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# ON SOME PROPERTIES OF $(H,E)$-IMPLICATIONS. DISTRIBUTIVITIES WITH T-NORMS AND T-CONORMS.

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**Abstract**

In this paper some basic properties of $(h,e)$-implications are studied. This kind of implications has been recently introduced (see [29]). They are implications generated from an additive generator of a representable uninorm in a similar way of Yager's $f$- and $g$-implications which are generated from additive generators of continuous Archimedean t-norms and t-conorms, respectively. In addition, they satisfy a classical property of some types of implications derived from uninorms that is $I(e,y) = y$ for all $y \in [0,1]$. Moreover they are examples of fuzzy implications satisfying the exchange principle but not the law of importation for any t-norm, in fact for any function $F : [0,1]^2 \to [0,1]$. On the other hand, the distributivities with conjunctions and disjunctions (t-norms and t-conorms) are also studied leading to new solutions of the corresponding functional equations. Finally, it is proved that they do not intersect with any of the most used classes of implications.

## 1 Introduction

Fuzzy implications have become a keystone in fuzzy control and approximate reasoning, as well as in many fields where these theories apply. Although they were introduced to model fuzzy conditionals, they also manage backward and forward inferences in any fuzzy rules based system (see for instance [17], [19] or [27]). Moreover, fuzzy implications are also useful in fuzzy relational equations and fuzzy mathematical morphology ([27]), fuzzy DI-subsethood measures and image processing ([9] and [10]) and data mining ([36]) among many other fields. Consequently, many authors have focused their interest in the theoretical study of fuzzy implications. See for instance, the surveys [27] and [4], and also the recent book [3], exclusively devoted to fuzzy implications.

Fuzzy implications have been commonly obtained by combinations of some kinds of aggregation functions. Not only t-norms and t-conorms, but also copulas, quasi-copulas and even conjunctors in general ([14]), representable aggregation functions ([11]), and mainly uninorms ([1], [5], [13], [24], [32], [31]). However, there exists a different approach in order to obtain fuzzy implications based on the direct use of additive generating functions. In this way, Yager's $f$- and $g$-generated fuzzy implications ([35]) can be seen as implications generated from additive generators of continuous Archimedean t-norms or t-conorms, respectively. Another attempt was the use of multiplicative generators of t-conorms to propose a new class of fuzzy implications, called the $h$-generated implications ([7]). This family of implications is contained in the family of all $(S,N)$-implications obtained from continuous negations (see [2]).

In this work, we propose a new method to obtain fuzzy implications through the use of generators of representable uninorms, i.e., from continuous and strictly increasing functions $h : [0,1] \to [-\infty,\infty]$ with $h(0) = -\infty$, $h(1) = \infty$ and $h(e) = 0$ for some fixed $e \in (0,1)$. This new class of implications, called $(h,e)$-implications, satisfy a common property among the types of implications derived from uninorms, that is, $I(e,y) = y$ for all $y \in [0,1]$.

Another interesting property satisfied by these implications is that they are a whole class of implications that satisfy the exchange principle but not the law of importation for any t-norm $T$. Other properties are studied in detail, specially the distributivity. The distributivity properties with t-norms and t-conorms have been used in avoiding combinatory rule explosion in fuzzy systems ([12]), and for this reason many authors have dealt with them (see [32], [33], [34]). Finally, the intersections with the most known classes of fuzzy implications are established. In fact, it is proved that $(h,e)$-implications are a new class of implications literally since they do not intersect with any other known class.

The communication is organized as follows. In the next section we recall the basic definitions and properties of implications needed in the subsequent sections. In Section 3, we present the definition of $(h,e)$-implications and we discuss then, the properties satisfied by this new class of fuzzy implications. Section 4 is completely devoted to the study of the distributivity properties, and Section 5 deals with the intersections of these implications with the other families. The paper ends with some conclusions and future work.

## 2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms and t-conorms (all necessary results and notations can be found in [23]). To make this work self-contained, we recall here some of the concepts and results employed in the rest of the paper.

