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# HARMONIC IMPACT ON THE SURFACE OF A HALF-PLANE IN THE FRAMEWORK OF TIME-FRACTIONAL HEAT CONDUCTION

*YURIY POVSTENKO*

Abstract

The time-fractional heat conduction equation with the Caputo derivative is considered in a half-plane. The boundary value of temperature varies harmonically in time. The integral transform technique is used; the solution is obtained in terms of integral with integrand being the Mittag-Leffler functions. The particular case of solution corresponding to the classical heat conduction equation is discussed in details.

## 1. Introduction

Fractional calculus (the theory of integrals and derivatives of non-integer order) still attracts attention of many researchers due to various applications in physics, chemistry, rheology, engineering, geophysics, geology, biology, medicine etc. (see [5], [6], [9]–[12], [17]–[19], [21]–[23] and references therein).

In the present paper, we consider the time-fractional heat conduction equation with the Caputo derivative in a half-plane with the boundary value of temperature being the harmonic function in time. The significant difference between the standard heat conduction equation and the time-fractional heat conduction equation with the Caputo derivative of order $\alpha$ is discussed. The obtained solution develops the results of the previous study [14], [15]. The interested reader is also referred to the book [13], which systematically presents solutions to different initial and boundary value problems for the time-fractional diffusion equation.

## 2. Statement of the problem

The time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad 0 < \alpha \le 2\,, \tag{1}$$

is considered in a half-plane $0 < x < \infty,\ \ -\infty < y < \infty$. In equation (1), $T$ denotes temperature, $a$ is the thermal diffusivity coefficient, $\frac{\partial^\alpha T}{\partial t^\alpha}$ is the Caputo fractional derivative of order $\alpha$ [2], [4], [10]

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \quad n-1 < \alpha < n\,, \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function.

As equation (1) is considered in a bounded domain, the corresponding boundary condition should be imposed. For example, the Dirichlet boundary condition has the following form:

$$x = 0: \quad T = g(y,t). \tag{3}$$

The initial conditions read

$$t = 0: \ \ T = \varphi(x,y), \qquad 0 < \alpha \le 2, \tag{4}$$

$$t = 0: \ \ \frac{\partial T}{\partial t} = \psi(x,y), \quad 1 < \alpha \le 2. \tag{5}$$

Here $g(y,t)$, $\varphi(x,y)$, and $\psi(x,y)$ are the given functions.

Before analysis of the initial-boundary-value problem (1), (3)–(5) we will discuss the significant difference between the time-fractional heat conduction equation (1) and the standard heat conduction equation corresponding to the value $\alpha = 1$:

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{6}$$

## 3. Quasi-steady-state oscillations

In the paper [7] and later on in the book [8], Nowacki considered the parabolic heat conduction equation in different unbounded domains with a source term varying harmonically in time. Similarly, we can study Eq. (6) with the Dirichlet boundary condition

$$x = 0: \quad T = u_0\, \delta(y)\, \mathrm{e}^{i\omega t}\,, \tag{7}$$

where $\delta(y)$ is the Dirac delta function.

According to Nowacki's approach, we investigate Eq. (6) under boundary condition (7) without taking the initial conditions into account. Instead, according to this approach, it is assumed that temperature $T(x,y,t)$ can be presented as

$$T(x,y,t) = U(x,y)\, \mathrm{e}^{i\omega t}. \tag{8}$$

For the auxiliary function $U(x,y)$ we obtain

$$iωU = a\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \tag{9}$$

$$x = 0: \quad U = u_0\,\delta(y)\,. \tag{10}$$

To solve equation (9) under boundary condition (10) the integral transform technique will be used. Recall that the exponential Fourier transform [13], [20]

$$\mathcal{F}\{f(y)\} = \widetilde{f}(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{iy\eta}\,\mathrm{d}y \tag{11}$$

is used in the domain $-\infty < y < \infty$ and has the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\eta)\} = f(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}(\eta)\,\mathrm{e}^{-iy\eta}\,\mathrm{d}\eta. \tag{12}$$

