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Design of unknown input observers for non-linear stochastic systems and their application to robust fault diagnosis*

by

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Abstract: The paper deals with the problem of designing filters for non-linear discrete-time stochastic systems. In particular, it is shown how to design an unknown input filter for a single (constant) unknown input distribution matrix, which guarantees that the effect of a fault will not be decoupled from the residual. Subsequently, the problem of using one, fixed disturbance distribution matrix is eliminated by using the interacting multiple models algorithm to select an appropriate unknown input distribution matrix from a predefined set of matrices. The final part of the paper shows an illustrative example, which confirms the effectiveness of the proposed approach.

Keywords: fault diagnosis, unknown input, unscented Kalman filter, interactive multiple models, non-linear systems

Notation

$\boldsymbol{x}_k, \hat{\boldsymbol{x}}_k \in \mathbb{R}^n$	state vector and its estimate
$\boldsymbol{y}_k, \hat{\boldsymbol{y}}_k \in \mathbb{R}^m$	output vector and its estimate
$\boldsymbol{e}_k \in \mathbb{R}^n$	state estimation error
$\boldsymbol{z}_k \in \mathbb{R}^m$	output error (residual)
$\boldsymbol{u}_k \in \mathbb{R}^r$	input vector
$\boldsymbol{d}_k \in \mathbb{R}^q$	unknown input vector, $q \leq m$
$\boldsymbol{w}_k \in \mathbb{R}^n,\ \boldsymbol{v}_k \in \mathbb{R}^m$	process and measurement noise
$\boldsymbol{Q}_k,\ \boldsymbol{R}_k$	covariance matrices of $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$
$\boldsymbol{f}_k \in \mathbb{R}^s$	fault vector
$\boldsymbol{g}(\cdot),\ \boldsymbol{h}(\cdot)$	non-linear functions
$\boldsymbol{E} \in \mathbb{R}^{n \times q}$	unknown input distribution matrix
$\boldsymbol{L} \in \mathbb{R}^{n \times s}$	fault distribution matrix
$\boldsymbol{I}_n$	identity matrix of size n

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1. Introduction

There is no doubt that the theory (and practice, as a consequence) of fault diagnosis and control is well-developed and mature for linear systems only (Korbicz et al, 2004). There is also a number of different approaches that can be employed to settle the robustness problems regarding model uncertainty (Witczak, 2007). Such robustness is especially important in practical implementations where various sources of uncertainty may be present, e.g. differences between various copies of a given component, time-varying properties, noise, external disturbances, etc. As can be observed in the literature, observers (or filter in a stochastic framework) are commonly used in both control and fault diagnosis schemes of non-linear systems (see, e.g., Korbicz et al., 2004; Witczak, 2007; Ding, 2008; Puig, 2010; Tong et al., 2011; Kemir, 2011, and the references therein). Undoubtedly, the most common approach is to use robust observers, such as the Unknown Input Observer (UIO) (Frank and Marcu, 2000; Witczak, 2007), which can tolerate a degree of model uncertainty and hence increase the reliability of fault diagnosis. Although the origins of UIOs can be traced back to the early 1970s (see the seminal work of Wang et al.,1975), the problem of designing such observers is still of paramount importance both from the theoretical and practical viewpoints. A large amount of knowledge on using these techniques for model-based fault diagnosis has been accumulated through the literature for the last three decades (see Witczak, 2007, and the references therein). A number of approaches to non-linear fault diagnosis and the fault-tolerant control (FTC) was published during the last two decades. For an example, in Hammouri et al. (2002) the high gain observer for Lipschitz systems was applied for the purpose of fault diagnosis. One of the standard methods for observer design consists in using the non-linear change of coordinates to bring the original system into a linear one (or pseudo linear one). As indicated in the literature such approaches can be applied for fault diagnosis and the FTC (Hammouri et al., 1999; Kabore et Wang, 2001). It should be also noted that when the feasibility condition regarding the non-linear change of coordinates are not matched, then the celebrated Extended Kalman Filter (EKF) can be applied in both stochastic and deterministic context (see, e.g., Witczak, 2007).

Generally, the design problems regarding UIOs for non-linear systems can be divided into three distinct categories:

- *nonlinear state-transformation-based techniques:* apart from a relatively large class of systems for which they can be applied, even if the nonlinear transformation is possible it leads to another nonlinear system and hence the observer design problem remains open (see Frank and Marcu, 2000) and the references therein);
- *linearization-based techniques:* such approaches are based on a similar strategy as that for the Extended Kalman filter (EKF) (Korbicz et al., 2004). In Witczak (2007) the author proposed an extended unknown input observer for non-linear systems. He also proved that the proposed observer is convergent under certain conditions;
- *observers for particular classes of nonlinear systems:* for example Unknown

Input Observers for polynomial and bilinear systems or UIOs for Lipschitz systems (Chen et al., 2011; Koenig and Mammar, 2002; Pertew et al., 2005; Witczak, 2007).

In the light of the above discussion, it is clear that an accurate state estimation is extremely important for fault detection and control applications. However, estimation under noise and unknown inputs is very difficult.

In order to settle the above-mentioned challenges, the design problems regarding UIOs (undertaken within the framework of this paper) are divided into three distinct categories:

1. How to determine the unknown input distribution matrix, which will not decouple the effect of faults from the residual?
2. How to develop the possibly simple and reliable design procedure of the UIO for non-linear stochastic systems?
3. How to extend the approach developed for the constant unknown input distribution matrix into a set of predefined unknown input distribution matrices?

Concerning the first question, a partial answer can be found in Chen and Patton (1999). Indeed, the authors concentrate on the determination of the unknown input distribution matrix for linear systems but they do not answer the question when this matrix will cause the fault decoupling effect. Apart from the fact that there are approaches that can be used for designing UIOs for non-linear systems (listed before), the problem of determining unknown input distribution matrix for this class of systems remains untouched. In other words, the authors assume that this matrix is known, which apart from a relatively simple case, is never the truth. It should be also mentioned that it is usually assumed that the disturbance decoupling will not cause the decrease of fault diagnosis sensitivity or fault decoupling in the worst scenario. To tackle this problem within the framework of this paper, a numerical optimisation-based approach is proposed that can be used to estimate the unknown input distribution matrix, which does not cause the fault decoupling effect. As an answer to the second question, this work presents an alternative Unknown Input Filter (UIF) for non-linear systems, which is based on the general idea of the Unscented Kalman Filter (UKF) (Julier and Uhlman, 2004; Kandepu et al., 2008). This approach is based on a similar idea as that proposed by Witczak and Pretki (2007), Witczak (2007), but the structure of the scheme is different and instead of the EKF, the UKF is employed. To tackle the third problem, it is shown that the Interacting Multiple Model (IMM) algorithm can be employed to tackle the problem of selecting an appropriate unknown input distribution matrix from a predefined set. The proposed solutions can be perceived as an alternative to the Takagi-Sugeno-based approach presented in Uppal et al. (2006).

The paper is organised as follows. Section 2 presents a general way for decoupling the unknown input. In Section 3 a fault decoupling prevention condition is developed, while Section 4 introduces the idea of the UKF and shows how to use it in order to design an UIF. Section 5 shows a numerical procedure for determining the unknown input distribution matrix. The subsequent Section 6

extends the approach presented in Section 4 in a way that instead of a constant unknown input distribution matrix a predefined set of matrices is given. Finally, Section 7 presents a numerical example with an induction motor, which exhibits the performance of the solutions developed within the framework of this paper.

2. Unknown input decoupling

Let us consider the non-linear stochastic system given by the following equations:

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{E}\boldsymbol{d}_k + \boldsymbol{L}\boldsymbol{f}_k + \boldsymbol{w}_k, \tag{1}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}. \tag{2}$$

Note that the unknown input and fault distribution matrices denoted by $\boldsymbol{E}$ and $\boldsymbol{L}$ are constant in this section. This assumption will be relaxed in Section 6 where a set of predefined matrices will be used instead. Moreover, it should be mentioned that this paper focuses on the faults that can influence the state equation (1), such as actuator faults. The case of sensor faults is beyond the scope of this paper and will be investigated in future work.

The main problem is to design a filter insensitive to the influence of the unknown input (external disturbances and modeling errors) being sensitive to fault. The necessary condition for the existence of a solution to the unknown input de-coupling problem is as follows:

$$\operatorname{rank}(\boldsymbol{C}\boldsymbol{E}) = \operatorname{rank}(\boldsymbol{E}) = q, \tag{3}$$

(see Witczak, 2007, for a comprehensive explanation). If condition (3) is satisfied, then it is possible to calculate $\boldsymbol{H} = (\boldsymbol{C}\boldsymbol{E})^{+} = \left[(\boldsymbol{C}\boldsymbol{E})^{T}\boldsymbol{C}\boldsymbol{E}\right]^{-1}(\boldsymbol{C}\boldsymbol{E})^{T}$. Thus, by inserting (1) into (2) and then multiplying (2) by $\boldsymbol{H}$ it is straightforward to show that

$$\boldsymbol{d}_k = \boldsymbol{H}\left[\boldsymbol{y}_{k+1} - \boldsymbol{C}\left[\boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{L}\boldsymbol{f}_k + \boldsymbol{w}_k\right] - \boldsymbol{v}_{k+1}\right]. \tag{4}$$

Substitution of (4) into (1) for $\boldsymbol{d}_k$ gives

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1} + \bar{\boldsymbol{L}}\boldsymbol{f}_k + \bar{\boldsymbol{w}}_k, \tag{5}$$

where:

$$\bar{\boldsymbol{g}}\left(\cdot\right) = \boldsymbol{G}\boldsymbol{g}\left(\cdot\right),\ \bar{\boldsymbol{h}}(\cdot) = \boldsymbol{G}\boldsymbol{h}(\cdot)$$
$$\bar{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{H},\ \bar{\boldsymbol{w}}_k = \boldsymbol{G}\boldsymbol{w}_k - \boldsymbol{E}\boldsymbol{H}\boldsymbol{v}_{k+1},$$

and

$$\boldsymbol{G} = \boldsymbol{I} - \boldsymbol{E}\boldsymbol{H}\boldsymbol{C}.$$

Consequently, the general observer structure is:

$$\hat{\boldsymbol{x}}_{k+1} = \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) + \bar{\boldsymbol{h}}(\cdot) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1} + \boldsymbol{K}(\cdot), \tag{6}$$

where $\boldsymbol{K}(\cdot)$ is the state correction term. In order to make further deliberations more general, no particular form of $\boldsymbol{K}(\cdot)$ is assumed in the present and subsequent sections.