### 2.1 Fuzzy Negations

**Definition 1.** (see Definition 1.1 in [15] and Definition 11.3 in [23]) A decreasing function $N : [0,1] \to [0,1]$ is called a *fuzzy negation*, if $N(0) = 1$, $N(1) = 0$. A fuzzy negation $N$ is called

(i) *strict*, if it is strictly decreasing and continuous.

(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

### 2.2 Fuzzy Implications

**Definition 2.** (see Definition 1.15 in [15], Definition 1.1.1 in [3]) A binary operator $I : [0,1]^2 \to [0,1]$ is said to be an *implication function*, or an *implication*, if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition. We will denote by $\mathcal{FI}$ the set of all implications. Special interesting properties for implication functions are:

– The *exchange principle*,

$$I(x,I(y,z)) = I(y,I(x,z)), \quad \text{for all } x,y,z \in [0,1]. \quad (\mathbf{EP})$$

– The *law of importation* with a t-norm $T$,

$$I(T(x,y),z) = I(x,I(y,z)), \quad \text{for all } x,y,z \in [0,1]. \quad (\mathbf{LI})$$

– The *weak law of importation* with a conjunctive, commutative and nondecreasing function $F : [0,1]^2 \to [0,1]$, (see [28])

$$I(F(x,y),z) = I(x,I(y,z)), \quad \text{for all } x,y,z \in [0,1]. \quad (\mathbf{WLI})$$

– The *left neutrality principle*,

$$I(1,y) = y, \quad y \in [0,1]. \quad (\mathbf{NP})$$

– The *ordering property*,

$$x \leq y \Longleftrightarrow I(x,y) = 1, \quad \text{for all } x,y \in [0,1]. \quad (\mathbf{OP})$$

– The *identity principle*,

$$I(x,x) = 1, \quad x \in [0,1]. \quad (\mathbf{IP})$$

– The *contrapositive symmetry* with respect to a fuzzy negation $N$,

$$I(x,y) = I(N(y),N(x)), \quad \text{for all } x,y \in [0,1]. \quad (\mathbf{CP(N)})$$

**Definition 3.** (see Definition 1.14.15 in [3]) Let $I$ be a fuzzy implication. The function $N_I$ defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the *natural negation* associated to $I$.

The most usual kinds of implications are:

– $(S,N)$-implications derived from a t-conorm $S$ and a fuzzy negation $N$ by $I_{S,N}(x,y) = S(N(x),y)$ for all $x,y \in [0,1]$. If $N$ is strong, they are called $S$-implications.

– $R$-implications derived from a t-norm $T$ by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$$

for all $x,y \in [0,1]$.

– $QL$-operations derived from a t-conorm $S$, a t-norm $T$ and a fuzzy negation $N$ by $I_{QL}(x,y) = S(N(x),T(x,y))$ for all $x,y \in [0,1]$.

– $D$-operations derived from a t-conorm $S$, a t-norm $T$ and a fuzzy negation $N$ by $I_D(x,y) = S(T(N(x),N(y)),y)$ for all $x,y \in [0,1]$.

These models of fuzzy implications have their counterpart for uninorms (see [5], [13], [24], [31]) where conjunctive and disjunctive uninorms play the role of t-norms and t-conorms respectively. Thus $(U,N)$, $RU$, $QLU$ and $DU$-implications are defined. Next, the definitions of Yager's classes of fuzzy implications are given.

**Definition 4.** ([35], [3]) Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)), \quad x,y \in [0,1]$$

with the understanding $0 \cdot \infty = 0$, is called an *$f$-generated implication*. The function $f$ is called an $f$-generator of the function $I$. In this case, we will write $I_f$ instead of $I$.

**Definition 5.** ([35], [3]) Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x,y \in [0,1]$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *$g$-generated implication*, where the function $g^{(-1)}$ is the pseudo-inverse of $g$ given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0,g(1)], \\ 1 & \text{if } x \in [g(1),\infty]. \end{cases}$$

The function $g$ is called a $g$-generator of the function $I$. In this case, we will write $I_g$ instead of $I$.