The Fourier transform of the second derivative of a function has the following form

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(y)}{\mathrm{d}y^2}\right\} = -\eta^2\widetilde{f}(\eta). \tag{13}$$

Similarly, for the sin-Fourier transform we have [13], [20]:

$$\mathcal{F}\{f(x)\} = \widehat{f}(\xi) = \int_{0}^{\infty} f(x)\sin(x\xi)\,\mathrm{d}x\,, \tag{14}$$

$$\mathcal{F}^{-1}\{\widehat{f}(\xi)\} = f(x) = \frac{2}{\pi}\int_{0}^{\infty} \widehat{f}(\xi)\sin(x\xi)\,\mathrm{d}\xi. \tag{15}$$

The sin-Fourier transform is used in the domain $0 \le x < \infty$ for the Dirichlet boundary condition with the prescribed boundary value of a function, since for the second derivative of a function we get

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widehat{f}(\xi) + \xi\,f(x)\Big|_{x=0}. \tag{16}$$

Applying to equation (9) with the boundary condition (10) the integral transforms, we arrive at

$$\widehat{\widetilde{U}}(\xi,\eta) = \frac{u_0 a}{\sqrt{2\pi}}\,\frac{\xi}{a\,(\xi^2+\eta^2)+i\omega}\,. \tag{17}$$

To invert the integral transforms the following integrals [16] will be used:

$$\int_0^\infty \frac{x}{x^2+c^2}\,\sin(bx)\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{e}^{-bc}\,, \quad b>0, \quad \Re e\, c>0\,, \tag{18}$$

$$\int_0^\infty \exp\left(-p\sqrt{x^2+c^2}\right)\cos(bx)\,\mathrm{d}x = \frac{pc}{\sqrt{p^2+b^2}}\,K_1\left(c\sqrt{p^2+b^2}\right), \tag{19}$$
$$b>0\,, \quad \Re e\, p>0\,, \quad \Re e\, c>0\,,$$

where $K_1(z)$ is the modified Bessel function.

The solution reads

$$U(x,y) = \frac{u_0 x}{\pi\sqrt{x^2+y^2}}\sqrt{\frac{i\omega}{a}}\,K_1\left(\sqrt{\frac{i\omega}{a}}\sqrt{x^2+y^2}\right) \tag{20}$$

and finally

$$T(x,y,t) = \frac{u_0 x}{\pi\sqrt{x^2+y^2}}\sqrt{\frac{i\omega}{a}}\,K_1\left(\sqrt{\frac{i\omega}{a}}\sqrt{x^2+y^2}\right)\mathrm{e}^{i\omega t}. \tag{21}$$

## 4. The time-fractional heat conduction equation

It should be emphasized that Nowacki's approach is based on the well-known formula for integer order derivatives of exponential function

$$\frac{\mathrm{d}^n \mathrm{e}^{\lambda t}}{\mathrm{d}t^n} = \lambda^n\,\mathrm{e}^{\lambda t}. \tag{22}$$

For Caputo derivative of fractional (non-integer) order of exponential function we have [14], [15]

$$\frac{\mathrm{d}^\alpha \mathrm{e}^{\lambda t}}{\mathrm{d}t^\alpha} = \lambda^\alpha\,\mathrm{e}^{\lambda t}\,\frac{\gamma(n-\alpha,\lambda t)}{\Gamma(n-\alpha)}, \quad n-1<\alpha<n, \tag{23}$$

where $\gamma(a,x)$ is the incomplete gamma function [1]

$$\gamma(a,x) = \int_0^x \mathrm{e}^{-t}\,t^{a-1}\,\mathrm{d}t\,.$$

Hence, for fractional values of the order of derivative $\alpha$ formula (22) does not fulfill:

$$\frac{\mathrm{d}^\alpha \mathrm{e}^{\lambda t}}{\mathrm{d}t^\alpha} \neq \lambda^\alpha\,\mathrm{e}^{\lambda t}. \tag{24}$$

Therefore, assumption (8) cannot be used for the time-fractional heat conduction equation and the initial conditions should be taken into account.