Let us define a residual as a difference between the output of the system and the estimated output:

$$\begin{aligned}\boldsymbol{z}_{k+1} &= \boldsymbol{y}_{k+1} - \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1} = \\ &= \boldsymbol{C}(\bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) - \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) - \boldsymbol{K}(\cdot)) + \bar{\boldsymbol{f}}_k + C\bar{\boldsymbol{w}}_k + \boldsymbol{v}_{k+1},\end{aligned} \tag{7}$$

where

$$\bar{\boldsymbol{f}}_k = \boldsymbol{C}\bar{\boldsymbol{L}}\boldsymbol{f}_k = \boldsymbol{C}\left[\boldsymbol{I}_n - \boldsymbol{E}\left[(\boldsymbol{C}\boldsymbol{E})^T\boldsymbol{C}\boldsymbol{E}\right]^{-1}(\boldsymbol{C}\boldsymbol{E})^T\boldsymbol{C}\right]\boldsymbol{L}\boldsymbol{f}_k. \tag{8}$$

The objective of the subsequent section is to provide an answer to the following questions: How to determine the conditions under which the UIF will not decouple the fault from the residual? In other words, how to check whether $\bar{\boldsymbol{f}}_k$ defined by (8) will be different from zero when a fault occurs?

3. Preventing fault decoupling

It is usually assumed that the disturbance decoupling will not cause the decrease of fault diagnosis sensitivity or fault decoupling in the worst scenario. Such assumption, though, is a rather impractical tool in serious applications. Thus, to overcome such a challenging problem, the following theorem provides a simple rule for checking if the proposed unknown input observer does not decouple the effect of a fault from the residual. It relates the fault and unknown input distribution matrices, denoted by $\boldsymbol{E}$ and $\boldsymbol{L}$, respectively. Moreover, let us assume that the following rank condition is satisfied:

$$\operatorname{rank}(\boldsymbol{C}\boldsymbol{L}) = \operatorname{rank}(\boldsymbol{L}) = s. \tag{9}$$

Theorem 3.1. *The fault $\boldsymbol{f}_k$ will not be decoupled from the residual* (7) *if and only if the matrix*

$$[\boldsymbol{C}\boldsymbol{E}\,\boldsymbol{C}\boldsymbol{L}] \tag{10}$$

is a full-rank one.

Proof. Let us suppose (theoretically) that $\operatorname{rank}(\boldsymbol{C}\bar{\boldsymbol{L}}) = s$, then it can be shown that

$$\boldsymbol{f}_k = (\boldsymbol{C}\bar{\boldsymbol{L}})^{+}\bar{\boldsymbol{f}}_k \tag{11}$$

which means that there exists a unique relationship between $\boldsymbol{f}_k$ and $\bar{\boldsymbol{f}}_k$ and hence the fault will not be decoupled from the residual. Unfortunately, the subsequent

part of the proof shows that this cannot always be attained. Indeed, (8) can be written in an equivalent form

$$\bar{\boldsymbol{f}}_k = \left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CLf}_k. \tag{12}$$

Moreover, it can be observed that

$$\begin{aligned}&\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]^2\\ &= \boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T,\end{aligned} \tag{13}$$

which means that $\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T$ is an idempotent matrix. One of the fundamental properties of an idempotent matrix is that its rank is equal to the trace, i.e.:

$$\begin{aligned}&\operatorname{rank}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right)\\ &= \operatorname{trace}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right)\\ &= \operatorname{trace}(\boldsymbol{I}_m) - \operatorname{trace}\left(\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right)\\ &= m - \operatorname{trace}\left(\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CE}\right) = m - q.\end{aligned} \tag{14}$$

Thus, from (9) it is clear that

$$\begin{aligned}&\operatorname{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \leq\\ &\leq \min(m-q, s).\end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned}&\operatorname{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \geq\\ &\geq \operatorname{rank}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right) + \operatorname{rank}(\boldsymbol{CL}) - m =\\ &s - q.\end{aligned} \tag{16}$$

Finally,

$$\begin{aligned}&\max(s-q, 0) \leq \operatorname{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \leq\\ &\leq \min(m-q, s).\end{aligned} \tag{17}$$

Thus, it is necessary to find an alternative condition under which

$$\begin{aligned}\bar{\boldsymbol{f}}_k &= \boldsymbol{CLf}_k - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CLf}_k =\\ &= \boldsymbol{CLf}_k - \boldsymbol{CLf}_k = \boldsymbol{0}.\end{aligned} \tag{18}$$

Indeed, any vector $\boldsymbol{CLf}_k \in \text{col}(\boldsymbol{CE})$, where

$$\text{col}(\boldsymbol{CE}) = \{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} = \boldsymbol{CE\beta} \quad \text{for some} \quad \boldsymbol{\beta} \in \mathbb{R}^q\}, \tag{19}$$

can be written as

$$\boldsymbol{CLf}_k = \boldsymbol{CE}\tilde{\boldsymbol{f}}_k, \tag{20}$$

for some non-zero vector $\tilde{\boldsymbol{f}}_k$. As a consequence:

$$\begin{aligned}&\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CLf}_k = \\ &\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CE}\tilde{\boldsymbol{f}}_k = \boldsymbol{CE}\tilde{\boldsymbol{f}}_k = \boldsymbol{CLf}_k.\end{aligned} \tag{21}$$

From the above discussion, it is clear that the proposed unknown input observer will not decouple the fault effect from the residual iff $\boldsymbol{CLf}_k \notin \text{col}(\boldsymbol{CE})$, which is equivalent to:

$$\text{rank}\left([\boldsymbol{CE}\,\boldsymbol{CLf}_k]\right) = q + 1, \tag{22}$$

for all $f_{i,k} \neq 0$, $i = 1, \ldots, s$. It is clear that (22) is equivalent to the fact that the only solution to (for all $f_{i,k} \neq 0$, $i = 1, \ldots, s$)

$$\alpha_1(\boldsymbol{CE})_1 + \alpha_2(\boldsymbol{CE})_2 + \cdots + \alpha_q(\boldsymbol{CE})_q + \alpha_{q+1}\boldsymbol{CLf}_k = 0, \tag{23}$$

is for $\alpha_i = 0$, $i = 1, \ldots, q+1$. By the further expansion (23) to

$$\alpha_1(\boldsymbol{CE})_1 + \cdots + \alpha_q(\boldsymbol{CE})_q + \alpha_{q+1}f_{1,k}(\boldsymbol{CL})_1 + \cdots + \alpha_{q+1}f_{s,k}(\boldsymbol{CL})_s = 0, \tag{24}$$

it can be seen that the zero-valued solution to (24) is equivalent to the existance of a full-rank matrix (10), which completes the proof.

4. Unscented Kalman filter

As it was already mentioned, state estimation for non-linear stochastic systems is an extremely difficult and important problem for modern fault diagnosis and control systems (see the recent books in the domain for a complete survey and explanations: Ding, 2008, Ducard, 2009; Isermann, 2011; Mohmoud et al., 2003; Noura et al., 2009; Witczak, 2007). As can be observed in the literature, the most frequently used approach to state estimation of non-linear stochastic systems is to use the celebrated EKF. However, the linearised non-linear transformations of the state and/or output are reliable only if there is no excessive difference as to the local behaviour compared to the original non-linear transformation. If this is not the case, then the EKF will suffer from the divergence. However, there are works in which the authors use the EKF as a non-linear deterministic observer and in this case the process and measurement noise matrices are used as instrumental matrices that can significantly improve the convergence performance (see Witczak

and pretki, 2007; Witczak, 2007 for a comprehensive survey). Unfortunately, in the stochastic case $\boldsymbol{Q}$ and $\boldsymbol{R}$ have to play their primary role as covariance matrices.

As indicated in Julier and Uhlmann (2004), *it is easier to approximate a probability distribution than it is to approximate an arbitrary non-linear function or transformation.*

Bering in mind this sentence, the idea of an Unscented Transform (UT) was developed, and applied along with the celebrated Kalman filter in order to form the UKF. To make the paper self-contained, the subsequent points will describe the UT and the algorithm of the UKF.

Finally, it should be underline that the reader is refereed to Julier and Uhlmann (2004)(and the references therein) for a large number of practical examples showing the superiority of the UKF over the conventional EKF. Thus, the subsequent parts of the paper are focused on developing new UKF-based scheme rather then showing its superiority over the EKF.

4.1. Unscented transform

The unscented transform boils down to approximating the mean and covariance of the so-called sigma points after the non-linear transformation $\boldsymbol{h}(\cdot)$. The mean and covariance of sigma points are given as $\bar{\boldsymbol{x}}$ and $\boldsymbol{P}$, while the UT procedure is, Julier and Uhlmann (2004):

1. Generate k sigma points

$$\boldsymbol{X}_i, \quad i = 1, \ldots, k \tag{25}$$

   with $\bar{\boldsymbol{x}}$ and $\boldsymbol{P}$.

2. Obtain a non-linear transformation of each sigma point

$$\boldsymbol{X}_i^t = \boldsymbol{h}(\boldsymbol{X}_i), \quad i = 1, \ldots, k. \tag{26}$$

3. Calculate the weighted mean of the transformed points

$$\bar{\boldsymbol{x}}^t = \sum_{i=1}^{k} W^i \boldsymbol{X}_i^t. \tag{27}$$

4. Calculate the covariance of the transformed points

$$\boldsymbol{P}^t = \sum_{i=1}^{k} W^i \left[\boldsymbol{X}_i^t - \bar{\boldsymbol{x}}^t\right] \cdot \left[\boldsymbol{X}_i^t - \bar{\boldsymbol{x}}^t\right]^T. \tag{28}$$

Note that the sigma-points can be generated with various scenarios, Julier and Uhlmann (2004); Kandepu et al. (2008), and one of them will be described in the subsequent point. It should be also mentioned that in order to provide an unbiased estimate, Julier and Uhlmann (2004), the weights should satisfy

$$\sum_{i=1}^{k} W^i = 1. \tag{29}$$

4.2. The principle of the UKF

Let us consider a non-linear, discrete-time fault-free system, i.e. (1)–(2) for $\boldsymbol{f}_k = \boldsymbol{0}$:

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{E}\boldsymbol{d}_k + \boldsymbol{w}_k, \tag{30}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}. \tag{31}$$

As it was mentioned, the UKF, Kandepu et al. (2008) can be perceived a derivative-free alternative to the extended Kalman filter in the framework of state-estimation. The UKF calculates the mean and covariance of a random variable, which undergoes a non-linear transformation by utilizing a deterministic "sampling" approach. Generally, $2n+1$, *sigma* points are chosen based on a square-root decomposition of the prior covariance. These sigma points are propagated through the true nonlinearity, without any approximation, and then a weighted mean and covariance are taken, as described in Section 4.1.