Finally, it follows the definition of $h$-generated implications. Note that this class of implications, unlike $(h,e)$-implications presented in this paper, is based on the use of multiplicative generators of t-conorms.

**Definition 6.** ([7]) If $h : [0,1] \to [0,1]$ is a strictly decreasing and continuous function with $h(0) = 1$, then the function $I : [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = h^{(-1)}(x \cdot h(y)), \ \ x,y \in [0,1],$$

is a fuzzy implication, where $h^{(-1)}$ is the pseudo-inverse of $h$ given by

$$h^{(-1)}(x) = \begin{cases} h^{-1}(x) & \text{if } x \in [h(1),1], \\ 1 & \text{if } x \in [0,h(1)]. \end{cases}$$

## 3 $(h,e)$-Implications

Yager's implications are obtained through the use of additive generators of t-norms for $f$-generated implications and additive generators of t-conorms for $g$-generated implications. In [16], representable uninorms were introduced by analogy to the representation theorems for continuous Archimedean t-norms and t-conorms. This class of uninorms is defined by means of a continuous and strictly increasing function $h : [0,1] \to [-\infty,\infty]$ such that $h(0) = -\infty$, $h(e) = 0$ for an $e \in (0,1)$ and $h(1) = \infty$ which is unique up to a positive multiplicative constant. From this function $h$, several new classes of implications can be obtained in an analogous way to Yager's. In [29], $h$-implications and some modifications and generalizations were introduced as new classes of fuzzy implications generated from additive generators of representable uninorms.

From these new classes of implications, the class of $(h,e)$-implications is the only one that satisfies (NP$_e$), that is $I(e,y) = y$ for all $y \in [0,1]$ for some fixed $e \in (0,1)$. Among the classes of implications derived from uninorms, this property is quite interesting since it can be seen as the counterpart of (NP) for these classes of implications since $e$ plays the role of 1 as the neutral element of the uninorm. In this way, $RU$-implications, $e$-implications and pseudo-$e$-implications (see [20], [21]) satisfy this property. Moreover, the two latter classes include this property as an axiom in their definitions.

Therefore, $(h,e)$-implications deserve a deep study. In [29], only some initial properties of these implications were listed and few of them proved. So, for the sake of completeness, we recall here the definition and these basic properties including the proofs that were omitted in [29].

**Definition 7.** Let $h:[0,1]\to[-\infty,\infty]$ be a strictly increasing and continuous function with $h(0)=-\infty$, $h(e)=0$ for an $e\in(0,1)$ and $h(1)=+\infty$. The function $I:[0,1]^2\to[0,1]$ defined by

$$I(x,y)=\begin{cases}1 & \text{if } x=0,\\ h^{-1}\left(\frac{x}{e}\cdot h(y)\right) & \text{if } x>0 \text{ and } y\le e,\\ h^{-1}\left(\frac{e}{x}\cdot h(y)\right) & \text{if } x>0 \text{ and } y>e,\end{cases}$$

is called an $(h,e)$-implication. The function $h$ is called an $h$-generator of the function $I$. We will write in this case $I^{h,e}$ instead of $I$.

Note now that they are always fuzzy implications.

**Proposition 8.** If $h$ is an $h$-generator with respect to a fixed $e\in(0,1)$, then $I^{h,e}\in\mathcal{FI}$.

*Proof.* First of all, recall that $h$ is strictly increasing and consequently also $h^{-1}$. This fact is useful to prove (I1) and (I2).

