(H_{xu}+Qf_u)H_{uu}^{-1}(H_{ux}+f_u^\top Q)\delta x,\delta x\rangle \\ &+\langle(Qf_u+H_{xu})\delta u,\delta x\rangle+\langle(f_u^\top Q+H_{ux}\delta x,\delta u\rangle+\langle(H_{uu})^{-1}H_{uu}\delta u,H_{uu}\delta u\rangle \\ &=\langle(H_{uu})^{-1}((H_{ux}+f_u^\top Q)\delta x+H_{uu}\delta u),((H_{ux}+f_u^\top Q)\delta x+H_{uu}\delta u)\rangle \\ &=\langle(H_{uu})^{-1}\delta v,\delta v\rangle,\end{aligned}$$

where $\delta v=(H_{ux}+f_u^\top Q)\delta x+H_{uu}\delta u$.
Consequently, $2\Omega(\delta w)=\int_0^1\langle(H_{uu})^{-1}\delta v,\delta v\rangle\ \mathrm{d}t+\langle M\delta\xi,\delta\xi\rangle$. □

Assume that $M$ is nonnegative definite. Recall that $H_{uu}$ is positive definite, and then $(H_{uu})^{-1}$ is positive definite, too. Hence, $\Omega(\delta w)\geq 0\,\forall\,\delta w=(\delta x,\delta u)\in\Gamma$.

Suppose that $\Omega(\delta w)=0$ for some $\delta w=(\delta x,\delta u)\in\Gamma$. Then, given (10), both non negative terms $\int_0^1\langle(H_{uu})^{-1}\delta v,\delta v\rangle$ and $\langle M\delta\xi,\delta\xi\rangle$ are equal zero. Condition $\int_0^1\langle(H_{uu})^{-1}\delta v,\delta v\rangle\,\mathrm{d}t=0$ implies

$$\delta v=0, \text{ i.e. } (H_{ux}+f_u^\top Q)\delta x+H_{uu}\delta u=0.$$

Hence, $\delta u = -(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$. It follows that $\delta x$ is a solution to the homogeneous differential equation

$$\delta \dot{x} = f_x(\hat{w})\delta x - f_u(\hat{w})(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$$

Let us now assume that the conditions $\mathcal{E}\delta\xi = 0$, $\langle M\delta\xi, \delta\xi\rangle = 0$ imply that $\delta x(0) = 0$ or $\delta x(1) = 0$. Then, $\delta x = 0$ and hence $\delta u = 0$. Consequently, $\Omega(\delta w) > 0$ for all $\delta w \in \Gamma \setminus \{0\}$. Since $\Omega$ is a Legendre form, its positiveness on the subspace $\Gamma$ implies positive definiteness on $\Gamma$. Thus, we obtain the following result.

THEOREM 2.4 *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution $Q$ (with the entries belonging to $C^1$) of the Riccati matrix equation on $[0,1]$ such that*

*(a) the matrix $M$ is nonnegative definite;*

*(b) for all $\xi = (x_0, x_1) \in \mathbb{R}^{2n}$ the conditions $\mathcal{E}\xi = 0$, $\langle M\xi, \xi\rangle = 0$ imply that $x_0 = 0$ or $x_1 = 0$. Then, the quadratic form $\Omega$ is positive definite on the subspace $\Gamma$.*

### Other designations

Let $\Gamma$ have the form:

$$\Gamma = \Big\{\delta w = (\delta x, \delta u): \quad \delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t),\ \mathcal{E}\delta\xi = 0\Big\},$$

and

$$2\Omega(\delta w) = \langle N\delta\xi, \delta\xi\rangle + \int_0^1 \Big(\langle R(t)\delta x(t), \delta x(t)\rangle + 2\langle S(t)\delta u(t), \delta x(t)\rangle + \langle U(t)\delta u(t), \delta u(t)\rangle\Big)\,\mathrm{d}t, \tag{11}$$

where $\mathcal{E}$ and $N$ are constant matrices, $A(t)$, $B(t)$, $R(t)$, $S(t)$, $U(t)$ are matrices with continuous entries. Assume that the matrices $R(t)$ and $U(t)$ are symmetric and, moreover, the matrix $U(t)$ is positive definite for all $t \in [0,1]$, and the constant symmetric matrix $N$ of the order $2n$ has the form

$$N = \begin{pmatrix} N_{00} & N_{01} \\ N_{10} & N_{11} \end{pmatrix},$$

where $N_{00}$, $N_{01}$, $N_{10}$, $N_{11}$ are constant $n \times n$ matrices, $N_{00}$ and $N_{11}$ are symmetric, and $N_{10} = N_{01}^\top$. Previously, we had $A = f_x$, $B = f_u$, $R = H_{xx}$, $S = H_{xu}$,

$U = H_{uu}$. We can prove similar results for the new quadratic form and subspace in the same way as before. Now, the Riccati equation and the matrix $M$ are:

$$\dot{Q} + QA + A^\top Q + R - (S + QB)U^{-1}(S^\top + B^\top Q) = 0, \tag{12}$$

$$M = \begin{pmatrix} N_{00} + Q(0) & N_{01} \\ N_{10} & N_{11} - Q(1) \end{pmatrix}. \tag{13}$$

# 3. The single beam problem

## 3.1. Statement of the problem with one beam

Consider the following optimal control problem. Let $z(x)$ be a state variable, $f(x)$ be a control, where $x \in [0, l]$. Here $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$, $f \in \mathbb{R}^1$, $l > 0$. We assume that $z(x)$ is a continuously differentiable function and $f(x)$ is a continuous function. The control system has the form

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = \varphi(z(x)) + e_1 f(x), \quad x \in [0, l], \quad K(z(0), z(l)) = 0, \tag{14}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function, $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$, and $K : \mathbb{R}^{2n} \to \mathbb{R}^s$ is an affine function of its arguments $\zeta_0 := z(0)$ and $\zeta_l := z(l)$. Set $\zeta = (\zeta_0, \zeta_l)$. The cost that needs to be minimized is:

$$J = \int_0^l F(x, z(x), f(x))\,\mathrm{d}x, \tag{15}$$

where $F(x, z, f)$ is a twice continuously differentiable function. In this problem $l$ is not fixed, but satisfies the constraint

$$l \in [a, b], \quad \text{where} \quad 0 < a < b. \tag{16}$$

An arbitrary admissible process in this problem is defined by the triple $(l, z(\cdot), f(\cdot))$, where $z : [0, l] \to \mathbb{R}^n$, $f : [0, l] \to \mathbb{R}$. We will consider a fixed admissible process

$$(l^0, z^0(\cdot), f^0(\cdot)), \tag{17}$$

where $z^0$ and $f^0$ are defined on $[0, l^0]$.

Let us represent this problem as a problem on the interval $[0, 1]$. To do this, we use the following change of the independent variable $x$. Let $t \in [0, 1]$ be a new independent variable. We set

$$\tilde{x}(t) = lt, \quad t \in [0, 1].$$

Then, $\tilde{x} : [0,1] \to [0,l]$. We treat $\tilde{x}(t)$ as a new state variable. We also treat $l = \tilde{l}(t)$ as another state variable, constant on $[0,1]$. Hence

$$\frac{\mathrm{d}\tilde{l}(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{l}(t), \quad t \in [0,1], \quad \tilde{x}(0) = 0.$$

To any admissible process $(l, z, f)$ in the original problem, we associate the process $(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f})$ in the new problem by the formulas

$$\tilde{l}(t) = l,\ \tilde{x}(t) = lt,\ \tilde{z}(t) = z(\tilde{x}(t)) = z(lt),\ \tilde{f}(t) = f(\tilde{x}(t)) = f(lt)\, \forall\, t \in [0,1].$$

This is one-to-one correspondence. In what follows, we will continue to use the tilde for the variables in the interval $[0,1]$.

Thus, we obtain an autonomous problem with a new independent variable $t \in [0,1]$:

$$\frac{\mathrm{d}\tilde{l}(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{l}(t), \quad t \in [0,1], \tag{18}$$

$$\frac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t} = \tilde{l}(t)\big(\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t)\big), \quad t \in [0,1], \tag{19}$$

$$\tilde{x}(0) = 0, \quad K(\tilde{z}(0), \tilde{z}(1)) = 0, \tag{20}$$

$$-\tilde{l}(0) + a \le 0. \quad \tilde{l}(0) - b \le 0, \tag{21}$$

$$J = \int_0^1 \tilde{l}(t) F(\tilde{x}(t), \tilde{z}(t), \tilde{f}(t))\, \mathrm{d}t \to \min. \tag{22}$$

We study the local minimum at the point

$$(\tilde{l}^0(\cdot), \tilde{x}^0, \tilde{z}^0(\cdot), \tilde{f}^0(\cdot)), \tag{23}$$

such that

$$\tilde{l}^0(t) = l^0, \quad \tilde{x}^0(t) = l^0 t, \quad \tilde{z}^0(t) = z^0(l^0 t), \quad \tilde{f}^0(t) = f^0(l^0 t), \quad t \in [0,1].$$

This point corresponds to the process (17) in the original problem (14)-(16). Clearly, the minimum at (17) in problem (14)-(16) implies the minimum at (23) in problem (18)-(22), and vice versa.

### 3.2. Local minimum principle for problem with one beam

Denote by $\tilde{p}^z(t)$ the adjoint variable, which corresponds to the equation for $\tilde{z}$ in the new problem. We consider $\tilde{p}^z = (\tilde{p}^z_1, \ldots, \tilde{p}^z_n)$ as a row vector. We also introduce one-dimensional adjoint variables $\tilde{p}^x(t)$ and $\tilde{p}^l(t)$. The Hamiltonian and the endpoint Lagrange function are:

$$\tilde{H}(\tilde{l},\tilde{x},\tilde{z},\tilde{f},\tilde{p}^l,\tilde{p}^x,\tilde{p}^z,\alpha_0)=\tilde{p}^x\tilde{l}+\tilde{p}^z\tilde{l}\big(\varphi(\tilde{z})+e_1\tilde{f}\big)+\alpha_0\tilde{l}F(\tilde{x},\tilde{z},\tilde{f}),$$

$$\tilde{L}=\alpha_a(-\tilde{l}(0)+a)+\alpha_b(\tilde{l}(0)-b)+\beta_x\tilde{x}(0)+\beta K(\tilde{z}(0),\tilde{z}(1)).$$

Note that $\tilde{L}$ is an affine function of the endpoint values $\tilde{l}(0)$, $\tilde{x}(0)$, $\tilde{z}(0)$, $\tilde{l}(1)$, $\tilde{x}(1)$, $\tilde{z}(1)$ of the states $\tilde{l}$, $\tilde{x}$, and $\tilde{z}$, since $K$ is an affine function by assumption.

Let us write down the first-order necessary optimality conditions at the point (23) in problem (18)-(22). The partial derivatives of $\tilde{H}$ with respect to $\tilde{l},\tilde{x},\tilde{z},\tilde{f}$ have the form

$$\begin{aligned}
\tilde{H}_{\tilde{l}}&=\tilde{p}^x+\tilde{p}^z(\varphi(\tilde{z})+e_1\tilde{f})+\alpha_0F(\tilde{x},\tilde{z},\tilde{f}),\\
\tilde{H}_{\tilde{x}}&=\alpha_0\tilde{l}F_{\tilde{x}}(\tilde{x},\tilde{z},\tilde{f}),\\
\tilde{H}_{\tilde{z}}&=\tilde{p}^z\tilde{l}\varphi'(\tilde{z})^T+\alpha_0\tilde{l}F_{\tilde{z}}(\tilde{x},\tilde{z},\tilde{f}),\\
\tilde{H}_{\tilde{f}}&=\tilde{p}^z\tilde{l}e_1+\alpha_0\tilde{l}F_{\tilde{f}}(\tilde{x},\tilde{z},\tilde{f}).
\end{aligned}$$

Hence, the conditions of the local minimum principle at the point (23) in problem (18)-(22) are as follows.

(a) The nonnegativity conditions: $\alpha_0\ge 0,\quad \alpha_a\ge 0,\quad \alpha_b\ge 0.$
(b) The nontriviality condition: $\alpha_0+\alpha_a+\alpha_b+|\beta_x|+|\beta|>0.$
(c) The complementary slackness conditions: $\alpha_a(\tilde{l}^0(0)-a)=0$, $\alpha_b(\tilde{l}^0(0)-b)=0$.
(d) The adjoint equations:

$$-\frac{\mathrm{d}\tilde{p}^l(t)}{\mathrm{d}t}=\tilde{p}^x(t)+\tilde{p}^z(t)\big(\varphi(\tilde{z}^0(t))+e_1\tilde{f}^0(t)\big)+\alpha_0F(\tilde{x}^0(t),\tilde{z}^0(t),\tilde{f}^0(t)), \tag{24}$$

$$-\frac{\mathrm{d}\tilde{p}^x(t)}{\mathrm{d}t}=\alpha_0\tilde{l}^0F_{\tilde{x}}(\tilde{x}^0(t),\tilde{z}^0(t),\tilde{f}^0(t)), \tag{25}$$

$$-\frac{\mathrm{d}\tilde{p}^z(t)}{\mathrm{d}t}=\tilde{p}^z(t)\tilde{l}^0\varphi'(\tilde{z}^0(t))+\alpha_0\tilde{l}^0F_{\tilde{z}}(\tilde{x}^0(t),\tilde{z}^0(t),\tilde{f}^0(t)),\quad t\in[0,1]. \tag{26}$$

(e) The transversality conditions:

$$\begin{aligned}
-\tilde{p}^l(0)&=-\alpha_a+\alpha_b, & \tilde{p}^l(1)&=0,\\
-\tilde{p}^x(0)&=\beta_x, & \tilde{p}^x(1)&=0,\\
-\tilde{p}^z(0)&=\beta K_{\tilde{\zeta}_0}(\tilde{z}^0(0),\tilde{z}^0(1)), & \tilde{p}^z(1)&=\beta K_{\tilde{\zeta}_1}(\tilde{z}^0(0),\tilde{z}^0(1)),
\end{aligned}$$

where $\tilde{\zeta}_0=\tilde{z}(0)$, $\tilde{\zeta}_1=\tilde{z}(1)$.

(f) The condition $\tilde{H}_{\tilde{f}} = 0$: $\tilde{p}^z(t)\tilde{l}^0 e_1 + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0$. Since $\tilde{l}^0 > 0$ and $\tilde{p}^z(t)e_1 = \tilde{p}^z_1(t)$, we get

$$\tilde{p}^z_1(t) + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0,1].$$

(g) Finally, the condition $\tilde{H} = \mathrm{const}$ has the form: there exists a constant $\hat{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0 + \tilde{p}^z(t)\tilde{l}^0\big(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)\big) + \alpha_0 \tilde{l}^0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \tilde{c}_H \quad \forall\, t \in [0,1].$$

Denote the left hand side of this equality by $\tilde{H}(t)$. Dividing this equality by $\tilde{l}^0$, we obtain

$$\tilde{p}^x(t) + \tilde{p}^z(t)\big(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)\big) + \alpha_0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \frac{\tilde{c}_H}{\tilde{l}^0} \quad \forall\, t \in [0,1].$$

Integrating equation (24) over the interval $[0,1]$ and using the above condition, we get $\tilde{p}^l(0) - \tilde{p}^l(1) = \frac{\tilde{c}_H}{\tilde{l}^0}$. This, and the transversality conditions $-\tilde{p}^l(0) = -\alpha_a + \alpha_b$, $\tilde{p}^l(1) = 0$, give

$$\frac{\tilde{c}_H}{\tilde{l}^0} = \alpha_a - \alpha_b.$$

This relation means the following:

(1) If $a < \tilde{l}^0 < b$ then by the complementary slackness conditions (c) we have $\alpha_a = \alpha_b = 0$ and therefore $\tilde{c}_H = 0$.
(2) If $\tilde{l}^0 = a$, then by (c) we have $\alpha_b = 0$ and, therefore, $\tilde{c}_H = \alpha_a \tilde{l}^0 \geq 0$.
(3) If $\tilde{l}^0 = b$, then by (c) we have $\alpha_a = 0$ and, therefore, $\tilde{c}_H = -\alpha_b \tilde{l}^0 \leq 0$.
(4) Moreover, if $\tilde{c}_H > 0$, then $\alpha_a > 0$, and, therefore, by (c) $\tilde{l}^0 = a$; if $\tilde{c}_H < 0$, then $\alpha_b > 0$, and, therefore, by (c) $\tilde{l}^0 = b$.