The presented form of the UKF is based on the general structure of the unknown input observer (6), taking into account the fact that the output equation (31) is linear.

The UKF involves a recursive application of these sigma points to the state-space equations. The standard UKF implementation for state-estimation uses the following variable definitions:

- $W_0^m = \lambda/(n+\lambda)$,
- $W_0^c = \lambda/(n+\lambda) + (1-\alpha^2+\beta)$,
- $W_i^m = W_i^c = 1/\{2(n+\lambda)\}$,
- $\lambda = L(\alpha^2 - 1)$,
- $\eta = \sqrt{(n+\lambda)}$,

where W_i $(i = 0, \ldots, 2n)$ is a set of scalar weights, λ and η are scaling parameters. The constant α determines the spread of sigma points around $\hat{x}$ and is usually set to $10^{-4} \leq \alpha \leq 1$. β is used to incorporate prior knowledge of the distribution (for Gaussian distribution, $\beta = 2$ is an optimal choice). The UKF algorithm is as follows:

Initialize with:

$$\hat{\boldsymbol{x}}_0 = \mathcal{E}[\boldsymbol{x}_0] \quad \boldsymbol{P}_0 = \mathcal{E}[(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)^T], \tag{32}$$

for $k \in \{1, \ldots, \infty\}$
Calculate $2n+1$ sigma points:

$$\hat{\boldsymbol{X}}_{k-1} = [\hat{\boldsymbol{x}}_{k-1} \quad \hat{\boldsymbol{x}}_{k-1} + \eta S(1), \ldots, \hat{\boldsymbol{x}}_{k-1} + \eta S(n),$$
$$\hat{\boldsymbol{x}}_{k-1} - \eta S(1), \ldots, \hat{\boldsymbol{x}}_{k-1} - \eta S(n)], \tag{33}$$

where $\boldsymbol{S} = \sqrt{\boldsymbol{P}_{k-1}}$ and $\boldsymbol{S}(j)$ stands for the jth column of $\boldsymbol{S}$.

Time update equations:

$$\hat{\boldsymbol{X}}_{i,k|k-1} = \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{X}}_{i,k-1}\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1},\ i = 0, \ldots, 2n, \tag{34}$$

$$\hat{\boldsymbol{x}}_{k,k-1} = \sum_{i=0}^{2n} W_i^{(m)} \hat{\boldsymbol{X}}_{i,k|k-1}, \tag{35}$$

$$\begin{aligned}\boldsymbol{P}_{k,k-1} = \sum_{i=0}^{2n} W_i^{(c)} [\hat{\boldsymbol{X}}_{i,k|k-1} - \\ - \hat{\boldsymbol{x}}_{k,k-1}][\hat{\boldsymbol{X}}_{i,k|k-1} - \hat{\boldsymbol{x}}_{k,k-1}]^T + \boldsymbol{Q}.\end{aligned} \tag{36}$$

$$\tag{37}$$

Measurement update equations:

$$\begin{aligned}\boldsymbol{P}_{y_k y_k} &= \boldsymbol{C}\boldsymbol{P}_{k,k-1}\boldsymbol{C}^T + \boldsymbol{R}, \\ \boldsymbol{K}_k &= \boldsymbol{P}_{k,k-1}\boldsymbol{C}^T \boldsymbol{P}_{y_k y_k}^{-1}, \qquad (38) \\ \hat{\boldsymbol{y}}_{k,k-1} &= \boldsymbol{C}\hat{\boldsymbol{x}}_{k,k-1}, \qquad (39) \\ \hat{\boldsymbol{x}}_k &= \hat{\boldsymbol{x}}_{k,k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k,k-1}), \qquad (40) \\ \boldsymbol{P}_k &= [\boldsymbol{I}_n - \boldsymbol{K}_k\boldsymbol{C}]\,\boldsymbol{P}_{k,k-1}. \qquad (41)\end{aligned}$$

5. Determination of unknown input distribution matrix

As a result of the deliberations presented in the previous section, matrix $\boldsymbol{E}$ should satisfy the following conditions:

$$\operatorname{rank}(\boldsymbol{CE}) = \operatorname{rank}(\boldsymbol{E}) = q, \tag{42}$$

where

$$[\boldsymbol{CE}\ \boldsymbol{CL}] \tag{43}$$

should be a full rank one, which means that

$$\operatorname{rank}\left([\boldsymbol{CE}\ \boldsymbol{CL}]\right) = \min(m, s+q). \tag{44}$$

Thus, the set of matrices $\boldsymbol{E}$ satisfying (42) and (44) is given by

$$\mathbb{E} = \left\{\boldsymbol{E} \in \mathbb{R}^{n\times q} : \operatorname{rank}(\boldsymbol{CE}) = q \wedge \operatorname{rank}(\boldsymbol{E}) = q \wedge \operatorname{rank}\left([\boldsymbol{CE}\ \boldsymbol{CL}]\right) = \min(m, s+q)\right\}. \tag{45}$$

It should be strongly underlined that $\mathbb{E}$ is not convex, which significantly complicates the problem and limits the spectrum of possible approaches that can be used for settling the determination of the unknown input distribution matrix.

The subsequent part of this section presents a numerical algorithm that can be used for estimating the unknown input distribution matrix $\boldsymbol{E}$ based on a set of input-output measurements $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$.

To settle the problem of numerical estimation of $\boldsymbol{E}$, the following optimization criterion is assumed

$$\hat{\boldsymbol{E}} = \arg\min_{\boldsymbol{E}\in\mathbb{E}} J(\boldsymbol{E}), \tag{46}$$

with

$$J(\boldsymbol{E}) = \frac{1}{mn_t}\sum_{k=1}^{n_t} \boldsymbol{z}_k^T \boldsymbol{z}_k, \tag{47}$$

where $\boldsymbol{z}_k$ stands for the residual defined by (7) and $\hat{\boldsymbol{E}}$ is an estimate of $\boldsymbol{E}$.

It is important to underline that the computation of (47) requires the run of the proposed UIF for a given instance of the unknown input distribution matrix $\boldsymbol{E}$. The computation of the cost function (47) is a definitely most time consuming part of the proposed algorithm. On the other hand, the computation time and the resulting computational burden are not of paramount importance since the proposed algorithm performs off-line. Indeed, only the result of the proposed algorithm, being an estimate of the unknown input distribution matrix $\boldsymbol{E}$ is utilised on-line for unknown input decoupling.

The outline of the proposed algorithm:

Step 1: Obtain the fault-free input-output data set from the system $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$.

Step 2: Initialise the algorithm with some initial value of $\boldsymbol{E}$ satisfying (42) and (43).

Step 3: Use an optimisation strategy to find an estimate of $\boldsymbol{E}$ for which (47) reaches its minimum and conditions (42) and (43) are satisfied.

Similarly as in the case of (8), i.e. by following with $\tilde{\boldsymbol{d}}_k$ in a similar way as with $\bar{\boldsymbol{f}}_k$ in (8), it can be shown that the fault-free residual is:

$$\begin{aligned}\boldsymbol{z}_{k+1} &= \boldsymbol{y}_{k+1} - \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1} = \\ &= \boldsymbol{C}\left(\bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) - \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) - \boldsymbol{K}(\cdot)\right) + \tilde{\boldsymbol{d}}_k + \boldsymbol{C}\bar{\boldsymbol{w}}_k + \boldsymbol{v}_{k+1},\end{aligned} \tag{48}$$

where

$$\tilde{\boldsymbol{d}}_k = \boldsymbol{C}\left[\boldsymbol{I}_n - \hat{\boldsymbol{E}}\left[(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\hat{\boldsymbol{E}}\right]^{-1}(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\right]\bar{\boldsymbol{d}}_k. \tag{49}$$

Alternatively, assuming $\bar{\boldsymbol{d}}_k = \boldsymbol{E}\boldsymbol{d}_k$, it can be expressed by

$$\tilde{\boldsymbol{d}}_k = \boldsymbol{C}\left[\boldsymbol{I}_n - \hat{\boldsymbol{E}}\left[(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\hat{\boldsymbol{E}}\right]^{-1}(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\right]\boldsymbol{E}\boldsymbol{d}_k. \tag{50}$$

Following the same line of reasoning as in the proof of Theorem 1, it can be shown that for any vector $\boldsymbol{C}\boldsymbol{E}\boldsymbol{d}_k \in \mathrm{col}(\boldsymbol{C}\hat{\boldsymbol{E}})$ the effect of an unknown input $\tilde{\boldsymbol{d}}_k$ will be decoupled from the residual, i.e. $\tilde{\boldsymbol{d}}_k = \boldsymbol{0}$.

Based on the above deliberations, it seems that an alternative approach is:

Step 0: Obtain the fault-free input-output data set from the system $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$.

Step 1: Estimate $\bar{\boldsymbol{d}}_k$ for $k = 1, \ldots, n_t$ with, e.g., an augmented UKF.

Step 2: Find a basis of $[\bar{\boldsymbol{d}}_1, \ldots, \bar{\boldsymbol{d}}_{n_t}]$ (e.g. an orthonormal basis), which will constitute an estimate of $\boldsymbol{E}$.

Apart from the unquestionable appeal of the above algorithm it does not take into account that conditions (42) and (43) must be satisfied. On the other hand, it was empirically proven that due to the process and measurement noise, an accurate estimation of $\bar{\boldsymbol{d}}_k$ (for $k = 1, \ldots, n_t$) is impossible, and hence, Step 2 of the above algorithm cannot be realised with expected results.