In what follows, Eq. (1) will be considered under the boundary condition (7) and zero initial conditions

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{25}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2. \tag{26}$$

To solve this initial-boundary-value problem the integral transforms technique will be used. Recall that the Caputo fractional derivative for its Laplace transform rule requires the knowledge of the initial values of the function and its integer derivatives of order $k = 1, 2, \ldots, n-1$ [2], [4], [10]:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n, \tag{27}$$

where the asterisk denotes the transform, $s$ is the Laplace transform variable.

The sin-Fourier transform with respect to the spatial coordinate $x$, the exponential Fourier transform with respect to the spatial coordinate $y$, and the Laplace transform with respect to time $t$ give:

$$\widehat{\widehat{T}}^*(\xi, \eta, s) = \frac{u_0 a}{\sqrt{2\pi}} \frac{\xi}{s^\alpha + a\left(\xi^2 + \eta^2\right)} \frac{1}{s - i\omega}. \tag{28}$$

First of all, we will analyze the classical heat conduction equation ($\alpha = 1$). In this case

$$\begin{aligned} \widehat{\widehat{T}}^*(\xi, \eta, s) &= \frac{u_0 a}{\sqrt{2\pi}} \frac{\xi}{s + a\left(\xi^2 + \eta^2\right)} \frac{1}{s - i\omega} \\ &= \frac{u_0 a}{\sqrt{2\pi}} \frac{\xi}{a\left(\xi^2 + \eta^2\right) + i\omega} \frac{1}{s - i\omega} \\ &\quad - \frac{u_0 a}{\sqrt{2\pi}} \frac{\xi}{a\left(\xi^2 + \eta^2\right) + i\omega} \frac{1}{s + a\left(\xi^2 + \eta^2\right)} \end{aligned} \tag{29}$$

and

$$\begin{aligned} T(x, y, t) &= \frac{u_0 x}{\pi\sqrt{x^2 + y^2}} \sqrt{\frac{i\omega}{a}}\, K_1\left(\sqrt{\frac{i\omega}{a}}\sqrt{x^2 + y^2}\right) \mathrm{e}^{i\omega t} \\ &\quad - \frac{u_0 a}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \mathrm{e}^{-a\left(\xi^2 + \eta^2\right)t} \frac{\xi}{a\left(\xi^2 + \eta^2\right) + i\omega} \sin(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\,. \end{aligned} \tag{30}$$

The first term in (30) coincides with the solution (21) and represents the steady state oscillations, whereas the second term tends to zero for large values of time and describes the transient process.

Now we return to solving the time-fractional heat conduction equation. To invert the Laplace transform in Eq. (28), the following formula (see [2], [3], [4], [10])

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\}=t^{\beta-1}\,E_{\alpha,\beta}(-bt^\alpha) \tag{31}$$

is used. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function in two parameters $\alpha$ and $\beta$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\,,\qquad \alpha>0\,,\quad \beta>0\,,\quad z\in C\,. \tag{32}$$

Inversion of all the integral transforms with taking the convolution theorem into account result in the sought-for solution

$$\begin{aligned}T(x,y,t)=\frac{au_0}{\pi^2}\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\tau^{\alpha-1}E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)\tau^\alpha\right]\\ \times\,\mathrm{e}^{i\omega(t-\tau)}\cos(y\eta)\,\sin(x\xi)\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\tau\,.\end{aligned} \tag{33}$$

Now we pass to polar coordinates in the $(\xi,\eta)-$plane:

$$\xi=\rho\cos\theta,\quad \eta=\rho\sin\theta\,. \tag{34}$$

In this case equation (33) can be rewritten as

$$\begin{aligned}T(x,y,t)=\frac{2au_0}{\pi^2}\int_0^t\int_0^{\infty}\rho^2\,\tau^{\alpha-1}\mathrm{e}^{i\omega(t-\tau)}E_{\alpha,\alpha}\left(-a\rho^2\tau^\alpha\right)\\ \times\int_0^{\pi/2}\cos(y\rho\sin\theta)\,\sin(x\rho\cos\theta)\,\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\tau\,.\end{aligned} \tag{35}$$