Note that the transversality condition $\tilde{p}^x(1) = 0$ and the adjoint equation (25) imply

$$\tilde{p}^x(t) = \alpha_0 \tilde{l}^0 \int_t^1 F_{\tilde{x}}(\tilde{x}^0(\tau), \tilde{z}^0(\tau), \tilde{f}^0(\tau))\, \mathrm{d}\tau, \quad t \in [0,1]. \tag{27}$$

Thus, we obtain the following result. If (23) is a local minimum in problem (18)–(22), then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{s\top}$, and a continuously differentiable function $\tilde{p}^z(t)$ such that the following system of optimality conditions holds:

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = \tilde{l}^0\big(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)\big), \quad t \in [0,1], \quad K(\tilde{z}^0(0), \tilde{z}^0(1)) = 0,$$

$$-\ \frac{\mathrm{d}\tilde{p}^z}{\mathrm{d}t} = \tilde{p}^z(t)\tilde{l}^0 \varphi'(\tilde{z}^0(t)) + \alpha_0 \tilde{l}^0 F_{\tilde{z}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad t \in [0,1],$$

$$-\ \tilde{p}^z(0) = \beta K_{\tilde{\zeta}_0}(\tilde{z}^0(0), \tilde{z}^0(1)), \quad \tilde{p}^z(1) = \beta K_{\tilde{\zeta}_1}(\tilde{z}^0(0), \tilde{z}^0(1)),$$

$$\tilde{p}^z_1(t) + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0,1].$$

These conditions imply the condition of constancy of the Hamiltonian: there exists a constant $\tilde{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0+\tilde{p}^z(t)\tilde{l}^0\big(\varphi(\tilde{z}^0(t))+e_1\tilde{f}^0(t)\big)+\alpha_0\tilde{l}^0F(\tilde{x}^0(t),\tilde{z}^0(t),\tilde{f}^0(t))=\tilde{c}_H \quad \forall\, t\in[0,1],$$

where $\tilde{p}^x(t)$ is defined by (27).

Moreover, the following is true. If $a<\tilde{l}^0<b$, then $\tilde{c}_H=0$. If $\tilde{l}^0=a$, then $\tilde{c}_H\geq 0$; if $\tilde{c}_H>0$, then $\tilde{l}^0=a$. If $\tilde{l}^0=b$, then $\tilde{c}_H\leq 0$; if $\tilde{c}_H<0$, then $\tilde{l}^0=b$. We now represent this system in an equivalent way on the interval $[0,l^0]$.

Let us introduce a function $p^z:[0,l^0]\to\mathbb{R}^{n\top}$ such that $\tilde{p}^z(t)=p^z(\tilde{x}^0(t))=p^z(l^0t)$, that is, $p^z(x)=\tilde{p}^z\left(\frac{x}{l^0}\right)$, $x\in[0,l^0]$. Then $\frac{\mathrm{d}\tilde{p}^z}{\mathrm{d}t}=\frac{\mathrm{d}p^z}{\mathrm{d}x}l^0$. Hence, the adjoint equation for $\tilde{p}^z$ takes the form

$$-\frac{\mathrm{d}p^z(x)}{\mathrm{d}x}=p^z(x)\varphi'(z^0(x))+\alpha_0F_z(x^0(t),z^0(t),f^0(t)), \quad x\in[0,l^0].$$

So, the obtained result has the following formulation on the interval $[0,l^0]$. In this formulation we replace $p^z$ with $p$, and we also replace $(l^0,z^0(\cdot),f^0(\cdot))$ with $(l,z(\cdot),f(\cdot))$, omitting the superscript zero.

Theorem 3.1 *If $(l,z(\cdot),f(\cdot))$ is a local minimum in problem (14)-(16), then there exist a number $\alpha_0\geq 0$, a row vector $\beta\in\mathbb{R}^{s\top}$, and a continuously differentiable function $p:[0,l]\to\mathbb{R}^{n\top}$ such that the following system of optimality conditions holds:*

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x}=\varphi(z(x))+e_1f(x), \quad x\in[0,l], \quad l\in[a,b], \quad K(z(0),z(l))=0,$$

$$-\quad \frac{\mathrm{d}p(x)}{\mathrm{d}x}=p(x)\varphi'(z(x))+\alpha_0F_z(x,z(x),f(x)), \quad x\in[0,l],$$

$$-\quad p(0)=\beta K_{\zeta_0}(z(0),z(1)), \quad p(l)=\beta K_{\zeta_1}(z(0),z(l)),$$

$$p_1(x)+\alpha_0F_f(x,z(x),f(x))=0, \quad x\in[0,l].$$

*These conditions imply the condition of constancy of the Hamiltonian: there exists a constant $c_H$ such that*

$$p^x(x)+p(x)\big(\varphi(z(x))+e_1f(x)\big)+\alpha_0F(x,z(x),f(x))=c_H \quad \forall\, x\in[0,l],$$

*where $p^x(x)=\alpha_0\int_x^l F_x(y,z(y),f(y))\,\mathrm{d}y$, $x\in[0,l]$. Moreover, the following is true. If $a<l<b$, then $c_H=0$. If $l=a$, then $c_H\geq 0$; if $c_H>0$, then $l=a$. If $l=b$, then $c_H\leq 0$; if $c_H<0$, then $l=b$.*

Since $p^x(l)=0$ and $c_H=H(l)$, we get

$$c_H=p(l)\big(\varphi(z(l))+e_1f(l)\big)+\alpha_0F(l,z(l),f(l)).$$

This formula does not use the adjoint variable $p^x$.

In what follows, we consider the case of

$$F(x,z,f) = \frac{1}{2}|z - z^*(x)|^2 + \frac{1}{2}(f - f^*(x))^2, \tag{28}$$

where $|z| = \sqrt{\langle z, z\rangle}$ and $z^*(x)$ and $f^*(x)$ are twice continuously differentiable functions defined on $[0,b]$.

### 3.3. Second-order sufficient conditions for problem with one beam

For problem (18)-(22) on $[0,1]$ with the function $F$, defined by formula (28), we formulate sufficient second-order conditions for a weak local minimum at the point $\tilde{w}(\cdot) = (\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$.

Now suppose that the normal case holds for this point. Therefore, there are a row vector $\beta \in \mathbb{R}^{s\top}$ and a continuously differentiable function $\tilde{p} : [0,1] \to \mathbb{R}^{n\top}$ such that the necessary optimality conditions in Section 3.2 are satisfied with $\alpha_0 = 1$. In problem (18)-(22) on $[0,1]$, by definition $\tilde{\xi} = (\tilde{l}(0), \tilde{x}(0), \tilde{z}(0); \tilde{l}(1), \tilde{x}(1), \tilde{z}(1))$. Since $\tilde{L}$ is an affine function of $\tilde{\xi}$, we have $\tilde{L}_{\tilde{\xi}\tilde{\xi}} = 0$. Since $\alpha_0 = 1$, we have

$$\tilde{H}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z) = \tilde{p}^x \tilde{l} + \tilde{p}^z \tilde{l}\big(\varphi(\tilde{z}) + e_1 \tilde{f}\big) + \tilde{l} F(\tilde{x}, \tilde{z}, \tilde{f}).$$

Recall that $\tilde{H}_{\tilde{f}} = \tilde{p}^z \tilde{l} e_1 + \alpha_0 \tilde{l}(\tilde{f} - f^*(\tilde{x}))$. Consequently, $\tilde{H}_{\tilde{f}\tilde{f}} = \tilde{l}$. Since $\tilde{l} = l \geq a > 0$, the strengthened Legendre condition is satisfied.

Let us write down the definition of the critical cone $\tilde{\mathcal{C}}$. Equations in variations for the system

$$\frac{\mathrm{d}\tilde{l}}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \tilde{l}, \quad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = \tilde{l}(\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t))$$

at the point $\tilde{w}$ have the form

$$\delta\dot{\tilde{l}} = 0, \quad \delta\dot{\tilde{x}}(t) = \delta\tilde{l}, \quad \delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1 \delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t)\big)\delta\tilde{l}.$$

The endpoint conditions $\tilde{x}(0) = 0$ and $K(\tilde{z}(0), \tilde{z}(1)) = 0$ imply the following conditions in the critical cone

$$\delta\tilde{x}(0) = 0, \quad K'(\tilde{\zeta})\delta\tilde{\zeta} = 0,$$

where $\tilde{\zeta} = (\tilde{z}(0), \tilde{z}(1))$, $\delta\tilde{\zeta} = (\delta\tilde{z}(0), \delta\tilde{z}(1))$.

Further, recall that

$$\frac{\tilde{c}_H}{\tilde{l}} = \alpha_a - \alpha_b.$$

The initial conditions $-\tilde{l}(0) + a \leq 0$ and $\tilde{l}(0) - b \leq 0$ imply:

- if $a < \tilde{l} < b$, i.e., these constraints are not active, then $\tilde{c}_H = 0$, and we have no conditions on $\delta\tilde{l}(0)$,
- if $a = \tilde{l}$ and, therefore, $\tilde{l} < b$, then the following conditions are satisfied: $\delta\tilde{l}(0) \geq 0$, $\tilde{c}_H\delta\tilde{l}(0) = 0$,
- if $\tilde{l} = b$ and, therefore, $\tilde{l} > a$, then the following conditions are satisfied: $\delta\tilde{l}(0) \leq 0$, $\tilde{c}_H\delta\tilde{l}(0) = 0$.

Consequently,

$$\begin{aligned}
&\tilde{\mathcal{C}} = \Big\{\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) : \delta\dot{\tilde{l}} = 0 \\
&\delta\dot{\tilde{x}}(t) = \delta\tilde{l},\ \delta\tilde{x}(0) = 0,\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0,\ c_H\delta\tilde{l}(0) = 0, \\
&\delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\hat{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}, \\
&\tilde{l} = a \implies \delta\tilde{l}(0) \geq 0; \quad \tilde{l} = b \implies \delta\tilde{l}(0) \leq 0\Big\}.
\end{aligned}$$

As stated in Remark 2, if $\Omega$ is positive definite on $\tilde{\mathcal{C}}$, then it is positive definite on $(-\tilde{\mathcal{C}})$. Only one of the two conditions $\tilde{l} = a$ or $\tilde{l} = b$ could be realized. Therefore, the conditions $\tilde{l} = a \implies \delta\tilde{l}(0) \geq 0$; $\tilde{l} = b \implies \delta\tilde{l}(0) \leq 0$ in the definition of $\tilde{\mathcal{C}}$ can be omitted. More precisely, we can replace $\tilde{\mathcal{C}}$ with a subspace

$$\begin{aligned}
\tilde{\Sigma} = \Big\{&\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) :\ \delta\dot{\tilde{l}} = 0,\ \delta\dot{\tilde{x}}(t) = \delta\tilde{l},\ \delta\tilde{x}(0) = 0,\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0, \\
&\tilde{c}_H\delta\tilde{l}(0) = 0,\ \delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\hat{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}\ \Big\}.
\end{aligned}$$

Note that if $\tilde{c}_H \neq 0$, then in the definition of $\tilde{\Sigma}$ we have $\delta\tilde{l}(0) = 0$, which gives $\delta\tilde{l} = 0$, and this means that $\delta\tilde{x} = 0$. In this case,

$$\begin{aligned}
\tilde{\Sigma} = \Big\{&\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) : \\
&\delta\tilde{l} = 0,\ \delta\tilde{x} = 0,\ \delta\dot{\tilde{z}}(t) = \tilde{l}\varphi'(\hat{z}(t))\delta\tilde{z}(t) + \tilde{l}e_1\delta\tilde{f}(t),\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0\Big\}.
\end{aligned}$$

Let us write down the quadratic form $\tilde{\Omega}$. Since $\alpha_0 = 1$,

$$\begin{aligned}
&\tilde{H}_{\tilde{l}} = \tilde{p}^x + \tilde{p}^z(\varphi(\tilde{z}) + e_1\tilde{f}) + F(\tilde{x}, \tilde{z}, \tilde{f}), \\
&\tilde{H}_{\tilde{x}} = \tilde{l}F_{\tilde{x}} = -\tilde{l}\big(\tilde{z} - z^*(\tilde{x})\big)^{\top}(z^*)'(\tilde{x}) - \tilde{l}(\tilde{f} - f^*(\tilde{x}))(f^*)'(\tilde{x}), \\
&\tilde{H}_{\tilde{z}} = \tilde{p}^z\tilde{l}\varphi'(\tilde{z}) + \tilde{l}\big(\tilde{z} - z^*(\tilde{x})\big)^{\top}, \\
&\tilde{H}_{\tilde{f}} = \tilde{p}^z\tilde{l}e_1 + \tilde{l}F_{\tilde{f}}(\tilde{x}, \tilde{z}, \tilde{f}) = \tilde{p}^z\tilde{l}e_1 + \tilde{l}(\tilde{f} - f^*(\tilde{x})).
\end{aligned}$$

Once again we emphasize that we consider $z$, $\tilde{z}$, $z^*$ as column vectors, and $p^z$, $\tilde{p}^z$, $\tilde{H}_{\tilde{z}}$ as row vectors. Therefore, $\big(\tilde{z} - z^*(\tilde{x})\big)^{\top}(z^*)'(\tilde{x}) = \sum_i \big(\tilde{z}_i - z_i^*(\tilde{x})\big)(z_i^*)'(\tilde{x})$.

The second-order partial derivatives have the form

$$
\begin{aligned}
&\tilde H_{\tilde l\tilde l}=0,\\
&\tilde H_{\tilde l\tilde x}=\tilde H_{\tilde x\tilde l}=-\big(\tilde z-z^*(\tilde x)\big)^\top(z^*)'(\tilde x)-(\tilde f-f^*(\tilde x))(f^*)'(\tilde x),\\
&\tilde H_{\tilde l\tilde z}=\tilde H_{\tilde z\tilde l}^\top=\tilde p^z\varphi'(\tilde z)+\big(\tilde z-z^*(\tilde x)\big)^\top,\\
&\tilde H_{\tilde l\tilde f}=\tilde H_{\tilde f\tilde l}=\tilde p_1^z+\tilde f-f^*(\tilde x),\\
&\tilde H_{\tilde x\tilde x}=\tilde l[(z^*)'(\tilde x)]^\top(z^*)'(\tilde x)-\tilde l(\tilde z-z^*(\tilde x))^\top(z^*)''(\tilde x)\\
&\qquad\quad+\tilde l[(f^*)'(\tilde x)]^2-\tilde l(\tilde f-f^*(\tilde x))(f^*)''(\tilde x),\\
&\tilde H_{\tilde x\tilde z}=\tilde H_{\tilde z\tilde x}^\top=-\tilde l[(z^*)'(\tilde x)]^\top,\quad \tilde H_{\tilde x\tilde f}=\tilde H_{\tilde f\tilde x}=-\tilde l(f^*)'(\tilde x),\\
&\tilde H_{\tilde z\tilde z}=\tilde p^z\tilde l\varphi'(\tilde z)^\top+\tilde lI_n,\quad \tilde H_{\tilde z\tilde f}=\tilde H_{\tilde f\tilde z}^\top=0,\\
&\tilde H_{\tilde f\tilde f}=\tilde l.
\end{aligned}
$$

Here $I_n$ is the identity matrix of size $n$ and

$$\big(\tilde z-z^*(\tilde x)\big)^\top(z^*)''(\tilde x)=\sum_i\big(\tilde z_i-z_i^*(\tilde x)\big)(z_i^*)''(\tilde x).$$

By denoting $\tilde w=(\tilde l,\tilde x,\tilde z,\tilde f)$, we get

$$
\begin{aligned}
&\langle\tilde H_{ww}(\tilde l,\tilde x,\tilde z,\tilde f,\tilde p^l,\tilde p^x,\tilde p^z)\delta w,\delta w\rangle\\
&=\tilde H_{\tilde l\tilde l}(\delta\tilde l)^2+\tilde H_{\tilde x\tilde x}(\delta\tilde x)^2+\langle\tilde H_{\tilde z\tilde z}\delta\tilde z,\delta\tilde z\rangle+\tilde H_{\tilde f\tilde f}(\delta\tilde f)^2\\
&+2\tilde H_{\tilde l\tilde x}\delta\tilde x\cdot\delta\tilde l+2\tilde H_{\tilde l\tilde z}\delta\tilde z\cdot\delta\tilde l+2\tilde H_{\tilde l\tilde f}\delta\tilde f\cdot\delta\tilde l\\
&+2\tilde H_{\tilde x\tilde z}\delta\tilde z\cdot\delta\tilde x+2\tilde H_{\tilde x\tilde f}\delta\tilde f\cdot\delta\tilde x+2\tilde H_{\tilde f\tilde z}\delta\tilde z\cdot\delta\tilde f.
\end{aligned}
$$

Using the above formulas, we obtain

$$
\begin{aligned}
&\quad\Big\langle\tilde H_{\tilde w\tilde w}(\tilde l(t),\tilde x(t),\tilde z(t),\tilde f(t),\tilde p^l(t),\tilde p^x(t),\tilde p^z(t))\delta\tilde w(t),\delta\tilde w(t)\Big\rangle\\
=&\quad\tilde l\Big([(z^*)'(\tilde x(t))]^\top(z^*)'(\tilde x(t))-(\tilde z(t)-z^*(\tilde x(t)))^\top(z^*)''(\tilde x(t))\\
+&\quad[(f^*)'(\tilde x(t))]^2-(\tilde f(t)-f^*(\tilde x(t)))(f^*)''(\tilde x(t))\Big)(\delta\tilde x(t))^2\\
+&\quad\tilde l\big\langle\big(\tilde p^z(t)\varphi''(\tilde z(t))+I_n\big)\delta\tilde z(t),\delta\tilde z(t)\big\rangle+\tilde l\big(\delta\tilde f(t)\big)^2\\
-&\quad2\Big(\big(\tilde z(t)-z^*(\tilde x(t))\big)(z^*)'(\tilde x(t))+\big(\tilde f(t)-f^*(\tilde x(t))\big)(f^*)'(\tilde x(t))\Big)\delta\tilde x(t)\cdot\delta\tilde l\\
+&\quad2\big(\tilde p^z(t)\varphi'(\tilde z(t))+\big(\tilde z(t)-z^*(\tilde x(t))\big)^\top\big)\delta\tilde z(t)\cdot\delta\tilde l\\
+&\quad2\big(\tilde p_1^z(t)+\tilde f(t)-f^*(\tilde x(t))\big)\delta\tilde f(t)\cdot\delta\tilde l\\
-&\quad2\tilde l\cdot[(z^*)'(\tilde x(t))]^\top\delta\tilde z(t)\cdot\delta\tilde x(t)-2\tilde l\cdot(f^*)'(\tilde x(t))\delta\tilde f(t)\cdot\delta\tilde x(t).
\end{aligned}
$$

Recall that here $\tilde l=l=\text{const}>0$. Since $\tilde L_{\tilde\xi\tilde\xi}=0$, the quadratic form $\tilde\Omega$ is:

$$\tilde\Omega(\delta\tilde w)=\int_0^1\langle\tilde H_{\tilde w\tilde w}(\tilde l,\tilde x,\tilde z,\tilde f,\tilde p^l,\tilde p^x,\tilde p^z)\delta\tilde w,\delta\tilde w\rangle\,\mathrm{d}t.$$

Thus, we obtain the following result: *if there exists a constant* $\tilde{c}_\Omega > 0$ *such that*

$$\tilde{\Omega}(\delta\tilde{w}) \geq \tilde{c}_\Omega((\delta\tilde{l})^2 + |\delta\tilde{z}(0)|^2 + \|\delta\tilde{f}\|_2^2) \quad \forall\, \delta\tilde{w} \in \tilde{\Sigma},$$

*then the quadruple* $(\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$ *is a weak local minimum in problem (18)-(22) on* $[0,1]$.