Thus, the only fruitful conclusion is that an estimate of $\boldsymbol{E}$ is not unique, which will undoubtedly facilitate the performance of the optimisation-based approach presented in the subsequent part of this section.

Taking into account all the above-mentioned difficulties, it is proposed to use the Adaptive Random Search algorithm (ARS), Walter and Pronzato (1996); Witczak (2007), to solve (46). The algorithm has proven to be very reliable in various global optimisation problems, which also justifies its application for this particular task.

The search process of the ARS can be split into two phases. The first phase (variance-selection phase) consists in selecting an element from the sequence

$$\{\sigma^{(i)}\}, \qquad i = 1, \ldots, i_{max} \tag{51}$$

where $\sigma^{(1)}$ stands for an initial standard deviation selected by the designer (forming the covariance matrix $\Sigma = \sigma \boldsymbol{I}_{n \times q}$, where $n \times q$ is the number of elements of $\boldsymbol{E}$), and

$$\sigma^{(i)} = 10^{(-i+1)} \sigma^{(1)}. \tag{52}$$

In this way, the range of σ ensures both proper exploration properties over the search space and a sufficient accuracy of optimum localization. Larger values of σ decrease the possibility of getting stuck in a local minimum. The second phase (variance-exploration phase) is dedicated to exploring the search space with the use of σ obtained from the first phase and consists in repetitive random perturbation of the best point obtained in the first phase. The scheme of the ARS algorithm is as follows:

0. Input data:

- $\sigma^{(1)}$ – the initial standard deviation;
- $j_{\max}$ – the number of iteration in each phase;
- $i_{\max}$ – the number of standard deviations (σ^i) changes;
- $k_{\max}$ – the global number of algorithm runs;
- $\boldsymbol{E}^{(0)}$ – the initial value of the unknown input distribution matrix.

1. Initialize

(1.1) Generate $\boldsymbol{E}_{\text{best}} \rightarrow \boldsymbol{E}^0$, satisfying (42) and (43), $k \rightarrow 1$, $i \rightarrow 1$.

2. Variance-selection phase

(2.1) $j \rightarrow 1, \boldsymbol{E}^{(j)} \rightarrow \boldsymbol{E}^{(0)}$ and $\sigma^{(i)} \rightarrow 10^{(-i+1)} \sigma^{(1)}$.

(2.2) Perturb $\boldsymbol{E}^{(j)}$ to get a new trial point $\boldsymbol{E}_{+}^{(j)}$ satisfying (42) and (43).
(2.3) If $J(\boldsymbol{E}_{+}^{(j)}) \leq J(\boldsymbol{E}^{(j)})$ then $\boldsymbol{E}^{(j+1)} \to \boldsymbol{E}_{+}^{(j)}$
else $\boldsymbol{E}^{(j+1)} \to \boldsymbol{E}^{(j)}$.
(2.4) If $J(\boldsymbol{E}_{+}^{(j)}) \leq J(\boldsymbol{E}_{\text{best}})$ then
$\boldsymbol{E}_{\text{best}} \to \boldsymbol{E}_{+}^{(j)}, i_{\text{best}} \to i$.
(2.5) If $(j \leq j_{\max}/i)$ then $j \to j+1$ and go to (2.2).
(2.6) If $(i < i_{\max})$ then set $i \to i+1$ and go to (2.1).
3. Variance-exploration phase
(3.1) $j \to 1, \boldsymbol{E}^{(j)} \to \boldsymbol{E}_{\text{best}}, i \to i_{\text{best}}$
and $\sigma^{(i)} \to 10^{(-i+1)}\sigma^{(1)}$.
(3.2) Perturb $\boldsymbol{E}^{(j)}$ to get a new trial point $\boldsymbol{E}_{+}^{(j)}$ satisfying (42) and (43).
(3.3) If $J(\boldsymbol{E}_{+}^{(j)}) \leq J(\boldsymbol{E}^{(j)})$ then $\boldsymbol{E}^{(j+1)} \to \boldsymbol{E}_{+}^{(j)}$
else $\boldsymbol{E}^{(j+1)} \to \boldsymbol{E}^{(j)}$.
(3.4) If $J(\boldsymbol{E}_{+}^{(j)}) \leq J(\boldsymbol{E}_{\text{best}})$ then $\boldsymbol{E}_{\text{best}} \to \boldsymbol{E}_{+}^{(j)}$.
(3.5) If $(j \leq j_{\max})$ then $j \to j+1$ and go to Step 3.2.
(3.6) If $(k \to k_{\max})$ then STOP.
(3.7) $k \to k+1, \boldsymbol{E}^{(0)} \to \boldsymbol{E}_{\text{best}}$ and resume from (2.1).

The perturbation phase (points 2.2 and 3.2 of the algorithm) is realised according to

$$\boldsymbol{E}_{+}^{(j)} = \boldsymbol{E}^{(j)} + \boldsymbol{Z}, \tag{53}$$

where each element of $\boldsymbol{Z}$ is generated according to $\mathcal{N}(0, \sigma^i)$. When the newly generated $\boldsymbol{E}^{(j)}$ does not satisfy (42) and (43), then the perturbation phase (53) is repeated.

It should be also noted that for some $\boldsymbol{E}^{(j)}$ the proposed UIF may diverge, e.g. due to the loss of observability or a large mismatch with the real system. A simple remedy is to assume a bound (possibly large) ζ on $J(\boldsymbol{E}^{(j)})$, which means that when this bound is exceeded, then UIF is terminated and $J(\boldsymbol{E}^{(j)}) = \zeta$.

6. Design of UIF with a varying unknown input distribution matrix

The UIF proposed in this section is designed in such a way that it will be able to tackle the problem of automatically changing the unknown input distribution matrices according to the system behaviour. In other words, the user can design a number of such matrices in order to cover different operating conditions. Thus, having such a set of matrices, it is possible to design a bank of the UIFs and the algorithm should use them to obtain the best unknown input decoupling and state estimation. In order to realise this task, the Interacting Multiple-Model (IMM) approach (Blom and Bar–Shalom, 1988) is used. The subsequent part of this section shows a comprehensive description of the UIF and IMM.

The IMM solution consists of a filter for each disturbance matrix (corresponding to a particular model of the system), an estimate mixer at the input of the

filters, and an estimate combiner at the output of the filters. The IMM works as a recursive estimator. In each recursion it has four steps:

1. interacting or mixing of the model-conditional estimates, in which the input to the filter matched to a certain mode is obtained by mixing the estimates of all filters from the previous time instant under the assumption that this particular mode is in effect at the present time;
2. model-conditional filtering, performed in parallel for each mode;
3. model probability update, based on the model-conditional innovations and likelihood functions;
4. estimate combination, which yields the overall state estimate as the probabilistically weighted sum of the updated state estimates of all the filters.

The probability of a mode in effect plays a key role in determining the weights in the combination of the state estimates and covariances for the overall state estimate. Fig. 1 shows the block diagram of the classic IMM algorithm, where:

- $\hat{\boldsymbol{x}}_{k+1|k+1}$ is the state estimate for time k using measurements through time $(k+1|k+1)$ based on N models;
- $\hat{\boldsymbol{x}}^{j}_{k+1|k+1}$ is the state estimate for time k using measurements through time $(k+1|k+1)$ based on model j;
- Λ^{j}_{k} is the model likelihood at time k based on model j;
- μ_k is the vector of model probabilities at time k when all the likelihoods Λ^{j}_{k} have been considered at model probability update.

With the assumption that model switching is governed by an underlying Markov chain, an interacting mixer at the input of the N filters uses the model probabilities μ_k and the model switching probabilities p_{ij} to compute a mixed (initial or a priori) estimate $\hat{\boldsymbol{X}}^{0j}_{k|k}$ for N filters. Interacting mixer blends the previous state estimates based on N models to obtain new state estimates for input into each model. The mixing gains $\mu^{i|j}_{k-1|k-1}$ are computed from the previous model probabilities μ^{i}_{k-1} and the model switching probabilities p_{ij} in the model probability update. At the beginning of a filtering cycle, all filters use a priori mixed estimate $\hat{\boldsymbol{X}}^{0j}_{k-1|k-1}$ and the current measurement $\boldsymbol{y}_k$ to compute a new estimate $\hat{\boldsymbol{X}}^{j}_{k|k}$ and likelihood Λ^{j}_{k} for jth model filter. The likelihoods, prior model probabilities, and model switching probabilities are then used by model probability update to compute new model probabilities. The overall state estimate $\hat{\boldsymbol{X}}_{k|k}$ is then computed at an estimate combiner with the new state estimates and their probabilities.

The algorithm presented below is a combination of the UIF and the IMM and constitutes a solution to the challenging problem of designing UIF for a set of predefined unknown input distribution matrices $\{\boldsymbol{E}_j\}^{N}_{j=1}$.

Step 1: Mixing State Estimates

The filtering process starts with 'a priori' state estimates $\hat{\boldsymbol{X}}^{j}_{k-1|k-1}$, state error covariances $\boldsymbol{P}_{k-1|k-1}$ and the associated probabilities μ^{j}_{k-1} for each jth filter model corresponding to the jth unknown input distribution matrix. The initial or mixed state estimate and covariance for model j at time k is