Substitution of $v=\sin\theta$ with taking into account that [16]

$$\int_0^1\sin\left(p\sqrt{1-x^2}\right)\cos\left(qx\right)\,\mathrm{d}x=\frac{\pi}{2}\,\frac{p}{\sqrt{p^2+q^2}}\,J_1\left(\sqrt{p^2+q^2}\right), \tag{36}$$

where $J_1(z)$ is the Bessel function of the first kind, gives

$$\begin{aligned}T(x,y,t)=\frac{au_0}{\pi}\,\cos\varphi\int_0^t\int_0^{\infty}\rho^2\,E_{\alpha,\alpha}\left(-a\rho^2\tau^\alpha\right)\\ \times\,J_1(r\rho)\,\tau^{\alpha-1}\mathrm{e}^{i\omega(t-\tau)}\,\mathrm{d}\rho\,\mathrm{d}\tau\,.\end{aligned} \tag{37}$$

The obtained solution can be useful not only in the case of harmonic surface impact but also for a function $g(y,t)$ periodic in time. Expanding the boundary function in the Fourier series, the solution can be obtained as the result of superposition of successive harmonic terms.

## References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover, New York, 1972.

[2] R. Gorenflo, F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, In: A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 223–276. Springer, Wien, 1997.

[3] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*. Springer, Berlin, 2014.

[4] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[5] R. L. Magin, *Fractional Calculus in Bioengineering*. Begell House Publishers, Connecticut, 2006.

[6] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Imperial College Press, London, 2010.

[7] W. Nowacki, *State of stress in an elastic space due to a source of heat varying harmonically as function of time*. Bull. Acad. Polon. Sci. Sér. Sci. Techn. **5** (1957), 145–154.

[8] W. Nowacki, *Thermoelasticity*, 2nd edn. PWN-Polish Scientific Publishers, Warsaw and Pergamon Press, Oxford, 1986.

[9] K. B. Oldham, J. Spanier, *The Fractional Calculus*. Academic Press, New York, 1974.

[10] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[11] Y. Povstenko, *Fractional heat conduction equation and associated thermal stresses*. J. Thermal Stresses **28** (2005), 83–102.

[12] Y. Povstenko, *Fractional Thermoelasticity*. Springer, New York, 2015.

[13] Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*. Birkhäser, New York, 2015.

[14] Y. Povstenko, *Harmonic impact in the plane problem of fractional thermoelasticity*, In: 11th International Congress on Thermal Stresses, 5-9 June 2016, Salerno, Italy.

[15] Y. Povstenko, *Fractional heat conduction in a space with a source varying harmonically in time and associated thermal stresses*. J. Thermal Stresses, 2016; doi: 10.1080/01495739.2016.1209991.

[16] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and Series, Vol. 1: Elementary Functions*. Gordon and Breach Science Publishers, Amsterdam, 1986.

[17] Yu. N. Rabotnov, *Creep Problems in Structural Members*, North-Holland Publishing Company, Amsterdam, The Netherlands, 1969.

[18] Yu. A. Rossikhin, M. V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*. Appl. Mech. Rev. **50** (1997), 15–67.

[19] Yu. A. Rossikhin, M. V. Shitikova, *Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results*. Appl. Mech. Rev. **63** (2010), 010801, 52 pp.

[20] I. N. Sneddon, *The Use of Integral Transforms*. McGraw-Hill, New York, 1972.

[21] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers*. Springer, Berlin, 2013.

[22] B. J. West, M. Bologna, P. Grigolini, *Physics of Fractals Operators*. Springer, New York, 2003.

[23] G. M. Zaslavsky,. *Chaos, fractional kinetics, and anomalous transport*. Phys. Rep. **371** (2002), 461–580.

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*Yuriy Povstenko*
Jan Długosz University in Częstochowa,
Faculty of Mathematics and Natural Sciences,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
*E-mail address*: j.povstenko@ajd.czest.pl