Now let us rewrite the obtained sufficient second-order condition in terms of the independent variable $x \in [0, l]$. Let $\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) \in \tilde{\Sigma}$.

Introduce $\delta z(x)$ such that $\delta z(\tilde{x}(t)) = \delta z(lt) = \delta\tilde{z}(t)$, that is $\delta z(x) = \delta\tilde{z}\left(\frac{x}{l}\right)$. Then, $\delta\dot{\tilde{z}}(t) = (\delta z)'(\tilde{x}(t))l$, where $(\delta z)'(x) = \frac{\mathrm{d}z(x)}{\mathrm{d}x}$. Define $\delta l$ such that $\delta\tilde{l} = l\delta l$, that is $\delta l = \frac{\delta\tilde{l}}{l}$. Similarly, we define $\delta f(x) = \delta\tilde{f}\left(\frac{x}{l}\right)$, $\quad \delta x(x) = \delta\tilde{x}\left(\frac{x}{l}\right)$. Then the equation

$$\delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}$$

takes the form

$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l$$

and the subspace $\tilde{\Sigma}$ in the new variables reads as follows

$$\Sigma = \Big\{\delta w = (\delta l, \delta x, \delta z, \delta f) : (\delta l)' = 0,\ (\delta x)'(x) = \delta l,\ \delta x(0) = 0,\ K'(\zeta)\delta\zeta = 0,\\ c_H\delta l(0) = 0,\ (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l\Big\},$$

where $\zeta = (z(0), z(l))$, $\delta\zeta = (\delta z(0), \delta z(l))$. Recall that $\delta x$ and $\delta l$ are one-dimensional, $\delta l = \mathrm{const}$ and $\delta x = x\delta l$. Therefore,

$$\Sigma = \Big\{\delta w = (\delta l, \delta x, \delta z, \delta f) : (\delta l)' = 0,\ \delta x(x) = x \cdot \delta l,\ K'(\zeta)\delta\zeta = 0,\ c_H\delta l(0) = 0,\\ (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l\Big\}.$$

Let us rewrite the quadratic form $\tilde{\Omega}$ in the new variables. Recall that $\tilde{l} = l$. If $x = \tilde{x}(t) = lt$, then $\mathrm{d}x = l\,\mathrm{d}t$ and $\tilde{z}(t) = z(x)$, $\tilde{f}(t) = f(x)$, $\delta\tilde{l} = l\delta l$, $\delta z(x) = \delta\tilde{z}(t)$, $\delta x(x) = \delta\tilde{x}(t)$, $\delta f(x) = \delta\tilde{f}(t)$.

Therefore, we have

$$
\begin{aligned}
&\tilde{l}\Big([(z^*)'(\tilde{x}(t))]^\top (z^*)'(\tilde{x}(t)) - (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)''(\tilde{x}(t)) \\
&\qquad +[(f^*)'(\tilde{x}(t))]^2 - (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)''(\tilde{x}(t))\Big)(\delta\tilde{x}(t))^2 \,\mathrm{d}t \\
&= \Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\
&\qquad +[(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\Big)(\delta x(x))^2 \,\mathrm{d}x, \\
&\big\langle \big(\tilde{p}^z(t)\tilde{l}\varphi''(\tilde{z}(t)) + \tilde{l}I_n\big)\delta\tilde{z}(t), \delta\tilde{z}(t)\big\rangle \,\mathrm{d}t \\
&= \big\langle \big(p^z(x)\varphi''(z(x)) + I_n\big)\delta z(x), \delta z(x)\big\rangle \,\mathrm{d}x, \\
&l(\delta\tilde{f}(t))^2 \,\mathrm{d}t = (\delta f(x))^2 \,\mathrm{d}x, \\
&-2\Big(\big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)^\top (z^*)'(\tilde{x}(t)) + \big(\tilde{f}(t) - f^*(\tilde{x}(t))(f^*)'(\tilde{x}(t))\Big)\delta\tilde{x}(t)\cdot\delta\tilde{l}\cdot\,\mathrm{d}t \\
&= -2\Big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + \big(f(x) - f^*(x)\big)(f^*)'(x)\Big)\delta x(x)\cdot\delta l\cdot\,\mathrm{d}x, \\
&2\Big(\big(\tilde{p}^z(t)\varphi'(\tilde{z}(t)) + \big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)^\top\Big)\delta\tilde{z}(t)\cdot\delta\tilde{l}\cdot\,\mathrm{d}t \\
&= 2\Big(\big(p^z(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\Big)\delta z(x)\cdot\delta l\cdot\,\mathrm{d}x, \\
&2\big(\tilde{p}_1^z(t) + \tilde{f}(t) - f^*(\tilde{x}(t))\big)\delta\tilde{f}(t)\cdot\delta\tilde{l}\cdot\,\mathrm{d}t \\
&= 2\big(p_1^z(x) + f(x) - f^*(x)\big)\delta f(x)\cdot\delta l\cdot\,\mathrm{d}x \\
&\qquad -2\tilde{l}[(z^*)'(\tilde{x}(t))]^\top\delta\tilde{z}(t)\cdot\delta\tilde{x}(t)\cdot\,\mathrm{d}t - 2\tilde{l}(f^*)'(\tilde{x}(t))\cdot\delta\tilde{f}(t)\cdot\delta\tilde{x}(t)\cdot\,\mathrm{d}t \\
&= -2[(z^*)'(x)]^\top\delta z(x)\cdot\delta x(x)\cdot\,\mathrm{d}x - 2(f^*)'(x)\delta f(x)\cdot\delta x(x)\cdot\,\mathrm{d}x.
\end{aligned}
$$

Consequently,

$$
\tilde{\Omega}(\delta\tilde{w}) = \Omega(\delta w),
$$

where

$$
\begin{aligned}
\Omega(\delta w) \quad = \quad & \textstyle\int_0^l \big\{\big([(z^*)'(x)]^\top (z^*)'(x) - \big(z(x) - z^*(x)\big)^\top (z^*)''(x) \\
& +[(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\big)(\delta x(x))^2 \\
& +\big\langle\big(p^z\varphi''(z) + I_n\big)\delta z(x), \delta z(x)\big\rangle + (\delta f(x))^2 \\
& -2\big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x)\big)\delta x(x)\cdot\delta l \\
& +2\big(p^z(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\big)\delta z(x)\cdot\delta l \\
& \\
& +2\big(p_1^z(x) + f(x) - f^*(x)\big)\delta f(x)\cdot\delta l \\
& -2[(z^*)'(x)]^\top\delta z(x)\cdot\delta x(x) - 2(f^*)'(x)\delta f(x)\cdot\delta x(x)\big\}\,\mathrm{d}x.
\end{aligned}
$$

Below we replace $p^z$ with $p$, omitting the superscript $z$. Since $\delta l \in \mathbb{R}$ is a constant, $\delta x = x \cdot \delta l$, and $\delta f$ is one-dimensional, we obtain

$$\begin{aligned}\Omega(\delta w) \;=\;& (\delta l)^2 \int_0^l \big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\ & +[(f^*)'(x)]^2 - (f(x) - f^*(x)(f^*)''(x)\big) x^2 \,\mathrm{d}x \\ & + \int_0^l \big\langle \big(p(x)\varphi''(z(x)) + I_n\big)\delta z(x), \delta z(x)\big\rangle \,\mathrm{d}x + \int_0^l (\delta f(x))^2 \,\mathrm{d}x \\ & -2(\delta l)^2 \int_0^l \Big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x)\Big) x \,\mathrm{d}x \\ & +2\delta l \int_0^l \Big(p(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\Big)\delta z(x) \,\mathrm{d}x \\ & +2\delta l \int_0^l \big(p_1(x) + f(x) - f^*(x)\big)\delta f(x) \,\mathrm{d}x \\ & -2\delta l \int_0^l \big((z^*)'(x)\big)^\top \delta z(x) x \,\mathrm{d}x - 2\delta l \int_0^l (f^*)'(x)\delta f(x) x \,\mathrm{d}x.\end{aligned} \tag{29}$$

This quadratic form is independent of $\delta x$. We can exclude $\delta x$ from the definition of $\Sigma$ as well. Therefore, the quadratic form $\Omega$ is considered on a subspace, which we still denote by $\Sigma$ (we also keep the notation $\delta w$ for the shorter collection $(\delta l, \delta z, \delta f)$):

$$\begin{gathered}\Sigma := \Big\{\delta w = (\delta l, \delta z, \delta f) : \ (\delta l)' = 0,\ c_H \delta l = 0,\ K'(\zeta)\delta\zeta = 0, \\ (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l \ \Big\}.\end{gathered}$$

Thus, we obtain the following result:

THEOREM 3.2 *Let an admissible triple $(l, z(\cdot), f(\cdot))$ satisfy the first order necessary optimality conditions of Theorem 3.1 in problem (14)-(16) with the corresponding multipliers $\alpha_0 = 1, \beta, p(\cdot)$. Suppose there exists a constant $c_\Omega > 0$ such that*

$$\Omega(\delta w) \geq c_\Omega \Big((\delta l)^2 + |\delta z(0)|^2 + \|\delta f\|_2^2\Big) \quad \forall\, \delta w \in \Sigma.$$

*Then the triple $(l, z(\cdot), f(\cdot))$ is a weak local minimum in problem (14)-(16).*

### 3.4. Matrix Riccati equation for one beam: case $C_H \neq 0$

In this case, as we know, the condition $C_H > 0$ implies $l = a$, and the condition $C_H < 0$ implies $l = b$. Then, in the definition of $\Sigma$, we have $\delta l = 0$, so that we can put

$$\Sigma := \Big\{\delta w = (\delta z, \delta f) : \quad (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x), \quad K'(\zeta)\delta\zeta = 0 \Big\}.$$

Since $\delta l = 0$, the quadratic form reduces to

$$\Omega(\delta w) = \int_0^l \Big\langle \Big(p(x)\varphi''(z(x)) + I_n\Big)\delta z(x), \delta z(x)\Big\rangle \,\mathrm{d}x + \int_0^l (\delta f(x))^2 \,\mathrm{d}x.$$

We study the question of the positive definiteness of $\Omega$ on $\Sigma$ in terms of the solution of the matrix Riccati equation. Obviously, the strengthened Legendre condition is satisfied.

Upon comparing the differential equation in the definition of $\Sigma$ with the equation $(\delta z)' = A\delta z + B\delta f$ (see the end of Section 2.3), we obtain

$$A = \varphi'((z(x)), \quad B = e_1 = (1, 0, \dots, 0)^\top.$$

Comparing $\Omega$ with (2.3), we get

$$R = p(x)\varphi''(z(x)) + I_n, \quad S = 0, \quad U = 1.$$

Consequently,

$$(S + QB)U^{-1}(S^\top + B^\top Q) = Qe_1e_1^\top Q = \begin{pmatrix} Q_{11} \\ \dots \\ Q_{1n} \end{pmatrix} \begin{pmatrix} Q_{11} & \dots & Q_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} Q_{11}Q_{11} & \dots & Q_{11}Q_{1n} \\ \dots & \dots & \dots \\ Q_{1n}Q_{11} & \dots & Q_{1n}Q_{1n} \end{pmatrix} = ||Q_{1i}Q_{1j}||_{i,j=1}^n.$$

Thus, the Riccati equation (12) reduces to the following

$$\frac{\mathrm{d}}{\mathrm{d}x}Q + QA + A^\top Q + R - Qe_1e_1^\top Q = 0, \quad x \in [0, l]. \tag{30}$$

where

$$A = \varphi'((z(x)), \qquad R = p(x)\varphi''(z(x)) + I_n,$$
$$e_1 = (1, 0 \dots, 0)^T, \quad Qe_1e_1^TQ = ||Q_{1i}Q_{1j}||_{i,j=1}^n.$$

The matrix $M$ has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

To this Riccati equation, one can add the initial condition

$$Q(0) = I_n,$$

where $I_n$ is the identity matrix of order $n$.

Similarly to Theorem 2.4 the following theorem holds.

THEOREM 3.3 *Assume that the strengthened Legendre condition is satisfied, $C_H \neq 0$, and there exists a symmetric solution $Q$ (with the entries belonging to $C^1$) of the Riccati matrix equation (30) on $[0, l]$ such that*

*(a) the matrix $M$ is nonnegative definite;*

*(b) for all $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^{2n}$ the conditions $K'(\zeta)\zeta = 0, \quad \langle M\zeta, \zeta\rangle = 0$ imply that $\zeta_0 = 0 \quad$ or $\quad \zeta_1 = 0$. Then the quadratic form $\Omega$ is positive definite on the subspace $\Sigma$.*

### 3.5. Matrix Riccati equation for one beam: case $C_H = 0$

In this more complicated case, we have

$$\Sigma := \Big\{ \delta w = (\delta l, \delta z, \delta f) : (\delta l)' = 0, \quad K'(\zeta)\delta\zeta = 0,$$
$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l \Big\}.$$

Consider again the sufficient condition for the positive definiteness of the quadratic form $\Omega$ on the subspace $\Sigma$. Now $\Sigma$ is defined by a linear system of differential equations

$$\begin{cases} (\delta l)' = 0, \\ (\delta z)' = (\varphi((z(x)) + e_1 f(x))\delta l + \varphi'((z(x))\delta z(x) + e_1 \delta f(x). \end{cases}$$

In the sequel, we denote

$$X = \begin{pmatrix} l \\ z \end{pmatrix} = \begin{pmatrix} l \\ z_1 \\ \dots \\ z_n \end{pmatrix} \in \mathbb{R}^{n+1}, \quad \delta X = \begin{pmatrix} \delta l \\ \delta z \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z_1 \\ \dots \\ \delta z_n \end{pmatrix} \in \mathbb{R}^{n+1},$$

$$w = \begin{pmatrix} X \\ f \end{pmatrix} = \begin{pmatrix} l \\ z \\ f \end{pmatrix} \in \mathbb{R}^{n+2}, \quad \delta w = \begin{pmatrix} \delta X \\ \delta f \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z \\ \delta f \end{pmatrix} \in \mathbb{R}^{n+2}.$$

Let us represent the above system in matrix form $(\delta X)' = A\delta X + B\delta f$, where $A$ is a $(n+1) \times (n+1)$ matrix, and $B$ is a $(n+1) \times 1$ matrix such that

$$A = \begin{pmatrix} 0 & 0_n^\top \\ \varphi(z(x)) + e_1 f(x) & \varphi'((z(x)) \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \ 0_n^\top = (0, \dots, 0) \in \mathbb{R}^{n\top}.$$

It is convenient to present

$$A := \begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix}, \quad \text{where} \quad A_{zl} = \varphi(z(x)) + e_1 f(x), \quad A_{zz} = \varphi'((z(x)).$$