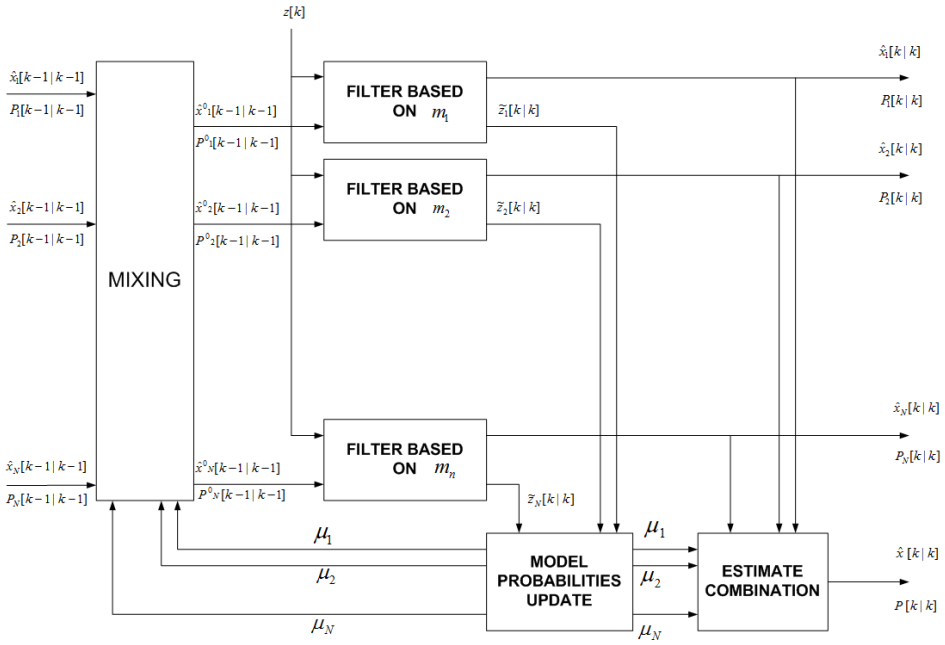

Figure 1. IMM algorithm

computed as

$$\bar{c_j} = \sum_{i=1}^{N} p_{ij}\mu_{k-1}^{i} \tag{54}$$

$$\mu_{k-1|k-1}^{i|j} = \frac{1}{\bar{c_j}} p_{ij}\mu_{k-1}^{i} \tag{55}$$

$$\hat{\boldsymbol{X}}_{k-1|k-1}^{0j} = \sum_{i=1}^{N} \hat{\boldsymbol{X}}_{k-1|k-1}^{i}\mu_{k-1|k-1}^{i|j} \tag{56}$$

$$\boldsymbol{P}_{k-1|k-1}^{0j} = \sum_{i=1}^{N}[\boldsymbol{P}_{k-1|k-1}^{i} + (\hat{\boldsymbol{X}}_{k-1|k-1}^{i} - \hat{\boldsymbol{X}}_{k-1|k-1}^{0j}) \cdot (\hat{\boldsymbol{X}}_{k-1|k-1}^{i} - \hat{\boldsymbol{X}}_{k-1|k-1}^{0j})^{T}]\mu_{k-1|k-1}^{i|j} \tag{57}$$

where p_{ij} is the assumed transition probability for switching from model i to model j, and $\bar{c_j}$ is a normalization constant. For every state estimate $\hat{\boldsymbol{X}}_{k|k}^{i}$ and $\hat{\boldsymbol{X}}_{k-1|k-1}^{i}$, there is a corresponding covariance $\boldsymbol{P}_{k|k}^{i}$ and $\boldsymbol{P}_{k-1|k-1}^{i}$.

Step 2: Model-Conditioned Update

Calculate sigma points (for each jth model):

$$\hat{\boldsymbol{X}}^{j}_{k-1} = [\hat{\boldsymbol{X}}^{0j}_{k-1|k-1} \;\; \hat{\boldsymbol{X}}^{0j}_{k-1|k-1} + \eta\sqrt{\boldsymbol{P}^{0j}_{k-1|k-1}} \;\; \hat{\boldsymbol{X}}^{0j}_{k-1|k-1} - \eta\sqrt{\boldsymbol{P}^{0j}_{k-1|k-1}}] \tag{58}$$

Time update (for each jth model):

$$\hat{\boldsymbol{X}}^{j}_{i,k|k-1} = \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{X}}^{j}_{i,k-1}\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1},\; i = 0,\ldots,2n, \tag{59}$$

$$\hat{\boldsymbol{x}}^{j}_{k,k-1} = \sum_{i=0}^{2n} W_i^{(m)}\,\hat{\boldsymbol{X}}^{j}_{i,k|k-1}, \tag{60}$$

$$\boldsymbol{P}^{j}_{k,k-1} = \sum_{i=0}^{2n} W_i^{(c)}[\hat{\boldsymbol{X}}^{j}_{i,k|k-1} - \\ - \hat{\boldsymbol{x}}^{j}_{k,k-1}][\hat{\boldsymbol{X}}^{j}_{i,k|k-1} - \hat{\boldsymbol{x}}^{j}_{k,k-1}]^T + \boldsymbol{Q}. \tag{61}$$

$$\tag{62}$$

Measurement update equations:

$$\begin{aligned}
\boldsymbol{P}^{j}_{y_k y_k} &= \boldsymbol{C}\boldsymbol{P}^{j}_{k,k-1}\boldsymbol{C}^T + \boldsymbol{R} \\
\boldsymbol{K}^{j}_{k} &= \boldsymbol{P}^{j}_{k,k-1}\boldsymbol{C}^T\boldsymbol{P}^{-1\;(j)}_{y_k y_k}, && (63)\\
\hat{\boldsymbol{y}}^{j}_{k,k-1} &= \boldsymbol{C}\hat{\boldsymbol{x}}^{j}_{k,k-1}, && (64)\\
\boldsymbol{z}^{j}_{k} &= \boldsymbol{y}_k - \hat{\boldsymbol{y}}^{j}_{k,k-1}, && (65)\\
\hat{\boldsymbol{x}}^{j}_{k|k} &= \hat{\boldsymbol{x}}^{j}_{k,k-1} + \boldsymbol{K}^{j}_{k}\boldsymbol{z}^{j}_{k} && (66)\\
\boldsymbol{P}^{j}_{k|k} &= \left[\boldsymbol{I}_n - \boldsymbol{P}^{j}_{k|k-1}\boldsymbol{K}_k\boldsymbol{C}\right]\boldsymbol{P}^{j}_{k|k-1}. && (67)
\end{aligned}$$

Step 3: Model likelihood computations

The likelihood of the jth model is computed with the filter residuals $\boldsymbol{z}^{j}_{k}$, the covariance of the filter residuals $\boldsymbol{P}^{j}_{y_k y_k}$ and the assumption of Gaussian statistics. The likelihood of the jth model and model probabilities update are as follows

$$\begin{aligned}
\Lambda^{j}_{k} &= \frac{1}{\sqrt{|2\pi\boldsymbol{P}^{j}_{y_k y_k}|}}\exp[-0.5(z^{j}_{k})^T(\boldsymbol{P}^{j}_{y_k y_k})^{-1}z^{j}_{k}] \\
c &= \sum_{i=1}^{N}\Lambda^{i}_{k}\bar{c_i} \\
\mu^{j}_{k} &= \frac{1}{c}\Lambda^{j}_{k}\bar{c_j}.
\end{aligned}$$

Step 4: Combination of State Estimates

The state estimate $\hat{\boldsymbol{x}}_{k|k}$ and the covariance $\boldsymbol{P}_{k|k}$ for IMM filter are obtained from a probabilistic sum of the individual filter outputs

$$\hat{\boldsymbol{x}}_{k|k} = \sum_{N}^{i=1} \hat{\boldsymbol{x}}^i_{k|k}\mu^i_k$$

$$\boldsymbol{P}_{k|k} = \sum_{N}^{i=1} \mu^i_k[\boldsymbol{P}^i_{k|k} + (\hat{\boldsymbol{x}}^i_{k|k} - \hat{\boldsymbol{x}}_{k|k})(\hat{\boldsymbol{x}}^i_{k|k} - \hat{\boldsymbol{x}}_{k|k})^T].$$

7. Experimental results

The objective of the subsequent parts of this section is to examine the proposed approaches with two exemplary systems, i.e. an induction motor and a two-tank system. In particular, the way of determining unknown input distribution matrix and 'switching' of these matrices will be illustrated with an induction motor. The two tank systems will be employed to show the performance of the proposed approach with respect to fault detection and isolation.

7.1. Estimation of E for an induction motor

The purpose of this section is to show the reliability and effectiveness of the proposed EUIO. The numerical example considered here is a fifth-order two-phase non-linear model of an induction motor, which has already been the subject of a large number of various control design applications (see Boutayeb and Aubry, 1999 and the references therein). The complete discrete-time model in a stator-fixed (a,b) reference frame is:

$$x_{1,k+1} = x_{1,k} + h\left(-\gamma x_{1k} + \frac{K}{T_r}x_{3k} + Kpx_{5k}x_{4k} + \frac{1}{\sigma L_s}u_{1k}\right), \tag{68}$$

$$x_{2,k+1} = x_{2,k} + h\left(-\gamma x_{2k} - Kpx_{5k}x_{3k} + \frac{K}{T_r}x_{4k} + \frac{1}{\sigma L_s}u_{2k}\right), \tag{69}$$

$$x_{3,k+1} = x_{3,k} + h\left(\frac{M}{T_r}x_{1k} - \frac{1}{T_r}x_{3k} - px_{5k}x_{4k}\right), \tag{70}$$

$$x_{4,k+1} = x_{4,k} + h\left(\frac{M}{T_r}x_{2k} + px_{5k}x_{3k} - \frac{1}{T_r}x_{4k}\right), \tag{71}$$

$$x_{5,k+1} = x_{5,k} + h\left(\frac{pM}{JL_r}(x_{3k}x_{2k} - x_{4k}x_{1k}) - \frac{T_L}{J}\right), \tag{72}$$

$$y_{1,k+1} = x_{1,k+1},\ y_{2,k+1} = x_{2,k+1}, \tag{73}$$

where $\boldsymbol{x}_k = [x_{1,k}, \ldots, x_{n,k}]^T = [i_{\mathrm{sak}}, i_{\mathrm{sbk}}, \psi_{\mathrm{rak}}, \psi_{\mathrm{rbk}}, \omega_k]^T$ represents the currents, the rotor fluxes, and the angular speed, respectively, while $\boldsymbol{u}_k = [u_{\mathrm{sak}}, u_{\mathrm{sbk}}]^T$ is the stator voltage control vector, p is the number of the pairs of poles, and T_L is

the load torque. The rotor time constant T_r and the remaining parameters are defined as:

$$T_r = \frac{L_r}{R_r},\ \sigma = 1 - \frac{M^2}{L_s L_r},\ K = \frac{M}{\sigma L_s L_r},\ \gamma = \frac{R_s}{\sigma L_s} + \frac{R_r M^2}{\sigma L_s L_r^2}, \tag{74}$$

where R_s, R_r and L_s, L_r are stator and rotor per-phase resistances and inductances, respectively, and J is the rotor moment inertia.
The numerical values of the above parameters are as follows: $R_s = 0.18\ \Omega$, $R_r = 0.15\ \Omega$, $M = 0.068$ H, $L_s = 0.0699$ H, $L_r = 0.0699$ H, $J = 0.0586$ kgm^2, $T_L = 10$ Nm, $p = 1$, and $h = 0.1$ ms. The input signals are:

$$u_{1,k} = 350\cos(0.03k), \quad u_{2,k} = 300\sin(0.03k). \tag{75}$$

Let us assume that the unknown input and its distribution matrix have the following form

$$\boldsymbol{E} = [1.2, 0.2, 2.4, 1, 1.6]^T, \tag{76}$$

$$\boldsymbol{d}_k = 0.3\sin(0.5\pi k)\cos(0.03\pi k), \tag{77}$$

while the noise covariance matrices are $\boldsymbol{Q} = 10^{-5}\boldsymbol{I}$ and $\boldsymbol{R} = 10^{-5}\boldsymbol{I}$, respectively. Note that the small values of the process and measurement noise are selected in order to clearly portray the effect of an unknown input.