Compare quadratic form (29) with the standard form (see (2.3)):

$$\Omega(\delta w) = \int_0^l \Big( \langle R\delta X, \delta X \rangle + 2(\delta X)^\top S \delta f + U(\delta f)^2 \Big) \, \mathrm{d}t,$$

where $R$ is the symmetric $(n+1) \times (n+1)$ matrix, $S \in \mathbb{R}^{n+1}$ is the column vector, $U$ is the number. Let us find the matrix $R$. Denote

$$R = \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix},$$

where $R_{ll}$, $R_{lz}$, $R_{zl} = R_{lz}^\top$, $R_{zz} = R_{zz}^\top$ are matrices of orders $1\times 1$, $1\times n$, $n\times 1$, $n\times n$, respectively. Then,

$$\langle R\delta X, \delta X\rangle = R_{ll}(\delta l)^2 + 2R_{lz}\delta z\delta l + \langle R_{zz}\delta z, \delta z\rangle.$$

Using (29), we obtain

$$\begin{aligned}\langle R\delta X, \delta X\rangle = \ & \Big[\Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x)-z^*(x))^\top (z^*)''(x) \\ & +[(f^*)'(x)]^2 - (f(x)-f^*(x))(f^*)''(x)\Big)x^2 \\ & -2\Big(\big(z(x)-z^*(x)\big)^\top (z^*)'(x) + (f(x)-f^*(x))(f^*)'(x)\Big)x\Big]\cdot(\delta l)^2 \\ & +\Big\langle \Big(p(x)\varphi''(z(x)) + I_n\Big)\delta z(x), \delta z(x)\Big\rangle \\ & +2\Big[\big(p(x)\varphi'(z(x)) + \big(z(x)-z^*(x)\big)^\top - x\big((z^*)'(x)\big)^\top\Big]\delta z(x)\cdot\delta l.\end{aligned}$$

Consequently,

$$\begin{aligned}R_{ll} = & \Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x)-z^*(x))^\top (z^*)''(x) \\ & +[(f^*)'(x)]^2 - (f(x)-f^*(x))(f^*)''(x)\Big)x^2 \\ & -2\Big(\big(z(x)-z^*(x)\big)^\top (z^*)'(x) + (f(x)-f^*(x))(f^*)'(x)\Big)x, \end{aligned} \tag{31}$$

$$R_{zz} = p(x)\varphi''(z(x)) + I_n, \tag{32}$$

$$R_{lz} = R_{zl}^\top = \big(p(x)\varphi'(z(x)) + \big(z(x)-z^*(x)\big)^\top - x\big((z^*)'(x)\big)^\top. \tag{33}$$

Further, $U = 1$, and finally, $S$ has the form

$$S = \begin{pmatrix} S_l \\ 0_n \end{pmatrix} \in \mathbb{R}^{n+1},$$

where $S_l = p_1(x) + f(x) - f^*(x) - x(f^*)'(x)$. Recall that the Riccati equation has the form

$$\frac{\mathrm{d}}{\mathrm{d}x}Q + QA + A^TQ + R - (S+QB)U^{-1}(S^T + B^TQ) = 0, \quad x\in[0,l],$$

where

$$Q(x) = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix}(x),$$

where

$$Q_{ll}(x)\in\mathbb{R}, \quad Q_{zl}(x) = \begin{pmatrix} Q_{z_1 l} \\ \dots \\ Q_{z_n l} \end{pmatrix}(x) \in \mathbb{R}^n,$$

$$Q_{lz}(x) = Q_{zl}^\top(x) = \begin{pmatrix} Q_{lz_1} & \dots & Q_{lz_n} \end{pmatrix}(x) \in \mathbb{R}^{n\top},$$

and

$$Q_{zz}(x) = \begin{pmatrix} Q_{z_1z_1} & \dots & Q_{z_1z_n} \\ \dots & \dots & \dots \\ Q_{z_nz_1} & \dots & Q_{z_nz_n} \end{pmatrix}(x)$$

is $n \times n$ symmetric matrix. Since $U = 1$, we have

$$(S + QB)U^{-1}(S^\top + B^\top Q) = (S + QB)(S + QB)^\top.$$

Further,

$$QB = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix}\begin{pmatrix} 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix}.$$

Hence

$$S + QB = \begin{pmatrix} S_l \\ 0 \end{pmatrix} + \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix}.$$

Consequently,

$$(S + QB)(S + QB)^\top = \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix}(Q_{lz}e_1 + S_l, e_1^\top Q_{zz})$$

$$= \begin{pmatrix} (Q_{lz}e_1 + S_l)^2 & (Q_{lz}e_1 + S_l)e_1^\top Q_{zz} \\ Q_{zz}e_1(Q_{lz}e_1 + S_l) & (Q_{zz}e_1)(e_1^\top Q_{zz}) \end{pmatrix}.$$

Moreover,

$$QA = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix}\begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix} = \begin{pmatrix} Q_{lz}A_{zl} & Q_{lz}A_{zz} \\ Q_{zz}A_{zl} & Q_{zz}A_{zz} \end{pmatrix},$$

$$A^\top Q = \begin{pmatrix} Q_{lz}A_{zl} & A_{zl}^\top Q_{zz} \\ A_{zz}^\top Q_{zl} & A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Here, $Q_{lz}A_{zl}$, $Q_{lz}A_{zz}$, $Q_{zz}A_{zl}$, $Q_{zz}A_{zz}$ are matrices of order $1 \times 1$, $1 \times n$, $n \times 1$, $n \times n$, respectively. Consequently,

$$QA + A^\top Q = \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Thus, according to (12), we obtain the matrix Riccati equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} + \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix} + \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix}$$
$$- \begin{pmatrix} (Q_{lz}e_1 + S_l)^2 & (Q_{lz}e_1 + S_l)e_1^\top Q_{zz} \\ (Q_{lz}e_1 + S_l)Q_{zz}e_1 & (Q_{zz}e)(e_1^\top Q_{zz}) \end{pmatrix} = 0, \quad x \in [0, l], \tag{34}$$

where the blocks of the matrix $R$ are determined by formulas (31)-(33). Further, the matrix $M$ has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

We set

$$\delta\xi = \begin{pmatrix} \delta X(0) \\ \delta X(l) \end{pmatrix}.$$

Then

$$\langle M\delta\xi, \delta\xi\rangle = \langle Q(0)\delta X(0), \delta X(0)\rangle - \langle Q(l)\delta X(l), \delta X(l)\rangle,$$

where

$$\delta X(0) = \begin{pmatrix} \delta l \\ \delta z(0) \end{pmatrix}, \quad \delta X(l) = \begin{pmatrix} \delta l \\ \delta z(l) \end{pmatrix}, \quad (\delta l)' = 0, \text{ i.e.,} \quad \delta l = \text{const}\,.$$

The condition $\mathcal{E}\delta\xi = 0$ in the definition of $\Sigma$ (see Section 2.3) means $K_{z_0}\delta z(0) + K_{z_l}\delta z(l) = 0$, $(\delta l)' = 0$. Consequently,

$$\delta X_0 := \delta X(0) = \begin{pmatrix} \delta l \\ \delta z(0) \end{pmatrix}, \quad \delta X_l := \delta X(l) = \begin{pmatrix} \delta l \\ \delta z(l) \end{pmatrix}.$$

Similarly to Theorem 2.4, the following theorem holds.

Theorem 3.4 *Assume that the strengthened Legendre condition is satisfied, $C_H = 0$, and there exists a symmetric solution $Q$ (with the entries belonging to $C^1$) of the Riccati matrix equation (3.5) on $[0, l]$ such that*

*(a) the matrix $M$ is nonnegative definite;*

*(b) for all pairs of vectors in $\mathbb{R}^{n+1}$*

$$\delta X_0 = \begin{pmatrix} \delta l \\ \delta z_0 \end{pmatrix}, \quad \delta X_l = \begin{pmatrix} \delta l \\ \delta z_l \end{pmatrix}$$

*the conditions $K_{z_0}\delta z_0 + K_{z_l}\delta z_l = 0$, $\delta l \in \mathbb{R}$, $\langle Q(0)\delta X_0, \delta X_0\rangle - \langle Q(l)\delta X_l, \delta X_l\rangle = 0$ imply that $\delta X_0 = 0$ or $\delta X_l = 0$. Then, the quadratic form $\Omega$ is positive definite on the subspace $\Sigma$.*

## 4. Numerical examples

### 4.1. Example 1

Consider a steady state scenario involving a single edge, governed by a semilinear differential equation. The control system has the form

$$z'(x) = \varphi(z(x)) + f(x), \quad x \in [0, l], \quad z(l) = 0,$$

where $z$ is one dimensional and $l$ is not fixed. The beam's behavior is described by the function

$$\varphi(z) = z - z^2.$$

Set

$$l^* = 1, \quad z^*(x) = -2 + x + x^2, \quad f^*(x) = 7 - 3x - 4x^2 + 2x^3 + x^4.$$

It is easy to check that the triple $(l^*, z^*, f^*(x))$ defines an admissible process of a given control system.

The cost functional is expressed as:

$$J = \frac{1}{2}\int_0^l \big((z(x) - z^*(x))^2 + (f(x) - f^*(x))^2\big)\,\mathrm{d}x + \frac{1}{2}(l - l^*)^2 \to \min.$$

The parameter $l$ is constrained, $l \in [\frac{1}{2}, \frac{3}{2}]$.

Obviously, $(l^*, z^*, f^*(x))$ is the solution to this problem. But assume that this solution is unknown and let us write down the necessary optimality conditions of Theorem 3.1.

Since $\int_0^l (x - l^*)\,\mathrm{d}x = \frac{1}{2}(l - l^*)^2 - \frac{1}{2}(l^*)^2$, we can consider the equivalent problem of minimizing the functional

$$J = \int_0^l F(x, z(x), f(x))\,\mathrm{d}x$$

with $F(x, z, f) = \frac{1}{2}\big((z - z^*(x))^2 + (f - f^*(x))^2\big) + x - l^*$.

Let the triple $(l, z(\cdot), f(\cdot))$ be a solution to this problem. Then, according to Theorem 3.1 there are numbers $\alpha_0 \geq 0$, $\beta$, and a continuously differentiable function $p : [0, l] \to \mathbb{R}$ such that

$$z'(x) = \varphi(z(x)) + f(x), \quad z(l) = 0,$$

$$-p'(x) = p(x)\varphi'(z(x)) + \alpha_0 F_z(x, z(x), f(x)), \quad x \in [0, l],$$

$$p(0) = 0, \quad p(l) = \beta,$$

$$p(x) + \alpha_0 F_f(x, z(x), f(x)) = 0, \quad x \in [0, l].$$

If $\alpha_0 = 0$, then $p(x) = 0$ and $\beta = 0$. Therefore, $\alpha_0 > 0$, and we put $\alpha_0 = 1$. Hence, taking into account the facts that $\varphi(z) = z - z^2$, $\varphi'(z) = 1 - 2z$, $F_z(x, z, f) = (z - z^*(x))$, and $F_f(x, z, f) = f - f^*(x)$, we get a system

$$\left.\begin{array}{l} z'(x) = z(x) - z^2(x) + f(x), \quad z(l) = 0, \\ -p'(x) = p(x)(1 - 2z(x)) + z - z^*(x), \quad p(0) = 0 \\ p(x) + f(x) - f^*(x) = 0. \end{array}\right\} \tag{35}$$

Theorem 3.1 gives one more necessary optimality condition for determining $(l, z(\cdot), f(\cdot))$. Recall that we are considering $l$ close to $l^* = 1$, which means $a < l < b$ with $a = 0.5$, $b = 1.5$. As we know, in this case $c_H = 0$. Since $\alpha_0 = 1$, this condition looks like

$$c_H = p^x(x) + p(x)\big(\varphi(z(x)) + f(x)\big) + F(x, z(x), f(x)) = 0 \quad \forall\, x \in [0, l],$$

where $p^x(x) = \alpha_0 \int_x^l F_x(y, z(y), f(y))\, \mathrm{d}y$, $x \in [0, l]$. Considering that $p^x(l) = 0$, $z(l) = 0$, and $-p(l) = f(l) - f^*(l)$, we get

$$\begin{aligned} 0 &= p(l)\big(\varphi(z(l)) + f(l)\big) + F(l, z(l), f(l)) \\ &= p(l)\big(\varphi(0) + f(l)\big) + F(l, 0, f(l)) \\ &= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}((f(l) - f^*(l))^2 + l - 1 \\ &= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1, \end{aligned}$$

that is

$$p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1 = 0. \tag{36}$$

Conditions (35) and (36) constitute a complete system of necessary optimality conditions for determining $(l, z(\cdot), f(\cdot))$. Obviously, the triple $p(x) = 0$, $f(x) = f^*(x)$, $z(x) = z^*(x)$ is a solution to this system.

We will now show numerical results for this problem. We conducted the computations using the finite element method and the Newton method to handle the nonlinear component. Here are the results. Fig. 2 illustrates the variation of the cost functional with respect to the length parameter. It is observed that the cost functional attains its minimum value at $l = 1 = l^*$, indicating the optimality of this length. This signifies that the length $l = 1$ is the optimal choice based on the minimization of the cost functional. In Fig. 3, we show the optimal control and state under the optimal length. Then, we computed the $L^2$ norm error between the numerical solution and the analytical solution to assess the accuracy of the results:

$$f_{err} = ||f(x) - f^*(x)|| = 1.6236e - 12,\ z_{err} = ||z(x) - z^*(x)|| = 4.5426e - 11.$$

### 4.2. Example 2

Consider the steady state for a single beam governed by the semilinear differential equation (Eq. 7). We present results from numerical simulations. The flexibility matrices are given by

$$\mathbf{C} = \operatorname{diag}\left(c_1, c_2, c_3, c_4, c_5, c_6\right)^{-1} = \operatorname{diag}\left(10^4, 10^4, 10^4, 500, 500, 500\right)^{-1}.$$

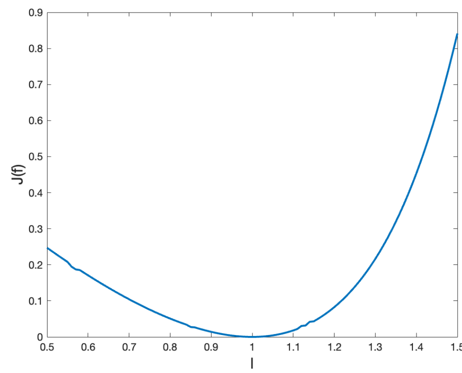


Figure 2. Cost with respect to $l$

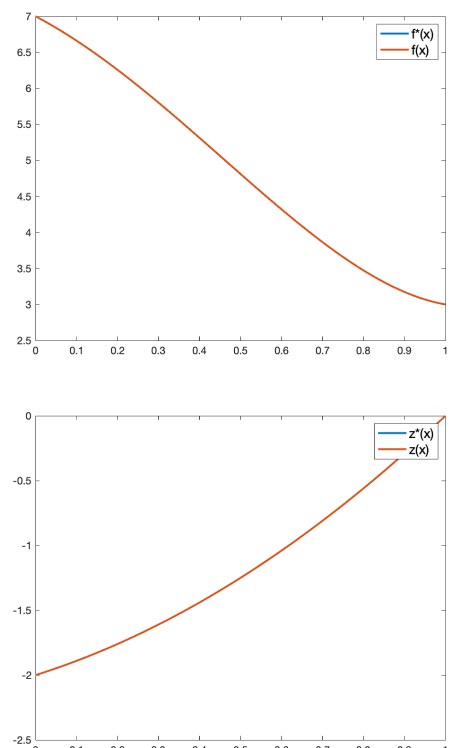


Figure 3. Optimal control and optimal state

In Eq. (14), upon setting $f^*(x) = -1$, the steady state values are $z_1^* = -x + 1, z_2^* = 0, \ldots, z_6^* = 0$. For IGEB model, the function

$$\varphi(r) = -\mathbf{E}(x)r + \mathbf{L}(r)\mathbf{C}r,$$

and its derivative is given by

$$\varphi'(r) = -\mathbf{E}(x) + (\mathbf{L}(r)\mathbf{C}r)' := -\mathbf{E}(x) + \bar{G}(r).$$

The optimality system of equations can be written as

$$\begin{cases} z'(x) = -\mathbf{E}(x)z(x) + \mathbf{L}(z)\mathbf{C}z(x) + e_1(f^* - p_1(x)), & x \in [0, l] \\ -p'(x) = -p(x)\mathbf{E}(x) + p(x)\bar{G}(z) + z(x) - z^*(x), & x \in [0, l] \\ z_i(l) = 0, & i = 1, 2, \ldots, 6 \\ p_i(0) = 0, & i = 1, 2, \ldots, 6. \end{cases}$$

The weak form of this system is given by:

$$\begin{cases} -\int_0^l \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x}, z \right\rangle dx - z(0)\psi(0) + \int_0^l \langle \mathbf{E}z, \psi \rangle \, dx - \int_0^l \langle \mathbf{CL}(z)z, \psi \rangle \, dx \\ \qquad + \int_0^l e_1 \langle p, \psi \rangle \, dx = \int_0^l e_1 \langle f^*, \psi \rangle \, dx, \; \forall \psi \in V_1 \\ \int_0^l \left\langle \frac{\mathrm{d}\eta}{\mathrm{d}x}, p \right\rangle dx - p(1)\eta(1) + \int_0^l \langle \mathbf{E}p, \eta \rangle \, dx - \int_0^l \left\langle \bar{G}(z)p, \eta \right\rangle dx \\ \qquad - \int_0^l \langle z, \eta \rangle \, dx = -\int_0^l \langle z^*, \eta \rangle \, dx, \; \forall \eta \in V_2, \end{cases} \tag{37}$$

where

$$V_1 := \{\psi \in H^1\left(0, l; \mathbb{R}^6\right), \psi(1) = 0\},$$

and

$$V_2 := \{\eta \in H^1\left(0, l; \mathbb{R}^6\right), \eta(0) = 0\}.$$

In numerical discretization, the interval $[0, l]$ is discretized into $N_x$ points $\{x_k\}_{k=1}^{N_x}$, where $x_1 = 0$ and $x_{N_x} = l$. Each subinterval ($\omega^e := [x_{2e-1}, x_{2e+1}]$) for $e \in \{1, 2, \ldots, N_e\}$ constitutes an element. These elements are defined by the points $x_{2e-1}$, $x_{2e}$, and $x_{2e+1}$ and have a uniform length $h_e = x_{2e+1} - x_{2e-1}$. It is important to note that $N_x = 2N_e + 1$.

We utilize $\mathbb{P}_2$ (quadratic) elements to define function spaces $V_{1,h}$ and $V_{2,h}$ as described below:

$$V_{1,h} := \left\{\psi \in C^0\left([0, l]; \mathbb{R}^{\mathbb{N}_6}\right) : \psi|_{\omega^e} \in \left(\mathbb{P}_2\right)^{\mathbb{N}_6} \text{ for all } e \in \{1, \ldots, N_e\}, \psi(1) = 0\right\},$$

$$V_{2,h} := \left\{\eta \in C^0\left([0, l]; \mathbb{R}^{\mathbb{N}_6}\right) : \eta|_{\omega^e} \in \left(\mathbb{P}_2\right)^{\mathbb{N}_6} \text{ for all } e \in \{1, \ldots, N_e\}, \eta(0) = 0\right\}.$$

The Lagrange finite element approximations for $z_i(x)$ and $p_i(x)$ are represented by the following expressions:

$$z_i(x) = \sum_{j=1}^{N_x} Z_{i,j}\psi_j(x), \quad p_i(x) = \sum_{j=1}^{N_x} P_{i,j}\eta_j(x),$$

where the unknown coefficients $Z_{i,j}$ denote the value of $z_i$ at the $\mathbb{P}_2$ basis function $\psi_j$, whose value is 1 at node $x_j$ and 0 at other nodes, and similarly for $P_{i,j}$. In the discretized system, we define the following matrices and vectors:

$$A_1 = \int_0^l \boldsymbol{\psi}\boldsymbol{\psi}^\top,\ A_2 = \int_0^l \boldsymbol{\psi}(\boldsymbol{\psi}')^\top,\ A_3[z] = \int_0^l z\boldsymbol{\psi}\boldsymbol{\psi}^\top,$$

$$\bar{A}_1 = \int_0^l \boldsymbol{\eta}\boldsymbol{\eta}^\top,\ \bar{A}_2 = \int_0^l \boldsymbol{\eta}(\boldsymbol{\eta}')^\top,\ \bar{A}_3[z] = \int_0^l z\boldsymbol{\eta}\boldsymbol{\eta}^\top,$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \cdots, \psi_{N_x})^\top$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_{N_x})^\top$. The matrix form of Eq. 37 can be written as

$$\begin{aligned} -K_{s,1}Z - M_{\mathbf{CL}(z)}Z + \bar{e}_1\bar{M}P &= \bar{e}_1\hat{F}, \\ K_{s,2}P - M_{G(z)}P - \bar{M}Z &= -\hat{Z}, \end{aligned} \tag{38}$$

i.e.,

$$\begin{pmatrix} -K_{s,1} & \bar{e}_1\bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix}\begin{pmatrix} Z \\ P \end{pmatrix} - \begin{pmatrix} M_{\mathbf{CL}(z)} & \\ & M_{G(z)} \end{pmatrix}\begin{pmatrix} Z \\ P \end{pmatrix} = \begin{pmatrix} \bar{e}_1\hat{F} \\ -\hat{Z} \end{pmatrix}, \tag{39}$$

where $\bar{e}_1 = \mathrm{diag}(\mathbb{I}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x})$. $\mathbb{O}_{N_x}$ is the matrix of zeros. Furthermore, the vectors $Z$ and $P$ are defined as:

$$\begin{aligned} Z &= (Z_{1,1}, \cdots, Z_{1,N_x}, \cdots, Z_{6,1}, \cdots, Z_{6,N_x})^\top, \\ P &= (P_{1,1}, \cdots, P_{1,N_x}, \cdots, P_{6,1}, \cdots, P_{6,N_x})^\top. \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{Z} &= \left(\hat{Z}_{1,1}, \cdots, \hat{Z}_{1,N_x}, \cdots, \hat{Z}_{6,1}, \cdots, \hat{Z}_{6,N_x}\right)^\top, \\ \hat{F} &= \left(\hat{F}_1, \cdots, \hat{F}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}\right)^\top, \end{aligned}$$

where $\hat{Z}_{i,j}$ represents the value of $z_i^*$ at the basis function $\psi_j$ and $\hat{F}_j$ represents the value of $f^*$ at the basis function $\eta_j$.

The other matrices are defined by:

$$K_{s,1} = \begin{pmatrix} A_2 & & & & & \\ & A_2 & & & & \\ & & A_2 & & & \\ & & & A_2 & & \\ & & A_1 & & A_2 & \\ & -A_1 & & & & A_2 \end{pmatrix},$$

$$K_{s,2} = \begin{pmatrix} \bar{A}_2 & & & & & \\ & \bar{A}_2 & & & & \\ & & \bar{A}_2 & & & \\ & & & \bar{A}_2 & & \\ & & -\bar{A}_1 & & \bar{A}_2 & \\ & \bar{A}_1 & & & & \bar{A}_2 \end{pmatrix},$$

$$M_{G(z)} = \begin{pmatrix} \mathbf{0} & c_6\bar{A}_3(z_6) & -c_5\bar{A}_3(z_5) & \mathbf{0} & -c_5\bar{A}_3(z_3) & c_6\bar{A}_3(z_2) \\ -c_6\bar{A}_3(z_6) & \mathbf{0} & c_4\bar{A}_3(z_4) & c_4\bar{A}_3(z_3) & \mathbf{0} & -c_6\bar{A}_3(z_1) \\ c_5\bar{A}_3(z_5) & -c_4\bar{A}_3(z_4) & 0 & -c_4\bar{A}_3(z_2) & c_5\bar{A}_3(z_1) & \mathbf{0} \\ \mathbf{0} & (c_3-c_2)\bar{A}_3(z_3) & (c_3-c_2)\bar{A}_3(z_2) & \mathbf{0} & (c_6-c_5)\bar{A}_3(z_6) & (c_6-c_5)\bar{A}_3(z_5) \\ (c_1-c_3)\bar{A}_3(z_3) & \mathbf{0} & (c_1-c_3)\bar{A}_3(z_1) & (c_4-c_6)\bar{A}_3(z_6) & \mathbf{0} & (c_4-c_6)\bar{A}_3(z_4) \\ (c_2-c_1)\bar{A}_3(z_2) & (c_2-c_1)\bar{A}_3(z_1) & \mathbf{0} & (c_5-c_4)\bar{A}_3(z_5) & (c_5-c_4)\bar{A}_3(z_4) & \mathbf{0} \end{pmatrix},$$

$$M_{\mathbf{CL}(z)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_5A_3(z_3) & c_6A_3(z_2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c_4A_3(z_3) & \mathbf{0} & -c_6A_3(z_1) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_4A_3(z_2) & c_5A_3(z_1) & \mathbf{0} \\ \mathbf{0} & -c_2A_3(z_3) & c_3A_3(z_2) & \mathbf{0} & -c_5A_3(z_6) & c_6A_3(z_5) \\ c_1A_3(z_3) & \mathbf{0} & -c_3A_3(z_1) & c_4A_3(z_6) & \mathbf{0} & c_6A_3(z_4) \\ -c_1A_3(z_2) & c_2A_3(z_1) & \mathbf{0} & -c_4A_3(z_5) & c_5A_3(z_4) & \mathbf{0} \end{pmatrix},$$

$$\bar{M} = \operatorname{diag}(A_1, A_1, A_1, A_1, A_1, A_1).$$

Denote

$$A = \begin{pmatrix} -K_{s,1} & \bar{e}_1\bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix},\ N_L(z) = \begin{pmatrix} M_{\mathbf{CL}(z)} & \\ & M_{G(z)} \end{pmatrix},$$

$$W = \begin{pmatrix} Z \\ P \end{pmatrix},\ \mathbf{F} = \begin{pmatrix} \bar{e}_1\hat{F} \\ -\hat{Z} \end{pmatrix}.$$

So, Eq. (39) becomes:

$$AW - N_L(z)W = \mathbf{F}, \tag{40}$$

where $N_L(z)$ represents the nonlinear component. The iterative process is governed by the equation:

$$AW^{[n+1]} - N_L(z^{[n]})W^{[n+1]} = \mathbf{F},$$

where the superscript $[n]$ denotes the $n$-th iteration. With an auxiliary function $S_n$:

$$S_n(\zeta) = A\zeta - N_L(z^{[n]})\zeta - \mathbf{F},$$

equation (40) becomes $S_n(W^{[n+1]}) = 0$. To find an approximate solution to $S_n(\zeta) = 0$, we employ the Newton-Raphson method, i.e., finding $\zeta$ such that $S_n(\zeta) = 0$, by means of the scheme:

$$\zeta_{n+1} = \zeta_n - \left(\operatorname{Jac} S_n\left(\zeta_n\right)\right)^{-1} S_n\left(\zeta_n\right),$$

where $\operatorname{Jac} S_n = A - N_L(z^{[n]})$.

For our problem, the initial data is set to zero. The following Algorithm 1 outlines the steps taken to approximate the solution to Eq. (37). The results of this iterative scheme are visually presented in Fig. 4. The diagrams in Fig. 4 demonstrate that the optimal state and control closely approach $z^*$ and $f^*$, respectively, when $l = l^*$. Furthermore, Fig. 5 illustrates the fact that the cost is convex with respect to the length of beam with the unique minimizer. The optimal design corresponds to the length $l = l^* = 1$.

**Algorithm 1:** Solve the obtained ODE for $W$

Set $\mathbf{C}$, $f^*$, $z^*$ ;
Given initial guesses $z^0$ ;
**while** *convergence* **do**
  $\zeta_{n+1} = \zeta_n - \left(\operatorname{Jac} S_n\left(\zeta_n\right)\right)^{-1} S_n\left(\zeta_n\right)$ ;
**end**
$W = \zeta_{n+1}$

## 5. Network modeling: optimal control problem P

### 5.1. Statment of problem $P$

Consider now the following optimal control problem that arises in network modeling. Let $z_i(x_i)$ be state variables, $f_i(x_i)$ be controls, where $x_i \in [0, l_i]$, $l_i > 0$, $i = 1, \ldots, m$. Here $z_i = (z_{i1}, \ldots, z_{in})^\top \in \mathbb{R}^n$, $f_i \in \mathbb{R}$, $i = 1, \ldots, m$. We assume that $z_i(x_i)$ are continuously differentiable functions and $f_i(x_i)$ are continuous functions, $i = 1, \ldots, m$.

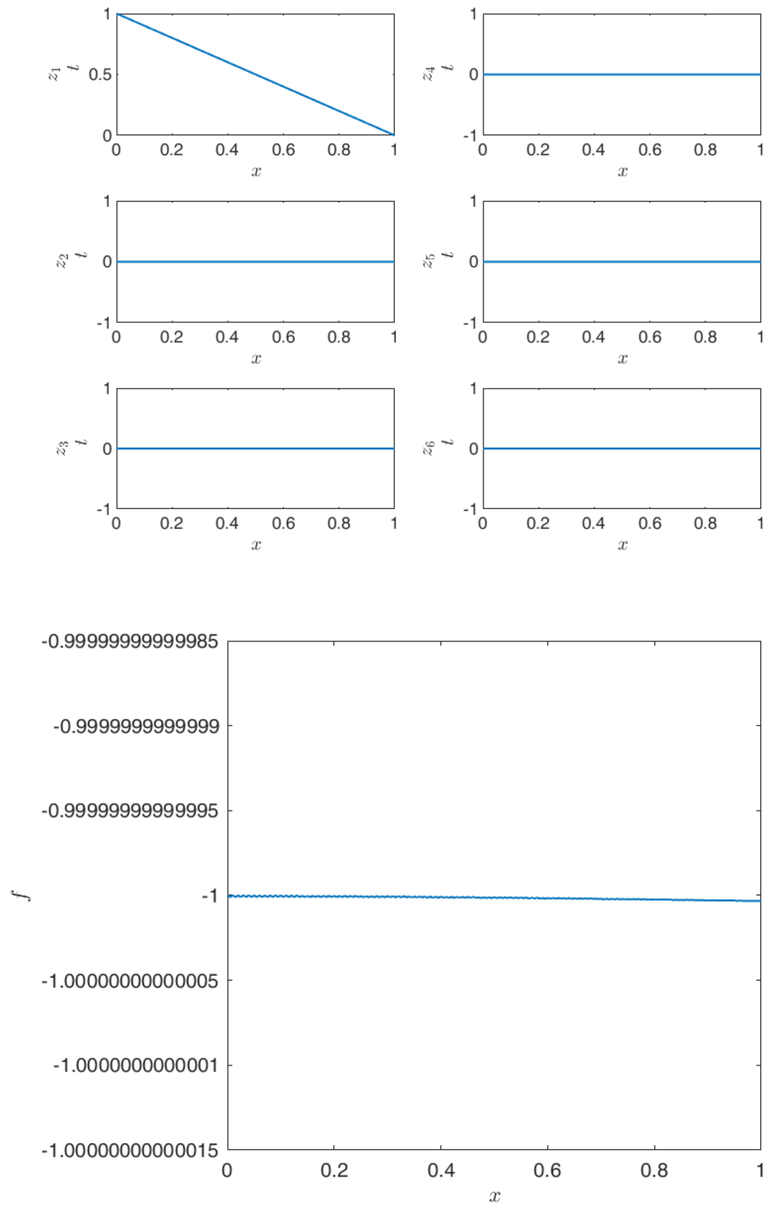


Figure 4. Optimal state $z$(top) and optimal force $f$(bottom) ($l = l^*$)

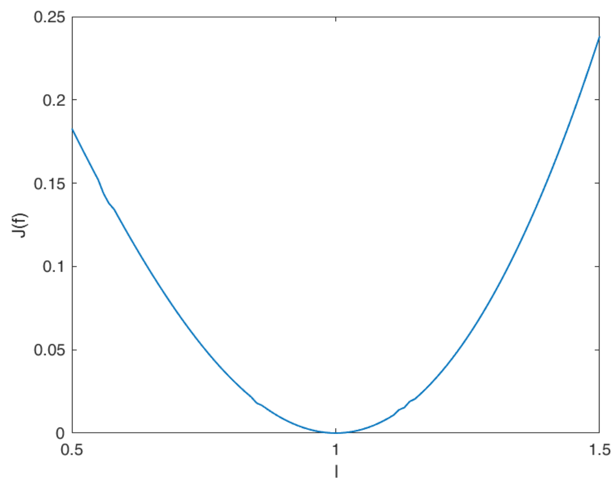


Figure 5. Cost with respect to $l$

**Problem** $P$**:** The control system has the form

$$\frac{\mathrm{d}z_i(x_i)}{\mathrm{d}x_i} = \varphi(z_i(x_i)) + e_1 f_i(x_i), \quad x_i \in [0, l_i], \quad i = 1, \dots, m, \tag{41}$$

where $e_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^n$, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function.

Additionally, there is a constraint:

$$K(z_1(0), z_1(l_1) \dots, z_m(0), z_m(l_m)) = 0, \tag{42}$$

where $K = (K_1, \dots, K_r) \in \mathbb{R}^r$.