Fig. 2 shows the residual $\boldsymbol{z}_k$ for randomly selected $\boldsymbol{E}$. From these results, it is evident that the estimation quality is very low and hence the residual is significantly different from zero, which may lead to the decrease of the fault detection abilities. In order to prevent such a situation, the algorithm presented in Section 5 was utilised with the following settings:

- $\sigma^{(1)}$ – the initial standard deviation;
- $j_{\max} = 20$ – the number of iteration in each phase;
- $i_{\max} = 5$ – the number of standard deviations (σ^i) changes;
- $k_{\max} = 50$;
- $\boldsymbol{E}^{(0)}$ – randomly selected.

The performance of the algorithm was tested for a set of $\sigma^{(1)}$, i.e., $\{1, 2, 3, 4, 5\}$. Note that $k_{\max} = 50$, i.e. each run of the algorithm was performed 50 times. As a result, the mean and standard deviation of the resulting $J(\boldsymbol{E})$ (see (47)) for each setting of $\sigma^{(1)}$ was calculated. The mean of $J(\boldsymbol{E})$ is presented in Fig. 3, while its standard deviation is portrayed in Fig. 4. From these results, it is evident that the smallest mean and standard deviation are obtained for $\sigma^{(1)} = 3$. This, of course, does not mean that this is a particular value $\sigma^{(1)} = 3$, which should be the best one for each example. However, it can be easily observed that for other $\sigma^{(1)}$, i.e. $\{1, 2, 4, 5\}$ the mean and standard deviation are also very small. Numerous numerical experiments confirm this property, i.e. this means that the proposed algorithm is not extremely sensitive to the initial value of $\sigma^{(1)}$.

As it was mentioned in the previous part of this paper, $\boldsymbol{E}$, which is able to decouple the unknown input, is not unique. Indeed, the estimate of $\boldsymbol{E}$, for which

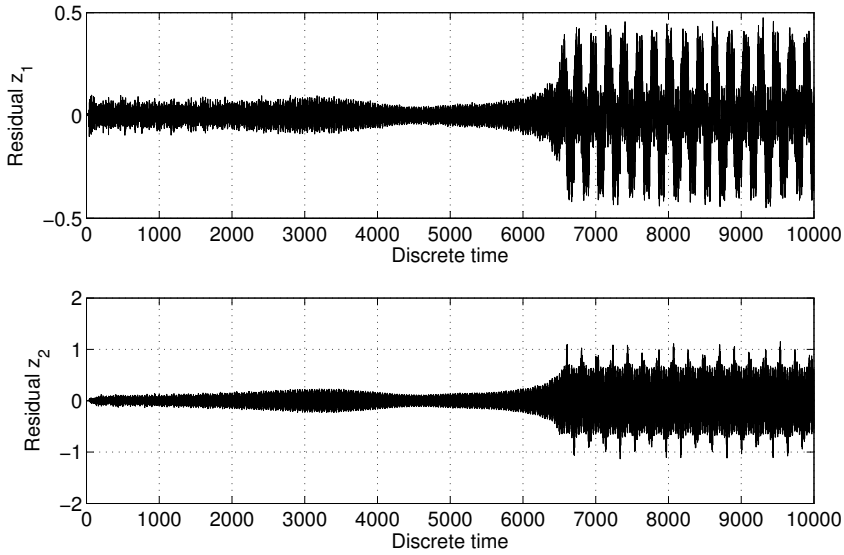

Figure 2. Residuals for randomly selected $\boldsymbol{E}$

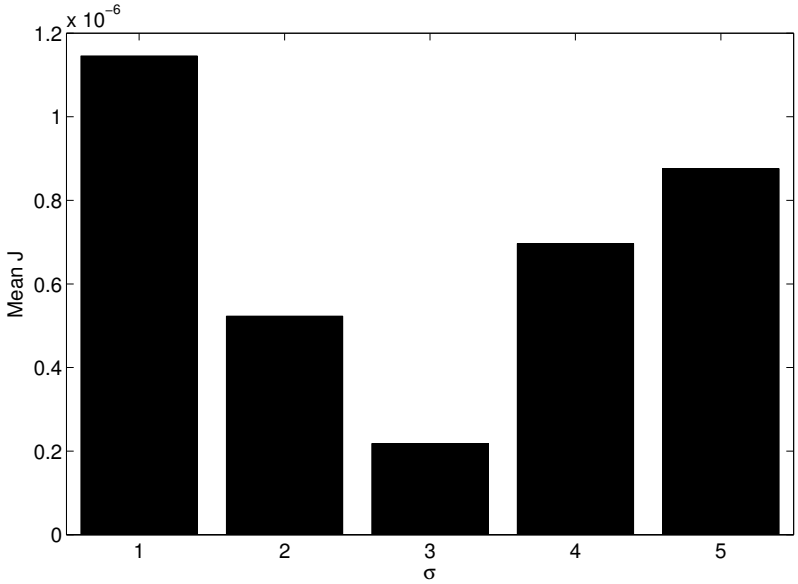

Figure 3. Mean of $J(\boldsymbol{E})$ for $\sigma^{(1)} = 1, \ldots, 5$

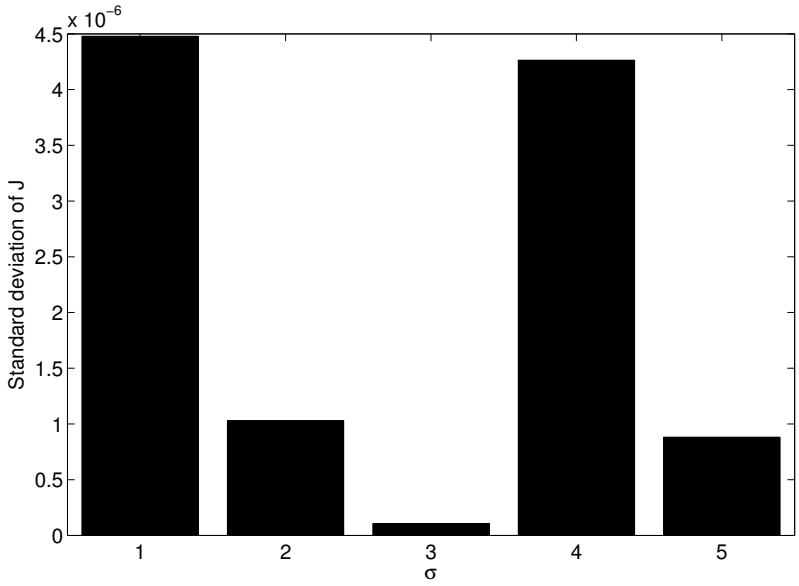

Figure 4. Standard deviation of $J(\boldsymbol{E})$ for $\sigma^{(1)} = 1, \ldots, 5$

$J(\boldsymbol{E})$ reaches its minimum is:

$$\hat{\boldsymbol{E}} = [0.3651, 0.0609, 0.7303, 0.3043, 0.4869]^T. \tag{78}$$

Figure 5 presents the residual for the obtained estimate. A direct comparison of Figs. 5 and 2 clearly shows the profits that can be gained while using the proposed algorithm.

8. The case of varying $\boldsymbol{E}$

Let us reconsider an example presented in the previous section. The unknown input is defined, as previously, by (77), but three different settings of the unknown input distribution matrix ($\boldsymbol{E}^j$) were employed during system simulation (the simulation time was 10000 samples):

$$\begin{aligned}
&\boldsymbol{E}^1 = [1.2, 0.2, 2.4, 1, 1.6]^T \text{ for } 0 \leq k < 2500,\\
&\boldsymbol{E}^2 = [0.2, 1.2, 2.4, 1, 1.6]^T \text{ for } 2500 \leq k < 5000\\
&\quad \text{and } 7500 \leq k < 10000,\\
&\boldsymbol{E}^3 = [2.1, 2.1, 2.1, 2.1, 2.1]^T \text{ for } 5000 \leq k < 7500.
\end{aligned}$$

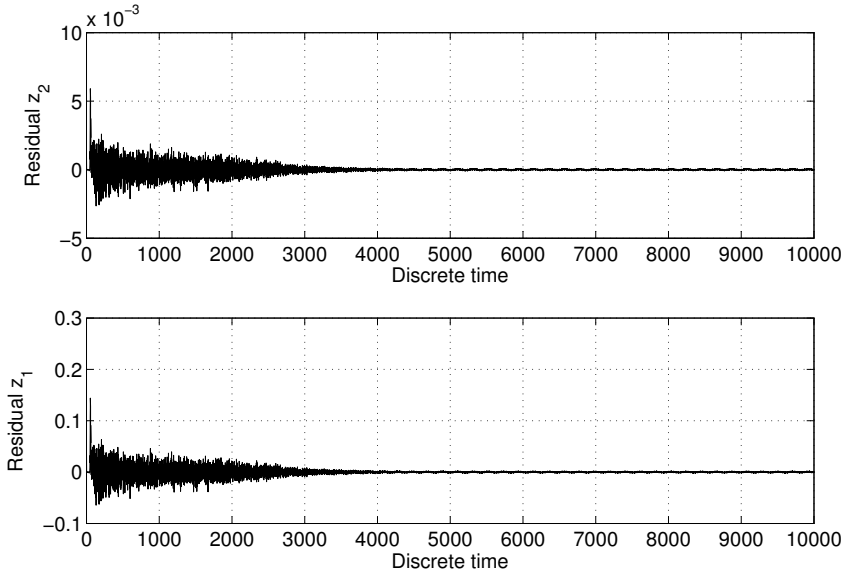

Figure 5. Residuals for the estimated $\boldsymbol{E}$

Contrarily to the above-described simulation scenario, it was assumed that the set of unknown input distribution matrices for the UIF is composed of:

$$E_1 = [0.2, 1.2, 2.4, 1, 1.6]^T,$$
$$E_2 = [0, 0.2, 2.4, 1, 0]^T,$$
$$E_3 = [2.1, 2.1, 2.1, 2.1, 2.1]^T,$$
$$E_4 = [1, 2, 3, 1, 0]^T,$$
$$E_5 = [1.2, 0.2, 2.4, 1, 1.6]^T.$$

This means that E_2 and E_4 should not be used by the UIF while E_1, E_3 and E_5 should be appropriately switched.