The cost to be minimized is:

$$J = \sum_{i=1}^{m} \int_0^{l_i} F_i(x_i, z_i(x_i), f_i(x_i))\, \mathrm{d}x_i, \tag{43}$$

where

$$F_i(x_i, z_i, f_i) = \frac{1}{2}|z_i - z_i^*(x_i)|^2 + \frac{1}{2}(f_i - f_i^*(x_i))^2, \quad x_i \in [0, l_i],$$

$z_i^*(\cdot)$ and $f_i^*(\cdot)$ are given twice continuously differentiable functions, $i = 1, \dots, m$. The lengths of the intervals $l_i$ satisfy the constraints

$$l_i \in [a, b], \quad \text{where} \quad 0 < a < b.$$

We assume that $z_i^*$ and $f_i^*$ are given on $[0,b]$, $i=1,\dots,m$.

We will be interested in optimality conditions in problem P for an admissible process

$$(z_i(x_i), f_i(x_i) \mid x_i \in [0,l_i])_{i=1}^m. \tag{44}$$

This is not a standard optimal control problem, since it has many independent variables $x_i$, and each independent variable changes over its own interval $[0,l_i]$. Now our goal is to represent this problem as a standard problem with one independent variable.

### 5.2. Change of independent variables $x_i$. Problem $\tilde{P}$ on the interval $[0,1]$

We shall rewrite this problem on the interval $[0,1]$. Let $t\in[0,1]$ be a new independent variable. We set $x_i=\tilde{x}_i(t)=l_i\cdot t$, $t\in[0,1]$, $i=1,\dots,m$. Then $\tilde{x}_i(t)\in[0,l_i]$, $i=1,\dots,m$. We consider each $x_i=\tilde{x}_i(t)$ as a new state variable, $i=1,\dots,m$. Moreover, we treat each $l_i=\tilde{l}_i(t)$ also as a new state variable, that is constant on $[0,1]$, $i=1,\dots,m$.

Therefore, we have

$$\frac{\mathrm{d}\tilde{l}_i(t)}{\mathrm{d}t}=0, \quad \frac{\mathrm{d}\tilde{x}_i(t)}{\mathrm{d}t}=\tilde{l}_i(t), \quad t\in[0,1], \quad \tilde{x}_i(0)=0, \quad i=1,\dots,m.$$

Further, we set $\tilde{z}_i(t)=z_i(\tilde{x}_i(t))=z_i(l_it)$, $\tilde{f}_i(t)=f_i(\tilde{x}_i(t))=f_i(l_it)$, $t\in[0,1]$, $i=1,\dots,m$. Then

$$\frac{\mathrm{d}\tilde{z}_i}{\mathrm{d}t}=\frac{\mathrm{d}z_i}{\mathrm{d}x_i}\tilde{l}_i, \quad i=1,\dots,m.$$

Also, note that

$$J:=\sum_{i=1}^m\int_0^{l_i}F_i(x_i,z_i(x_i),f_i(x_i))\,\mathrm{d}x_i=\sum_{i=1}^m\int_0^1\tilde{l}_i(t)F_i(\tilde{x}_i(t),\tilde{z}_i(t),\tilde{f}_i(t))\,\mathrm{d}t.$$

In what follows, we will continue to use the tilde for the variables in the interval $[0,1]$.

Thus, we get a **Problem** $\tilde{P}$ on [0,1]:

$$\frac{\mathrm{d}\tilde{l}_i(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t), \quad t \in [0,1], \quad i = 1,\dots,m, \tag{45}$$

$$\frac{\mathrm{d}\tilde{z}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t)\big(\varphi(\tilde{z}_i(t)) + e_1\tilde{f}_i(t)\big), \quad t \in [0,1], \quad i = 1,\dots,m, \tag{46}$$

$$\tilde{x}_i(0) = 0, \quad -\tilde{l}_i(0) + a \le 0, \quad \tilde{l}_i(0) - b \le 0, \quad i = 1,\dots,m, \tag{47}$$

$$K(\tilde{z}_1(0), \tilde{z}_1(1) \dots, \tilde{z}_m(0), \tilde{z}_m(1)) = 0, \tag{48}$$

$$J = \sum_{i=1}^{m} \int_0^1 \tilde{l}_i(t) F_i(\tilde{x}_i(t), \tilde{z}_i(t), \tilde{f}_i(t))\,\mathrm{d}t \to \min. \tag{49}$$

### 5.3. Local minimum principle for Problem $\tilde{P}$

The endpoint Lagrange function is:

$$\begin{aligned} \tilde{L} \quad &= \quad \textstyle\sum_{i=1}^m \alpha_{ai}(-\tilde{l}_i(0) + a) + \sum_{i=1}^m \alpha_{bi}(\tilde{l}_i(0) - b) \\ &+ \quad \textstyle\sum_{i=1}^m \beta_{xi}\tilde{x}_i(0) + \beta K(\tilde{z}_1(0), \tilde{z}_1(1) \dots, \tilde{z}_m(0), \tilde{z}_m(1)), \end{aligned}$$

where $\alpha_{ai}$, $\alpha_{bi}$, $\beta_{xi}$ are numbers, $\beta \in \mathbb{R}^{r\top}$ is a row vector of dimension $r$. The Hamiltonian is:

$$\tilde{H} = \sum_{i=1}^{m} \tilde{p}^{x_i}\tilde{l}_i + \sum_{i=1}^{m} \tilde{p}^{z_i}\tilde{l}_i\big(\varphi(\tilde{z}_i) + e_1\tilde{f}_i\big) + \alpha_0 \sum_{i=1}^{m} \tilde{l}_i F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i),$$

where $\alpha_0$, $\tilde{p}^{x_i}$ are numbers and $p^{z_i}$ are row vectors of dimension $n$. It is convenient to introduce

$$\tilde{H}^i := \tilde{l}_i\big(\tilde{p}^{x_i} + \tilde{p}^{z_i}\big(\varphi(\tilde{z}_i) + e_1\tilde{f}_i\big) + \alpha_0 F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i)\big), \quad i = 1,\dots,m.$$

Then $\tilde{H} = \sum\limits_{i=1}^{m} \tilde{H}^i$.

Let us write down the necessary first-order optimality conditions at an admissible point

$$(\tilde{l}_i(\cdot), \tilde{x}_i(\cdot), \tilde{z}_i(\cdot), \tilde{f}_i(\cdot))_{i=1}^m, \tag{50}$$

which corresponds to the process (44).

The partial derivatives of $\tilde{H}$ with respect to $\tilde{l}_i, \tilde{x}_i, \tilde{z}_i, \tilde{f}_i$ have the form

$$\tilde{H}_{\tilde{l}_i} = \tilde{p}^{x_i} + \tilde{p}^{z_i}\big(\varphi(\tilde{z}_i) + e_1\tilde{f}_i\big) + \alpha_0 F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i) = \tfrac{\tilde{H}^i}{\tilde{l}_i},$$

$$\begin{aligned}\tilde{H}_{\tilde{x}_i} &= \alpha_0\tilde{l}_i F_{i\tilde{x}_i}(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i)\\ &= -\alpha_0\tilde{l}_i\big(\tilde{z}_i - z_i^*(\tilde{x}_i)\big)^\top (z_i^*)'(\tilde{x}_i) - \alpha_0\tilde{l}_i(\tilde{f}_i - f_i^*(\tilde{x}_i))(f_i^*)'(\tilde{x}_i),\end{aligned}$$

$$\tilde{H}_{\tilde{z}_i} = \tilde{p}^{z_i}\tilde{l}_i\varphi'(\tilde{z}_i)^T + \alpha_0\tilde{l}_i\big(\tilde{z}_i - z_i^*(\tilde{x}_i)\big)^\top,$$

$$\tilde{H}_{\tilde{f}_i} = \tilde{p}^{z_i}\tilde{l}_i e_1 + \alpha_0\tilde{l}_i F_{\tilde{f}_i}(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i) = \tilde{p}^{z_i}\tilde{l}_i e_1 + \alpha_0\tilde{l}_i(\tilde{f}_i - f_i^*(\tilde{x}_i)).$$

We use these formulas below. In what follows we remember that $\tilde{l}_i(t) = l_i =$ const.

The LMP conditions at the point (50) in problem (45) -(49) are as follows.

(a) The nonnegativity conditions: $\alpha_0 \geq 0, \quad \alpha_{ai} \geq 0, \quad \alpha_{bi} \geq 0, \quad i = 1, \ldots, m.$

(b) The nontriviality condition: $\alpha_0 + \sum\limits_{i=1}^m \alpha_{ai} + \sum\limits_{i=1}^m \alpha_{bi} + \sum\limits_{i=1}^m |\beta_{xi}| + |\beta| > 0,$

(c) The complementarity conditions: $\alpha_{ai}(\tilde{l}_i(0) - a) = 0,\ \alpha_{bi}(\tilde{l}_i(0) - b) = 0,\ i = 1, \ldots, m.$

(d) The adjoint equations:

$$-\frac{\mathrm{d}\tilde{p}^{l_i}(t)}{\mathrm{d}t} = \tilde{p}^{x_i}(t) + \tilde{p}^{z_i}(t)\big(\varphi(\tilde{z}_i(t)) + e_1\tilde{f}_i(t)\big) + \alpha_0 F(\tilde{x}_i(t), \tilde{z}_i(t), \tilde{f}_i(t)), \tag{51}$$

$$\begin{aligned}-\frac{\mathrm{d}\tilde{p}^{x_i}(t)}{\mathrm{d}t} &= -\alpha_0 l_i\big(\tilde{z}_i(t) - z_i^*(\tilde{x}_i(t))\big)^\top (z_i^*)'(\tilde{x}_i(t))\\ &\quad -\alpha_0 l_i\big(\tilde{f}_i(t) - f_i^*(\tilde{x}_i(t))\big)(f_i^*)'(\tilde{x}_i(t)),\end{aligned} \tag{52}$$

$$-\frac{\mathrm{d}\tilde{p}^{z_i}(t)}{\mathrm{d}t} = \tilde{p}^{z_i}(t) l_i\varphi'(\tilde{z}_i(t)) + \alpha_0 l_i\big(\tilde{z}_i(t) - z_i^*(\tilde{x}_i(t))\big)^\top, \tag{53}$$

$$t \in [0,1], \quad i = 1, \ldots, m.$$

(e) The transversality conditions:

$$\begin{aligned}&-\tilde{p}^{l_i}(0) = -\alpha_{ai} + \alpha_{bi}, && \tilde{p}^{l_i}(1) = 0,\\ &-\tilde{p}^{x_i}(0) = \beta_{x_i}, && \tilde{p}^{x_i}(1) = 0,\\ &-\tilde{p}^{z_i}(0) = \beta K_{z_i(0)}, && \tilde{p}^{z_i}(1) = \beta K_{z_i(1)}, \quad i = 1, \ldots, m.\end{aligned}$$

By integrating equation (51) over $[0,1]$ and using the first two transversality conditions, we obtain

$$\alpha_{ai}-\alpha_{bi}=\int_0^1 \Big(\tilde{p}^{x_i}(t)+\tilde{p}^{z_i}(t)\big(\varphi(\tilde{z}_i(t))+e_1\tilde{f}_i(t)\big)+\alpha_0 F_i(\tilde{x}_i(t),\tilde{z}_i(t),\tilde{f}_i(t))\Big)\,\mathrm{d}t.$$

Multiplying this equality by $l_i$, we get

$$(\alpha_{ai}-\alpha_{bi})l_i=\int_0^1 \tilde{H}^i(t)\,\mathrm{d}t, \quad i=1,\dots,m,$$

where $\tilde{H}^i(t):=l_i\Big(\tilde{p}^{x_i}(t)+\tilde{p}^{z_i}(t)\big(\varphi(\tilde{z}_i(t))+e_1\tilde{f}_i(t)\big)+\alpha_0 F_i(\tilde{x}_i(t),\tilde{z}_i(t),\tilde{f}_i(t))\Big)$.

(f) The conditions $\tilde{H}_{\tilde{f}_i}=\tilde{p}^{z_i}(t)l_ie_1+\alpha_0 l_i\big(\tilde{f}_i(t)-f_i^*(\tilde{x}_i(t))\big)=0$, $i=1,\dots,m$.

Since $l_i>0$ and $\tilde{p}^{z_i}(t)e_1=\tilde{p}_1^{z_i}(t)$, we get

$$\tilde{p}_1^{z_i}(t)+\alpha_0\big(\tilde{f}_i(t)-f_i^*(\tilde{x}_i(t))\big)=0, \quad t\in[0,1], \quad i=1,\dots,m.$$

(g) Finally, the condition $\tilde{H}(t)=\text{const}$ has the form: there exists a constant $\hat{c}_H$ such that

$$\sum_{i=1}^m l_i\Big(\tilde{p}^{x_i}(t)+\tilde{p}^{z_i}(t)\big(\varphi(\tilde{z}_i(t))+e_1\tilde{f}_i(t)\big)+\alpha_0 F_i(\tilde{x}_i(t),\tilde{z}_i(t),\tilde{f}_i(t))\Big)=\tilde{c}_H \quad \forall\, t\in[0,1].$$

By integrating equation (g) over $[0,1]$ and using the condition $\tilde{H}=\sum_{i=1}^m \tilde{H}^i$, we get

$$\tilde{c}_H=\sum_{i=1}^m\int_0^1 \tilde{H}^i(t)\,\mathrm{d}t.$$

Equations

$$\frac{\mathrm{d}\tilde{l}_i}{\mathrm{d}t}=0,\ \frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}t}=\tilde{H}^i_{\tilde{p}^{x_i}},\ \frac{\mathrm{d}\tilde{z}_i}{\mathrm{d}t}=\tilde{H}^i_{\tilde{p}^{z_i}},\ -\frac{\mathrm{d}\tilde{p}^{x_i}}{\mathrm{d}t}=\tilde{H}^i_{\tilde{x}_i},\ -\frac{\mathrm{d}\tilde{p}^{z_i}}{\mathrm{d}t}=\tilde{H}^i_{\tilde{z}_i},\ \tilde{H}^i_{\tilde{f}_i}=0,$$

imply $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}^i(t)=0$, whence it follows that $\tilde{H}^i(t)=\text{const}\ \ \forall\, t\in[0,1]$, $i=1,\dots,m$. We set

$$\tilde{H}^i(t)=\tilde{c}_{H^i}, \quad t\in[0,1], \quad i=1,\dots,m.$$

Then

$$\int_0^1 \tilde{H}^i(t)\,\mathrm{d}t=\tilde{c}_{H^i}, \quad i=1,\dots,m.$$

Consequently,

$$\tilde{c}_H = \sum_{i=1}^{m} \tilde{c}_{H^i}, \quad \tilde{H}(t) = \tilde{c}_H, \quad t \in [0,1].$$

Using the relation $(\alpha_{ai} - \alpha_{bi}) l_i = \int_0^1 \tilde{H}^i(t)\,\mathrm{d}t$, we obtain

$$\tilde{c}_{H^i} = (\alpha_{ai} - \alpha_{bi}) l_i, \quad i = 1, \dots, m.$$

From these relations, together with the complementary slackness conditions (c), the following statements follow: for any $i = 1, \dots, m$ we have

(1) if $a < l_i < b$, then $\alpha_{ai} = \alpha_{bi} = 0$ and, therefore, $\tilde{c}_{H^i} = 0$,
(2) if $l_i = a$, then $\alpha_{bi} = 0$ and, therefore, $\tilde{c}_{H^i} = \alpha_{ai} l_i \geq 0$,
(3) if $l_i = b$, then $\alpha_{ai} = 0$ and, therefore, $\tilde{c}_{H^i} = -\alpha_{bi} l_i \leq 0$,
(4) moreover, if $\tilde{c}_{H^i} > 0$, then $\alpha_{ai} > 0$, and, therefore, $l_i = a$; if $\tilde{c}_{H^i} < 0$, then $\alpha_{bi} > 0$, and, therefore, $l_i = b$.

Thus, we obtain the following **result**.

If $(\tilde{l}_i(\cdot), \tilde{x}_i(\cdot), \tilde{z}_i(\cdot), \tilde{f}_i(\cdot))_{i=1}^m$ is a local minimum in problem $\tilde{P}$, then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{r\top}$ and continuously differentiable functions $\tilde{p}^{x_i}(t)$, $\tilde{p}^{z_i}(t)$, $t \in [0,1]$, $i = 1, \dots, m$ such that the following system of optimality conditions holds:

$$\frac{\mathrm{d}\tilde{l}_i(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t), \quad t \in [0,1], \quad i = 1, \dots, m,$$
$$\frac{\mathrm{d}\tilde{z}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t)\big(\varphi(\tilde{z}_i(t)) + e_1 \tilde{f}_i(t)\big), \quad t \in [0,1], \quad i = 1, \dots, m,$$
$$\tilde{x}_i(0) = 0, \quad -\tilde{l}_i(0) + a \leq 0, \quad \tilde{l}_i(0) - b \leq 0, \quad i = 1, \dots, m,$$
$$K(\tilde{z}_1(0), \tilde{z}_1(1) \dots, \tilde{z}_m(0), \tilde{z}_m(1)) = 0,$$
$$-\frac{\mathrm{d}\tilde{p}^{x_i}(t)}{\mathrm{d}t} = -\alpha_0 l_i \big(\tilde{z}_i(t) - z_i^*(\tilde{x}_i(t))\big)^\top (z_i^*)'(\tilde{x}_i(t))$$
$$-\alpha_0 l_i \big(\tilde{f}_i(t) - f_i^*(\tilde{x}_i(t))\big)(f_i^*)'(\tilde{x}_i(t)), \quad t \in [0,1], \ \tilde{p}^{x_i}(1) = 0, \ i = 1, \dots, m,$$
$$-\frac{\mathrm{d}\tilde{p}^{z_i}(t)}{\mathrm{d}t} = \tilde{p}^{z_i}(t) l_i \varphi'(\tilde{z}_i(t)) + \alpha_0 l_i \big(\tilde{z}_i(t) - z_i^*(\tilde{x}_i(t))\big)^\top, \ t \in [0,1], \ i = 1, \dots, m,$$
$$-\tilde{p}^{z_i}(0) = \beta K_{z_i(0)}, \quad \tilde{p}^{z_i}(1) = \beta K_{z_i(1)}, \quad i = 1, \dots, m,$$
$$\tilde{p}_1^{z_i}(t) + \alpha_0 \big(\tilde{f}_i(t) - f_i^*(\tilde{x}_i(t))\big) = 0, \quad t \in [0,1], \quad i = 1, \dots, m.$$

Further, each function

$$\tilde{H}^i(t) := l_i \Big(\tilde{p}^{x_i}(t) + \tilde{p}^{z_i}(t)\big(\varphi(\tilde{z}_i(t)) + e_1 \tilde{f}_i(t)\big) + \alpha_0 F_i(\tilde{x}_i(t), \tilde{z}_i(t), \tilde{f}_i(t))\Big),$$
$$i = 1, \dots, m$$

is constant on $[0,1]$. Set

$$\tilde{c}_{H^i} := \tilde{H}^i(t), \quad t \in [0,1], \quad i = 1, \dots, m. \tag{54}$$

Then, for every $i = 1, \dots, m$ the following is true:

if $a < l_i < b$, then $\tilde{c}_{H^i} = 0$,
if $l_i = a$, then $\tilde{c}_{H^i} \geq 0$,
if $l_i = b$, then $\tilde{c}_{H^i} \leq 0$.

Moreover, if $\tilde{c}_{H^i} > 0$, then $l_i = a$; if $\tilde{c}_{H^i} < 0$, then $l_i = b$.

Note that the equation (52) for $\tilde{p}^{x_i}$ and the transversality conditions for $\tilde{p}^{x_i}$ imply

$$\begin{aligned}\tilde{p}^{x_i}(t) = {} & -\alpha_0 l_i \int_t^1 \Big( \big(\tilde{z}_i(\tau) - z_i^*(\tilde{x}_i(\tau))\big)^\top (z_i^*)'(\tilde{x}_i(\tau)) + \big(\tilde{f}_i(\tau) \\ & - f_i^*(\tilde{x}_i(\tau))\big)(f_i^*)'(\tilde{x}_i(\tau))\Big)\,\mathrm{d}\tau,\end{aligned}$$

where $t \in [0,1]$. This is a complete information following from the LMP for Problem $\tilde{P}$.

The adjoint variable $\tilde{p}^{x_i}$ can be excluded from the system of optimality conditions. For this, we can use the formula $\tilde{c}_{H^i} = \tilde{H}^i(1)$, $i = 1, \dots, m$, following from (54). Since $\tilde{p}^{x_i}(1) = 0$, we get

$$\begin{aligned}& \tilde{c}_{H^i} = \tilde{H}^i(1) := l_i\Big(\tilde{p}^{z_i}(1)\big(\varphi(\tilde{z}_i(1)) + e_1 \tilde{f}_i(1)\big) + \alpha_0 F_i(\tilde{x}_i(1), \tilde{z}_i(1), \tilde{f}_i(1))\Big), \\ & i = 1, \dots, m.\end{aligned}$$

This allows us to define the sign of $\tilde{c}_{H^i}$ (only that sign is important) without using $\tilde{p}^{x_i}$.

Let us represent the above conditions using independent variables $x_i \in [0, l_i]$, $i = 1, \dots, m$.

Fix any $i \in \{1, \dots, m\}$. Recall that $x_i = l_i \cdot t$, $t \in [0,1]$, $\tilde{z}_i(t) = z_i(l_i t)$, $\tilde{f}_i(t) = f_i(l_i t)$. We set $p^{x_i}(x_i) = \tilde{p}^{x_i}(t)$, $p^{z_i}(x_i) = \tilde{p}^{z_i}(t)$, where $t = x_i / l_i$, $x_i \in [0, l_i]$. Then, it is easy to see that for a given $i$ we get a system on $[0, l_i]$, where $a \leq l_i \leq b$:

$$\begin{aligned}& \frac{\mathrm{d}z_i(x_i)}{\mathrm{d}x_i} = \varphi(z_i(x_i)) + e_1 f_i(x_i), \quad x_i \in [0, l_i], \\ & K(z_1(0), z_1(l_1) \dots, z_i(0), z_i(l_i), \dots, z_m(0), z_m(l_m)) = 0, \\ & -\frac{\mathrm{d}p^{x_i}(x_i)}{\mathrm{d}x_i} = -\alpha_0\big(z_i(x_i) - z_i^*(x_i)\big)^\top (z_i^*)'(x_i(x_i)) - \alpha_0\big(f_i(x_i) - f_i^*(x_i)\big)(f_i^*)'(x_i(t)), \\ & \qquad x_i \in [0, l_i], \\ & p^{x_i}(l_i) = 0, \\ & -\frac{\mathrm{d}p^{z_i}(x_i)}{\mathrm{d}x_i} = p^{z_i}(x_i)\varphi'(z_i(x_i)) + \alpha_0\big(z_i(x_i) - z_i^*(x_i)\big)^\top, \quad x_i \in [0, l_i], \\ & -p^{z_i}(0) = \beta K_{z_i(0)}, \quad p^{z_i}(l_i) = \beta K_{z_i(l_i)}, \\ & p_1^{z_i}(x_i) + \alpha_0\big(f_i(x_i) - f_i^*(x_i)\big) = 0, \quad x_i \in [0, l_i].\end{aligned}$$

Moreover, the function

$$H^i(x_i) := p^{x_i}(x_i) + p^{z_i}(x_i)\big(\varphi(z_i(x_i)) + e_1 f_i(x_i)\big) + \alpha_0 F_i(x_i, z_i(x_i), f_i(x_i))$$

is constant on $[0, l_i]$, where

$$p^{x_i}(x_i) = -\alpha_0 \int_{x_i}^{l_i} \Big( \big(z_i(x) - z_i^*(x)\big)^\top (z_i^*)'(x) + \big(f_i(x) - f_i^*(x)\big)(f_i^*)'(x) \Big)\,\mathrm{d}x,$$
$$x_i \in [0, l_i].$$

Set $c_{H^i} := H^i(x_i)$, $x_i \in [0, l_i]$. Then, for every $i = 1, \dots, m$ the following is true: if $a < l_i < b$, then $c_{H^i} = 0$; if $l_i = a$, then $c_{H^i} \geq 0$; if $l_i = b$, then $c_{H^i} \leq 0$. Moreover, if $c_{H^i} > 0$, then $l_i = a$; if $c_{H^i} < 0$, then $l_i = b$.

It is convenient to use formulas

$$c_{H^i} = H^i(l_i) = p^{z_i}(l_i)\big(\varphi(z_i(l_i)) + e_1 f_i(l_i)\big) + \alpha_0 F_i(l_i, z_i(l_i), f_i(l_i)), \quad i = 1, \dots, m,$$

which does not require calculations of $p^{x_i}(x_i)$.

Note that only the constraint

$$K(z_1(0), z_1(l_1) \dots, z_i(0), z_i(l_i), \dots, z_m(0), z_m(l_m)) = 0$$

and the corresponding transversality conditions

$$-p^{z_i}(0) = \beta K_{z_i(0)},\ p^{z_i}(l_i) = \beta K_{z_i(l_i)},\ i = 1, \dots, m$$

do not break up and unite system of necessary conditions for Problem $P$.

### 5.4. Problem $P_i$ and its relation to Problem $P$

Let $(\tilde{l}_i^0(\cdot), \tilde{x}_i^0(\cdot), \tilde{z}_i^0(\cdot), \tilde{f}_i^0(\cdot))_{i=1}^m$ be a solution to Problem $\tilde{P}$. Set $\tilde{\zeta}_i = (\tilde{z}_i(0), \tilde{z}_i(1))$, $\tilde{\zeta}_i^0 = (\tilde{z}_i^0(0), \tilde{z}_i^0(1))$, $i = 1, \dots, m$. Fix any $i$ and define the function

$$K^i(\tilde{\zeta}_i) := K(\tilde{\zeta}_1^0, \dots, \tilde{\zeta}_{i-1}^0, \tilde{\zeta}_i, \tilde{\zeta}_{i+1}^0, \dots, \tilde{\zeta}_m^0).$$

Consider the following **Problem** $\tilde{P}_i$ on [0,1]:

$$\begin{aligned}
&\tfrac{\mathrm{d}\tilde{l}_i(t)}{\mathrm{d}t} = 0, \quad \tfrac{\mathrm{d}\tilde{x}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t),\\
&\tfrac{\mathrm{d}\tilde{z}_i(t)}{\mathrm{d}t} = \tilde{l}_i(t)\big(\varphi(\tilde{z}_i(t)) + e_1 \tilde{f}_i(t)\big), \quad t \in [0,1],\\
&\tilde{x}_i(0) = 0, \quad -\tilde{l}_i(0) + a \leq 0, \quad \tilde{l}_i(0) - b \leq 0,\\
&K^i(\tilde{z}_i(0), \tilde{z}_i(1)) = 0,\\
&J_i = \textstyle\int_0^1 \tilde{l}_i(t) F_i(\tilde{x}_i(t), \tilde{z}_i(t), \tilde{f}_i(t))\,\mathrm{d}t \to \min.
\end{aligned}$$

The following assertion holds for any $i = 1, \dots, m$.

Lemma 1 *If a point* $(\tilde{l}_i^0(\cdot), \tilde{x}_i^0(\cdot), \tilde{z}_i^0(\cdot), \tilde{f}_i^0(\cdot))_{i=1}^m$ *is a solution to Problem* $\tilde{P}$*, then the point* $(\tilde{l}_i^0(\cdot), \tilde{x}_i^0(\cdot), \tilde{z}_i^0(\cdot), \tilde{f}_i^0(\cdot))$ *is a solution to Problem* $\tilde{P}_i$*.*

The proof is trivially carried out by contradiction.

The necessary first-order optimality conditions for Problem $\tilde{P}_i$ are given in Section 3.2. Sufficient second-order optimality conditions for Problem $\tilde{P}_i$ are given in Sections 3.3–3.5.

Similar relations between problems can be formulated on intervals $[0, l_i]$. Let

$$(l_i^0, z_i^0(\cdot), f_i^0(\cdot))_{i=1}^m$$

be a solution to Problem $P$.

Set $\zeta_i = (z_i(0), z_i(l_i))$, $\zeta_i^0 = (z_i^0(0), z_i^0(l_i))$, $i = 1, \ldots, m$. Fix any $i$ and define the function $K^i(\zeta_i) := K(\zeta_1^0, \ldots, \zeta_{i-1}^0, \zeta_i, \zeta_{i+1}^0, \ldots, \zeta_m^0)$. Consider the following **Problem** $P_i$ on $[0, l_i]$:

$$\begin{aligned}
&\tfrac{\mathrm{d}z_i(x_i)}{\mathrm{d}x_i} = \varphi(z_i(x_i)) + e_1 f_i(x_i), \quad x_i \in [0, l_i],\\
&a \le l_i \le b, \quad K^i(z_i(0), z_i(l_i)) = 0,\\
&J_i = \textstyle\int_0^{l_i} F_i(x_i, z_i(x_i), f_i(x_i))\,\mathrm{d}x_i \to \min .
\end{aligned}$$

For any $i = 1, \ldots, m$, the following assertion holds.

Lemma 2 *If* $(l_i^0, z_i^0(\cdot), f_i^0(\cdot))_{i=1}^m$ *is a solution to Problem* $P$*, then* $(l_i^0, z_i^0(\cdot), f_i^0(\cdot))$ *is a solution to Problem* $P_i$*.*

First and second order optimality conditions in Problem $P_i$ are given in Sections 3.2, 3.3–3.5.

### 5.5. Example

Consider the following control system, described by second-order differential equations for a three-star network:

$$\frac{\mathrm{d}^2 z_i(x_i)}{\mathrm{d}x_i^2} = \varphi_i(z_i(x_i), f_i(x_i)), \quad x_i \in [0, l_i], \quad i = 1, 2, 3.$$

The endpoint conditions are as follows:

$$z_1(0) = 0, \quad z_2(l_2) = 0, \quad z_3(l_3) = 0,$$

$$z_1(l_1) - z_2(0) = 0, \quad z_2(0) - z_3(0) = 0,$$

$$z_1'(l_1) - z_2'(0) - z_3'(0) = 0.$$

Here $z_i(x_i)$ are one-dimensional state variables, $f_i(x_i)$ are one-dimensional controls. We assume that $f_i(\cdot)$ are continuous functions, and $z_i(\cdot)$ are twice continuously differentiable functions, $i = 1, 2, 3$. We refer to Fig. 6 for visualization.

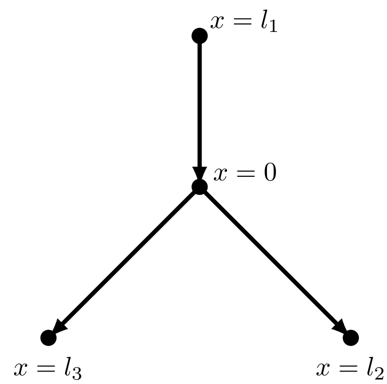


Figure 6. The three-star graph

Let us represent this system in an equivalent way as a system of first-order differential equations, introducing new one-dimensional state variables $y_i(x_i)$:

$$\frac{\mathrm{d}z_i(x_i)}{\mathrm{d}x_i} = y_i(x_i), \quad \frac{\mathrm{d}y_i(x_i)}{\mathrm{d}x_i} = \varphi_i(z_i(x_i), f_i(x_i)), \quad x_i \in [0, l_i], \quad i = 1, 2, 3, \tag{55}$$

$$z_1(0) = 0, \quad z_2(l_2) = 0, \quad z_3(l_3) = 0, \tag{56}$$

$$z_1(l_1) - z_2(0) = 0, \quad z_2(0) - z_3(0) = 0, \tag{57}$$

$$y_1(l_1) - y_2(0) - y_3(0) = 0. \tag{58}$$

In addition to this system, there is a constraint on the length of the intervals

$$l_1 + l_2 + l_3 = \mu, \tag{59}$$

where $l_i > 0$, $i = 1, 2, 3$, $\mu > 0$ is a given number. The cost that needs to be minimized is:

$$J = \sum_{i=1}^{3} \int_0^{l_i} \left( \frac{1}{2}(z_i(x_i) - z_i^*(x_i))^2 + \frac{1}{2}(f_i(x_i) - f_i^*(x_i))^2 \right) \mathrm{d}x_i \to \min, \tag{60}$$

where $z_i^*(\cdot)$ are given twice continuously differential functions, $f_i^*(\cdot)$ are given continuous functions, and $\varphi_i(\cdot,\cdot)$ are given Lipschitz continuous functions, $i = 1,2,3$. Problem (55)-(60) will be called Problem $P^E$.

This problem is not a special case of problem $P$ studied in this section, but we will show that the method of reduction to a standard optimal control problem used in this section can also be applied to problem $P^E$. Thus, this method has a much broader application than it was shown in Section 5. Here, we restrict ourselves to first-order optimality conditions only.