Fig. 6 shows the model probabilities corresponding to five unknown input distribution matrices. From these results, it is evident that the instrumental matrices E_1, E_3 and E_5 were switched correctly. Moreover, the probabilities corresponding to E_2 and E_4 are very low.

9. Fault detection and isolation for a two–tank system

The system being considered consists of two cylindrical tanks of the same diameter. They are linked to each other through the connecting cylindrical pipe (Fig. 7). The two-tank system can be perceived as a Single-Input Multi-Output

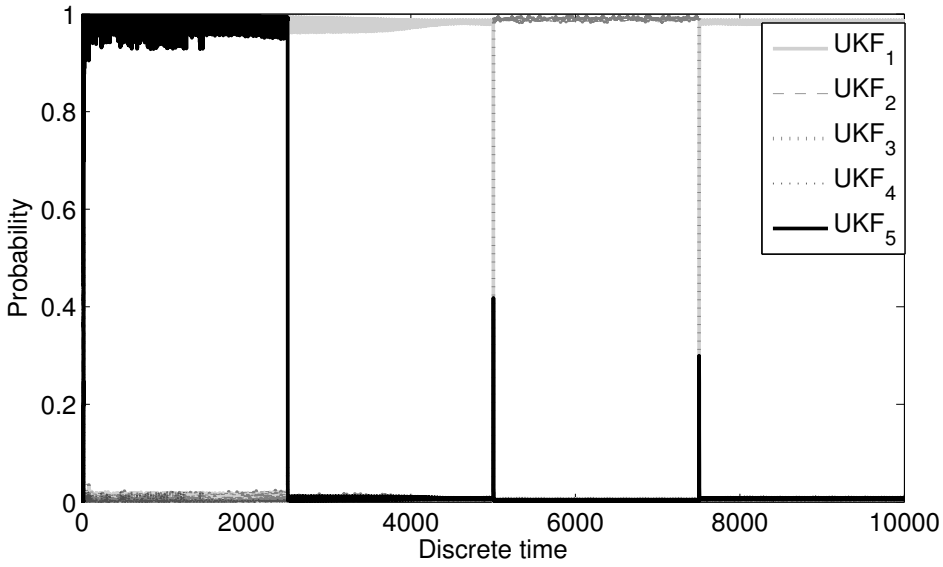

Figure 6. Model probabilities

(SIMO) system, where the input u is the water flow through the pump, while the outputs y_1 and y_2 are water levels in the first and second tank, respectively.

It is assumed that the system being considered can be affected by the following set of faults:

actuator fault: f_1 pump loss-of-effectiveness or leakage from the pump pipe,
process fault: f_2 connecting cylindrical pipe clogged,
sensor faults: f_3 water level sensor fault of the first tank, f_4 water level sensor fault of the second tank.

Once the fault description is provided, then a complete system description can be given as follows:

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{L}_1 \boldsymbol{f}_{a,k} \tag{79}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1} + \boldsymbol{L}_2 \boldsymbol{f}_{s,k+1} \tag{80}$$

where:

$$\boldsymbol{g}\left(\boldsymbol{x}_k\right) = \begin{bmatrix} -h\frac{K_1}{A_1}\sqrt{x_{1,k} - x_{2,k}} + x_{1,k} \\ h\frac{K_1}{A_2}\sqrt{x_{1,k} - x_{2,k}} - h\frac{K_2}{A_2}\sqrt{x_{2,k}} + x_{2,k} \end{bmatrix}, \tag{81}$$

$$\boldsymbol{h}(\boldsymbol{u}_k) = \left[h\frac{1}{A_1}u_k, 0\right]^T, \tag{82}$$

$$\boldsymbol{L}_1 = \begin{bmatrix} -\frac{h}{A_1} & \frac{h}{A_1} \\ 0 & \frac{-h}{A_2} \end{bmatrix} \tag{83}$$

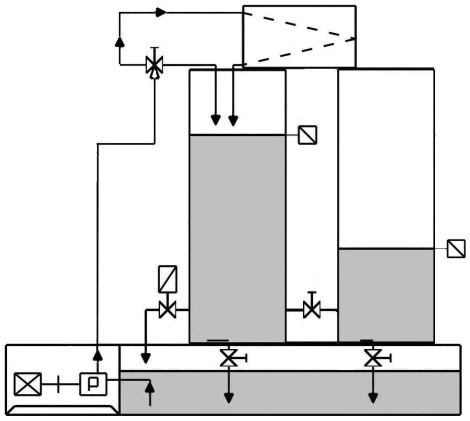

Figure 7. Schematic diagram of a two-tank system

$$\boldsymbol{L}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \boldsymbol{I}, \tag{84}$$

$$\boldsymbol{f}_{a,k} = [f_{1,k}, \sqrt{x_{1,k} - x_{2,k}} f_{2,k}]^T, \quad \boldsymbol{f}_{s,k} = [f_{3,k}, f_{4,k}]^T,$$

where $x_{1,k}$ and $x_{2,k}$ are the water levels in the first and second tank, respectively, A_1, A_2 stand for the cross-sections of the tanks, K_1 denotes the cross-section of the connecting pipe, K_2 is the cross-section of the outflow pipe from the second tank, and h is the sampling time.

The objective of the subsequent part of this section is to design UIF-based diagnostics filter, which will make it possible to detect and isolate the above mentioned faults.

Filter 1: In order to make the residual insensitive to f_1, it is proposed to use the developed UIF with the following settings

$$\boldsymbol{E} = \boldsymbol{L}_1^1, \quad \boldsymbol{d}_k = f_{1,k}, \quad \boldsymbol{L} = \boldsymbol{L}_1^2, \quad \boldsymbol{f}_k = f_{2,k}, \quad \boldsymbol{C}_k = [1,\, 0], \tag{85}$$

where $\boldsymbol{L}_1^i$ stands for ith column of $\boldsymbol{L}_1$. It is straightforward to examine that conditions (42) and (43) are satisfied, which means that the observer will be insensitive to $f_{1,k}$, while it will remain sensitive to $f_{2,k}$.

Filter 2: Similarly as in the *Observer 1* case, the residual generated by the *Observer 2* should be insensitive to $f_{2,k}$,

$$\boldsymbol{E} = \boldsymbol{L}_1^2, \quad \boldsymbol{d}_k = f_{2,k}, \quad \boldsymbol{L} = \boldsymbol{L}_1^1, \quad \boldsymbol{f}_k = f_{1,k}, \quad \boldsymbol{C}_k = [1,\, 0], \tag{86}$$

It is straightforward to examine that conditions (42) and (43) are satisfied, which means that the observer will be insensitive to $f_{2,k}$ while it will remain sensitive to $f_{1,k}$.

Filter 3: The observer should be insensitive to $f_{3,k}$, while it should be sensitive to $f_{4,k}$. This can be realised using the conventional UKF with

$$\boldsymbol{C} = [0,\, 1]. \tag{87}$$

Filter 4: The observer should be insensitive to $f_{4,k}$, while it should be sensitive to $f_{3,k}$. This can be realised using the conventional UKF with

$$\boldsymbol{C} = [1,\ 0]. \tag{88}$$

The main objective of this section is to show the testing results obtained with the proposed approach. To tackle this problem, a Matlab-based simulator of a two tank system was implemented. The simulator is able to generate the data for normal as well as for all faulty conditions $(f_1, \ldots, f_4)$ being considered. The observer-based fault diagnosis scheme was also implemented using Matlab. As a result, a complete scheme that is able to validate the performance of the proposed fault diagnosis strategy was developed. It should be also pointed out that the simulations were carried out using the following numerical parameters: $u_k = 2.56$, $h = 0.1$, $A_1 = 4.2929$, $A_2 = 4.2929$, $K_1 = 0.3646$, $K_2 = 0.2524$.

All fault scenarios where generated according to the following rule:

$$f_{i,k} = \begin{cases} \neq 0 & k = 300 \ldots 400 \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, \ldots, 4.$$

Moreover, y_1 and y_2 were corrupted by the measurement noise generated according to the normal distribution, i.e., $\mathcal{N}(\mathbf{0}, \mathrm{diag}(0.01, 0.01))$. Thus, the following settings of the instrumental matrices were employed: $\boldsymbol{R} = 0.1\boldsymbol{I}$ and $\boldsymbol{Q} = 0.1\boldsymbol{I}$.

Fig. 8 portrays the residual obtained with the four filters. As it can be observed all of them are very close to zero. Figs. 9–12 present the residuals for the faults f_1 to f_4 obtained with the four filters. The obtained results are summarised in the form of a diagnostic table presented in Table 2. It should be

Table 1. Diagnostic table

Filter	f_1	f_2	f_3	f_4
Filter 1	0	1	1	1
Filter 2	1	0	1	1
Filter 3	0	0	0	1
Filter 4	0	0	1	0

noticed that the residuals generated by *Filter 3* and *Filter 4* are insensitive to f_1 and f_2. Such a situation is caused by the fact that observers use a feedback from the system output and hence some damping effects may arise. This is the case in the presented situation. On the other hand, it was observed that the results of experiments can be consistent with the theoretical expectations when there is no measurement noise but this is rather an unreal situation. Irrespective of the presented results, the faults can still be isolated because they have unique signatures.