Problem $\tilde{P}^E$ on the interval $[0,1]$ has the form

$$
\begin{aligned}
&\dot{\tilde{l}}_i(t) = 0,\ \dot{\tilde{x}}_i(t) = \tilde{l}_i(t),\ \dot{\tilde{z}}_i(t) = \tilde{l}_i(t)\tilde{y}_i(t),\ \dot{\tilde{y}}_i(t) = \tilde{l}_i(t)\varphi_i(\tilde{z}_i(t),\tilde{f}_i(t)),\ i = 1,2,3,\\
&\tilde{x}_1(0) = 0, \quad \tilde{x}_2(0) = 0, \quad \tilde{x}_3(0) = 0,\\
&\tilde{z}_1(0) = 0, \quad \tilde{z}_2(1) = 0, \quad \tilde{z}_3(1) = 0,\\
&\tilde{z}_1(1) - \tilde{z}_2(0) = 0, \quad \tilde{z}_2(0) - \tilde{z}_3(0) = 0,\\
&\tilde{y}_1(1) - \tilde{y}_2(0) - \tilde{y}_3(0) = 0,\\
&\tilde{l}_1(0) + \tilde{l}_2(0) + \tilde{l}_3(0) = \mu,\\
&J = \sum_{i=1}^{3} \int_0^1 \Big(\tfrac{1}{2}(\tilde{z}_i(t) - z_i^*(\tilde{x}_i(t)))^2 + \tfrac{1}{2}(\tilde{f}_i(t) - f_i^*(\tilde{x}_i(t)))^2\Big)\tilde{l}_i(t)\,\mathrm{d}t \to \min.
\end{aligned}
$$

Let $(l_i^0, z_i^0(\cdot), y_i^0(\cdot), f_i^0(\cdot))_{i=1,2,3}$ be an optimal solution of problem $P^E$. Set

$$
\tilde{l}_i^0 = l_i^0, \quad \tilde{x}_i^0(t) = \tilde{l}_i^0 t, \quad \tilde{z}_i^0(t) = z_i^0(l_i^0 t), \quad \tilde{y}_i^0(t) = y_i^0(l_i^0 t), \quad \tilde{f}_i^0(t) = f_i^0(l_i^0 t).
$$

Then, $(\tilde{l}_i^0, \tilde{x}_i^0(\cdot), \tilde{z}_i^0(\cdot), \tilde{y}_i^0(\cdot), \tilde{f}_i^0(\cdot))_{i=1,2,3}$ is an optimal solution of Problem $\tilde{P}^E$. Let $\alpha_0 \geq 0$ be the cost Lagrange multiplier for this solution. Recall that $\alpha_0 = 1$ in the normal case and $\alpha_0 = 0$ in the abnormal case.

Let us write down the LMP conditions. Here,

$$
\begin{aligned}
\tilde{L} \;=\;& \gamma_1\tilde{x}_1(0) + \gamma_2\tilde{x}_2(0) + \gamma_3\tilde{x}_3(0) + \beta_1\tilde{z}_1(0) + \beta_2\tilde{z}_2(1) + \beta_3\tilde{z}_3(1)\\
&+\beta_4(\tilde{z}_1(1) - \tilde{z}_2(0)) + \beta_5(\tilde{z}_2(0) - \tilde{z}_3(0)) + \beta_6(\tilde{y}_1(1) - \tilde{y}_2(0) - \tilde{y}_3(0))\\
&+\delta(\tilde{l}_1(0) + \tilde{l}_2(0) + \tilde{l}_3(0) - \mu),
\end{aligned}
$$

$$
\tilde{H} = \sum_{i=1}^{3} \tilde{p}_{x_i}\tilde{l}_i + \sum_{i=1}^{3} \tilde{p}_{z_i}\tilde{l}_i\tilde{y}_i + \sum_{i=1}^{3} \tilde{p}_{y_i}\tilde{l}_i\varphi_i(\tilde{z}_i, \tilde{f}_i) + \alpha_0 \sum_{i=1}^{3} \tilde{l}_i F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i),
$$

where

$$
F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i) = \frac{1}{2}(\tilde{z}_i - z_i^*(\tilde{x}_i))^2 + \frac{1}{2}(\tilde{f}_i - f_i^*(\tilde{x}_i))^2.
$$

Set

$$\tilde{H}^i = \tilde{p}_{x_i}\tilde{l}_i + \tilde{p}_{z_i}\tilde{l}_i\tilde{y}_i + \tilde{p}_{y_i}\tilde{l}_i\varphi_i(\tilde{z}_i, \tilde{f}_i) + \alpha_0\tilde{l}_i F_i(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i),$$

Then $\tilde{H} = \sum_{i=1}^{3} \tilde{H}^i$.

The adjoint system and the condition $H^i_{f_i} = 0$ have the form:

$$\begin{aligned}
-\dot{\tilde{p}}_{l_i}(t) &= \tilde{p}_{x_i}(t) + \tilde{p}_{z_i}(t)\tilde{y}_i^0(t) + \tilde{p}_{y_i}(t)\varphi_i(\tilde{z}_i^0(t), \tilde{f}_i^0(t)) + \alpha_0 F_i(\tilde{x}_i^0(t), \tilde{z}_i^0(t), \tilde{f}_i^0(t)),\\
-\dot{\tilde{p}}_{x_i}(t) &= -\alpha_0\tilde{l}_i^0\big(\tilde{z}_i^0(t) - z_i^*(\tilde{x}_i^0(t))\big)(z_i^*)'(\tilde{x}_i^0(t))\\
&\quad -\alpha_0\tilde{l}_i^0\big(\tilde{f}_i^0(t) - f_i^*(\tilde{x}_i^0(t))\big)(f_i^*)'(\tilde{x}_i^0(t))\\
-\dot{\tilde{p}}_{z_i}(t) &= \tilde{p}_{y_i}(t)\tilde{l}_i^0\varphi_{i\tilde{z}_i}(\tilde{z}_i^0(t), \tilde{f}_i^0(t)) + \alpha_0\tilde{l}_i^0(\tilde{z}_i^0(t) - z_i^*(\tilde{x}_i^0(t)))\\
-\dot{\tilde{p}}_{y_i}(t) &= \tilde{p}_{z_i}(t)\tilde{l}_i^0,\\
p_{y_i}(t)\tilde{l}_i^0\varphi_{i\tilde{f}_i}&(\tilde{z}_i^0(t), \tilde{f}_i^0(t)) + \alpha_0\tilde{l}_i^0\big(\tilde{f}_i^0(t) - f_i^*(\tilde{x}_i^0(t))\big) = 0
\end{aligned}$$

for all $i = 1, 2, 3$.

Moreover, there exist constants $\tilde{c}_{H^i}$ such that

$$\begin{aligned}
&\tilde{H}^i(t) :=\\
&\tilde{l}_i^0\big(\tilde{p}_{x_i}(t) + \tilde{p}_{z_i}(t)\tilde{y}_i^0(t) + \tilde{p}_{y_i}(t)\varphi_i(\tilde{z}_i^0(t), \tilde{f}_i^0(t)) + \alpha_0 F_i(\tilde{x}_i^0(t), \tilde{z}_i^0(t), \tilde{f}_i^0(t))\big) = \tilde{c}_{H^i}
\end{aligned}$$

for all $t \in [0, 1]$, $i = 1, 2, 3$.

The transversality conditions are:

$$\begin{aligned}
-\tilde{p}_{l_1}(0) &= \delta, & \tilde{p}_{l_1}(1) &= 0,\\
-\tilde{p}_{x_1}(0) &= \gamma_1, & \tilde{p}_{x_1}(1) &= 0,\\
-\tilde{p}_{z_1}(0) &= \beta_1, & \tilde{p}_{z_1}(1) &= \beta_4,\\
-\tilde{p}_{y_1}(0) &= 0, & \tilde{p}_{y_1}(1) &= \beta_6,\\
-\tilde{p}_{l_2}(0) &= \delta, & \tilde{p}_{l_2}(1) &= 0,\\
-\tilde{p}_{x_2}(0) &= \gamma_2, & \tilde{p}_{x_2}(1) &= 0,\\
-\tilde{p}_{z_2}(0) &= -\beta_4 + \beta_5, & \tilde{p}_{z_2}(1) &= \beta_2,\\
-\tilde{p}_{y_2}(0) &= -\beta_6, & \tilde{p}_{y_2}(1) &= 0,\\
-\tilde{p}_{l_3}(0) &= \delta, & \tilde{p}_{l_3}(1) &= 0,\\
-\tilde{p}_{x_3}(0) &= \gamma_3, & \tilde{p}_{x_3}(1) &= 0,\\
-\tilde{p}_{z_3}(0) &= -\beta_5, & \tilde{p}_{z_3}(1) &= \beta_3,\\
-\tilde{p}_{y_3}(0) &= -\beta_6, & \tilde{p}_{y_3}(1) &= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\tilde{p}_{l_1}(0) = \tilde{p}_{l_2}(0) = \tilde{p}_{l_3}(0) = -\delta,\\
&\tilde{p}_{l_1}(1) = \tilde{p}_{l_2}(1) = \tilde{p}_{l_3}(1) = 0,\\
&\tilde{p}_{x_1}(1) = 0, \quad \tilde{p}_{x_2}(1) = 0, \quad \tilde{p}_{x_3}(1) = 0\\
&\tilde{p}_{y_1}(0) = 0, \quad \tilde{p}_{y_2}(1) = 0, \quad \tilde{p}_{y_3}(1) = 0,
\end{aligned}$$

$$\begin{aligned}
&\tilde{p}_{y_1}(1) = \tilde{p}_{y_2}(0) = \tilde{p}_{y_3}(0),\\
&\tilde{p}_{z_1}(1) - \tilde{p}_{z_2}(0) - \tilde{p}_{z_3}(0) = 0.
\end{aligned}$$

It follows from the condition of the constancy of the Hamiltonian $\tilde{H}^i$ and the transversality conditions for $\tilde{p}_{l_i}$ that $\tilde{c}_{H^i} = -\delta, \quad i = 1, 2, 3$. Consequently,

$$\tilde{c}_{H^1} = \tilde{c}_{H^2} = \tilde{c}_{H^3}.$$

The nontriviality condition means that $\alpha_0 = 1$ or not all adjoint variables are equal to zero.

Let us reformulate these conditions in the intervals $[0, l_i^0]$. We set

$$p_{x_i}(x_i) = \tilde{p}_{x_i}(t),\, p_{y_i}(x_i) = \tilde{p}_{y_i}(t), \quad p_{z_i}(x_i) = \tilde{p}_{z_i}(t), \text{ where } t = \frac{x_i}{l_i^0},\ x_i \in [0, l_i^0].$$

Then it is easy to see that for a given $i$ we get a system on the intervals $[0, l_i^0]$:

$$\begin{aligned}
&-(p_{x_i})'(x_i) = -\alpha_0\big(z_i^0(x_i) - z_i^*(x_i)\big)(z_i^*)'(x_i) - \alpha_0\big(f_i^0(x_i) - f_i^*(x_i)\big)(f_i^*)'(x_i),\\
&-(p_{z_i})'(x_i) = p_{y_i}(x_i)\varphi_{iz_i}(z_i^0(x_i), f_i^0(x_i)) + \alpha_0(z_i^0(x_i) - z_i^*(x_i))\\
&-(p_{y_i})'(x_i) = p_{z_i}(x_i),\\
&p_{y_i}(x_i)\varphi_{if_i}(z_i^0(x_i), f_i^0(x_i)) + \alpha_0\big(f_i^0(x_i) - f_i^*(x_i)\big) = 0,
\end{aligned}$$

where $x_i \in [0, l_i^0]$, $i = 1, 2, 3$. Moreover,

$$\left.\begin{aligned}
&p_{x_i}(l_i^0) = 0, \quad i = 1, 2, 3, \quad p_{y_1}(0) = 0, \quad p_{y_2}(l_2^0) = p_{y_3}(l_3^0) = 0,\\
&p_{y_1}(l_1^0) = p_{y_2}(0) = p_{y_3}(0), \quad p_{z_1}(l_1^0) - p_{z_2}(0) - p_{z_3}(0) = 0.
\end{aligned}\right\} \tag{61}$$

Set

$$p_i(x_i) := p_{y_i}(x_i), \quad x_i \in [0, l_i^0], \quad i = 1, 2, 3.$$

Then

$$p_i'(x_i) = -p_{z_i}(x_i), \quad x_i \in [0, l_i^0], \quad i = 1, 2, 3.$$

Thus, we obtain

$$\begin{aligned}
&(p_i)''(x_i) = p_i(x_i)\varphi_{iz_i}(z_i^0(x_i), f_i^0(x_i)) + \alpha_0(z_i^0(x_i) - z_i^*(x_i)),\\
&p_i(x_i)\varphi_{if_i}(z_i^0(x_i), f_i^0(x_i)) + \alpha_0\big(f_i^0(x_i) - f_i^*(x_i)\big) = 0,
\end{aligned}$$

where $x_i \in [0, l_i^0]$, $i = 1, 2, 3$.

The condition of the constancy of the Hamiltonian $\tilde{H}^i$ becomes

$$\begin{aligned}&H^i(x_i) :=\\ &p_{x_i}(x_i) + p_{z_i}(x_i)y_i^0(x_i) + p_{y_i}(x_i)\varphi_i(z_i^0(x_i), f_i^0(x_i)) + \alpha_0 F_i(x_i, z_i^0(x_i), f_i^0(x_i)).\end{aligned}$$

whence it follows that

$$H^i(x_i) = p_{x_i}(x_i) - p_i'(x_i)y_i^0(x_i) + p_i(x_i)\varphi_i(z_i^0(x_i), f_i^0(x_i)) + \alpha_0 F_i(x_i, z_i^0(x_i), f_i^0(x_i)),$$

and there exists a constant $c_{H^i}$ such that

$$H^i(x_i) = c_{H^i} \quad \forall x_i \in [0, l_i^0], \quad i = 1, 2, 3.$$

In particular, for $x_i = l_i^0$ we get

$$c_{H^i} = H^i(l_i^0) = -p_i'(l_i^0)y_i^0(l_i^0) + p_i(l_i^0)\varphi_i(z_i^0(l_i^0), f_i^0(l_i^0)) + \alpha_0 F_i(l_i^0, z_i^0(l_i^0), f_i^0(l_i^0)),$$
$$i = 1, 2, 3.$$

Here,

$$p_i(l_i^0) = p_{y_i}(l_i^0) = 0, \quad i = 1, 2.$$

Moreover,

$$c_{H^1} = c_{H^2} = c_{H^3}.$$

Note that the adjoint equation for $\tilde{p}_{x_i}$ and the condition $p_{x_i}(l_i^0) = 0$ give

$$p_{x_i}(x_i) = -\int_{x_i}^{l_i^0} \big(\alpha_0\big(z_i^0(x) - z_i^*(x)\big)(z_i^*)'(x) + \alpha_0\big(f_i^0(x) - f_i^*(x)\big)(f_i^*)'(x)\big)\big)\, \mathrm{d}x.$$

To the resulting system of necessary optimality conditions, we have to add the transversality conditions (61), which are equivalent to the system

$$p_1(0) = p_2(l_2^0) = p_3(l_3^0) = 0, \quad p_1(l_1^0) = p_2(0) = p_3(0), \quad p_1'(l_i^0) - p_2'(0) - p_3'(0) = 0.$$

Moreover, $\alpha_0 = 1$ or not all of $p_1, p_2, p_3$ are equal to zero.

To conclude this section, we write Problem $P_i^E$ for each beam. It has the form:

$$\begin{aligned}&\tfrac{\mathrm{d}z_i(x)}{\mathrm{d}x} = y_i(x), \quad \tfrac{\mathrm{d}y_i(x)}{\mathrm{d}x} = \varphi_i(z_i(x), f_i(x)), \quad x \in [0, l_i],\\ &K^i(l_i, z_i(0), y_i(0), z_i(l_i), y_i(l_i)) = 0,\\ &J_i = \textstyle\int_0^{l_i} \left(\frac{1}{2}(z_i(x) - z_i^*(x))^2 + \frac{1}{2}(f_i(x) - f_i^*(x))^2\right) \mathrm{d}x \to \min,\end{aligned}$$

where $l_i > 0$ is not fixed. Here $K^1 = 0$ means

$$z_1(0) = 0, \quad z_1(l_1) - z_2^0(0) = 0, \quad y_1(l_1) - y_2^0(0) - y_3^0(0) = 0, \quad l_1 + l_2^0 + l_3^0 = \mu,$$

$K^2 = 0$ means

$$z_2(l_2) = 0, \quad z_1^0(l_1^0) - z_2(0) = 0, \quad z_2(0) - z_3^0(0) = 0, \quad y_1^0(l_1^0) - y_2(0) - y_3^0(0) = 0,$$
$$l_1^0 + l_2 + l_3^0 = \mu,$$

and $K^3 = 0$ means

$$z_3(l_3) = 0, \quad z_2^0(0) - z_3(0) = 0, \quad y_1^0(l_1^0) - y_2^0(0) - y_3(0) = 0, \quad l_1^0 + l_2^0 + l_3 = \mu,$$

We know that $(l_i^0, z_i^0(\cdot), y_i^0(\cdot), f_i^0(\cdot))$ is the optimal solution to problem $P_i^E$, $i = 1, 2, 3$.

## 6. Conclusion

We consider the optimal control problem combined with the optimum design problem in an optimization problem for networks. The analysis of steady state model is useful for dynamic models with the so-called *control systems with turnpike property*. We refer the reader to the optimum design of the linear wave equation on networks in the forthcoming paper entitled *Network design and control. The turnpike property for wave equation*, by Martin Gugat, Meizhi Qian and Jan Sokolowski, where the bilevel optimization problems are considered. The results were presented at MMAR 2023 (Gugat, Qian and Sokołowski, 2023). The numerical analysis of the optimum design for nonlinear control systems on networks is still to be performed. Two main examples of such systems are networks of nonlinear elastic beams and gas transportation networks.

## Acknowledgments

This work was supported by China Scholarship Council (CSC) under Grant CSC No.202206140096.

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