10. Conclusions

The paper presents a complete design procedure of an unknown input filter for non-linear discrete-time stochastic systems. In particular, a system description is

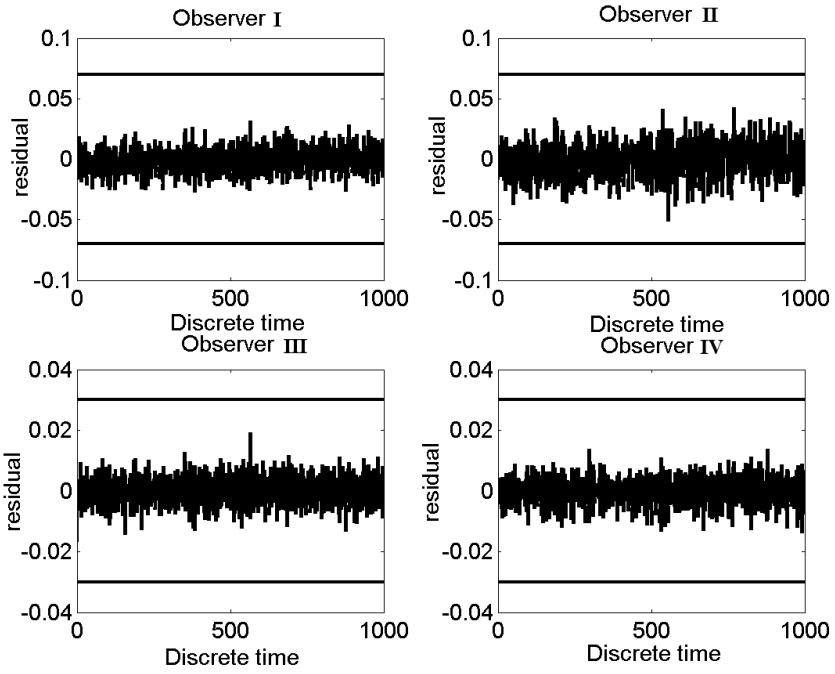

Figure 8. Residuals for fault-free case

provided which covers a large class of non-linear systems and the corresponding unknown input filter is proposed with a constant unknown input distribution matrix. Subsequently, a condition is determined under which the fault will not be decoupled from the residuals. Based on the achieved results, an algorithm for determining the unknown input distribution matrix is proposed. Finally, the interactive multiple model algorithm is used to extend the proposed approach to be applicable for a set of predefined unknown input distribution matrices. The final part of the paper presents comprehensive case studies regarding the practical application of the proposed approaches. These examples are the induction motor and the two-tank systems. In particular, based on an example with an induction motor the strategies of determining unknown input distribution matrix and the case with a set of predefined unknown input distribution matrices were considered. The abilities regarding the fault detection and isolation were illustrated with the two-tank systems. In all the cases, the proposed approaches exhibit their practical usefulness.

The future research direction will be oriented towards application of the proposed approaches to fault-tolerant control schemes.

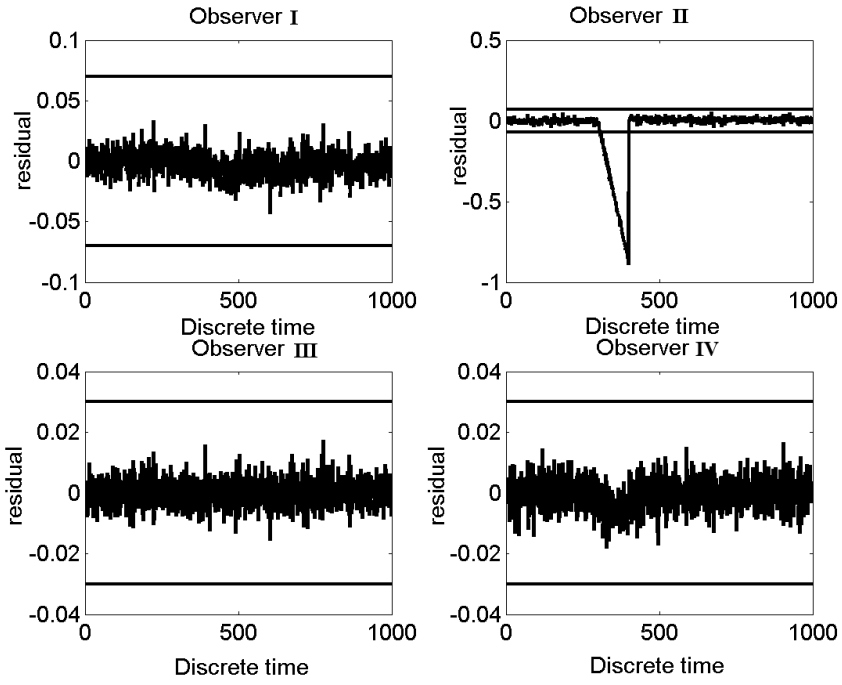

Figure 9. Residuals for fault f_1

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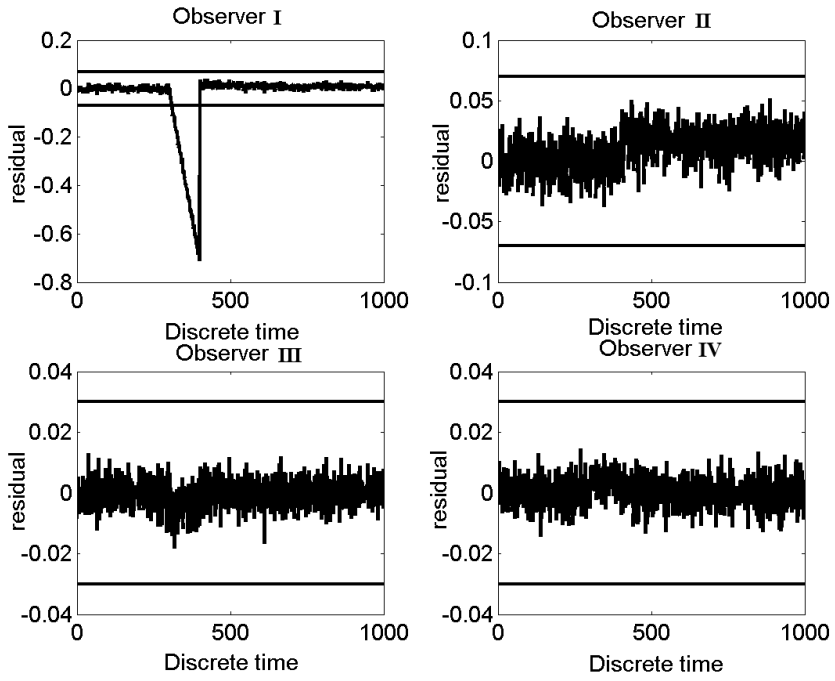

Figure 10. Residuals for fault f_2

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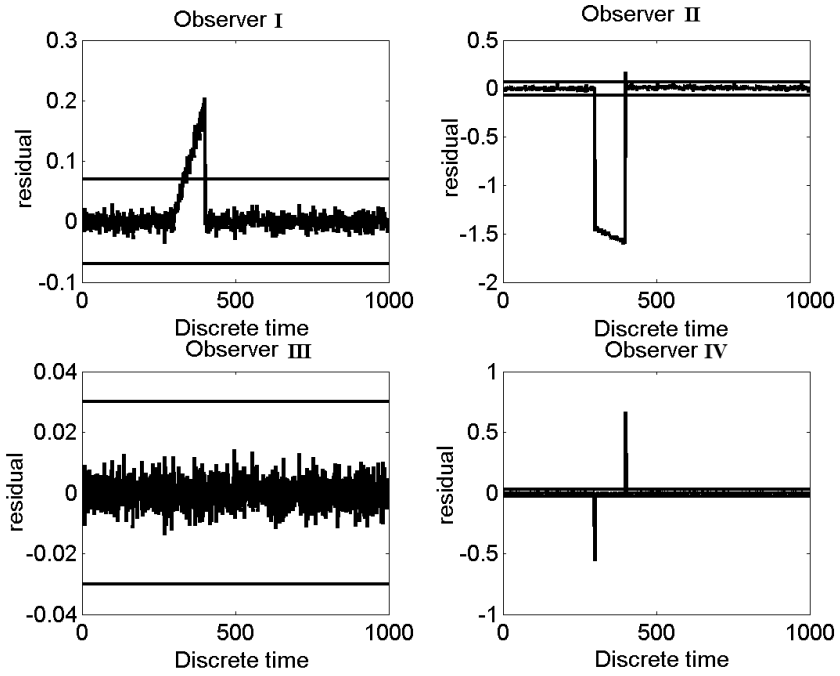

Figure 11. Residuals for fault f_3

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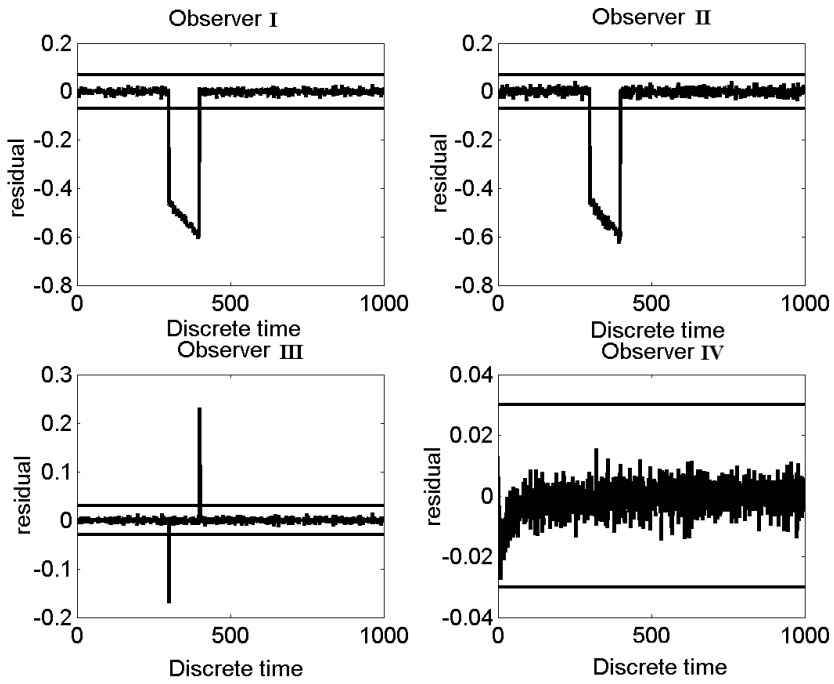

Figure 12. Residuals for fault f_4

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