- Consider $x_1\le x_2$. If $x_1=0$, $I^{h,e}(0,y)=1\ge I^{h,e}(x_2,y)$ for all $y\in[0,1]$. Otherwise,
  - If $y\le e$, then $h(y)\le 0$ and $\frac{x_1}{e}\cdot h(y)\ge\frac{x_2}{e}\cdot h(y)$. Consequently,
  $$\begin{aligned}I^{h,e}(x_1,y) &= h^{-1}\left(\tfrac{x_1}{e}\cdot h(y)\right)\\ &\ge h^{-1}\left(\tfrac{x_2}{e}\cdot h(y)\right)\\ &= I^{h,e}(x_2,y).\end{aligned}$$
  - If $y>e$, then $h(y)>0$ and $\frac{e}{x_1}\cdot h(y)\ge\frac{e}{x_2}\cdot h(y)$. Consequently,
  $$\begin{aligned}I^{h,e}(x_1,y) &= h^{-1}\left(\frac{e}{x_1}\cdot h(y)\right)\\ &\ge h^{-1}\left(\frac{e}{x_2}\cdot h(y)\right)\\ &= I^{h,e}(x_2,y).\end{aligned}$$

  I.e., $I^{h,e}$ satisfies (I1).
- Consider $y_1\le y_2$. If $x=0$, $I^{h,e}(0,y_1)=1=I^{h,e}(0,y_2)$. Otherwise,
  - If $y_1\le y_2\le e$, then $h(y_1)\le h(y_2)\le 0$ and
  $$\begin{aligned}I^{h,e}(x,y_1) &= h^{-1}\left(\tfrac{x}{e}\cdot h(y_1)\right)\\ &\le h^{-1}\left(\tfrac{x}{e}\cdot h(y_2)\right)=I^{h,e}(x,y_2).\end{aligned}$$
  - If $y_1\le e<y_2$, then $h(y_1)\le 0<h(y_2)$ and
  $$\begin{aligned}I^{h,e}(x,y_1) &= h^{-1}\left(\tfrac{x}{e}\cdot h(y_1)\right)\le h^{-1}(0)\\ &= e<h^{-1}\left(\tfrac{e}{x}\cdot h(y_2)\right)\\ &= I^{h,e}(x,y_2).\end{aligned}$$
  - If $e<y_1\le y_2$, then $h(y_2)\ge h(y_1)>0$ and
  $$\begin{aligned}I^{h,e}(x,y_1) &= h^{-1}\left(\tfrac{e}{x}\cdot h(y_1)\right)\\ &\le h^{-1}\left(\tfrac{e}{x}\cdot h(y_2)\right)=I^{h,e}(x,y_2).\end{aligned}$$

  I.e., $I^{h,e}$ satisfies (I2).

Finally, it satisfies the boundary conditions.

- $I^{h,e}(0,0)=1$ by construction.
- $I^{h,e}(1,1)=h^{-1}\left(\frac{e}{1}\cdot h(1)\right)=h^{-1}(\infty)=1$.
- $I^{h,e}(1,0)=h^{-1}\left(\frac{1}{e}\cdot h(0)\right)=h^{-1}(-\infty)=0$.

*Example 9.* From [16], an $h$-generator can be seen as the generator of a representable uninorm. Let us show the $(h,e)$-implications derived from the most usual generators of representable uninorms. The plot of these implications can be viewed in Figure 1 and Figure 2 .

(i) If we take the $h$-generator $h_1(x)=ln\left(\frac{x}{1-x}\right)$, which is the additive generator of the following conjunctive uninorm

$$U^c_{h_1}(x,y)=\begin{cases}0 & \text{if } (x,y)\in\{(0,1),(1,0)\},\\ \frac{xy}{(1-x)(1-y)+xy} & \text{otherwise,}\end{cases}$$

then we obtain the following implication

$$I^{h_1,e}(x,y)=\begin{cases}1 & \text{if } x=0,\\ \frac{y^{2x}}{(1-y)^{2x}+y^{2x}} & \text{if } x>0 \text{ and } y\le\frac{1}{2},\\ \frac{y^{\frac{1}{2x}}}{(1-y)^{\frac{1}{2x}}+y^{\frac{1}{2x}}} & \text{if } x>0 \text{ and } y>\frac{1}{2}.\end{cases}$$

(ii) Let us consider the $h$-generator $h_2(x)=ln\left(-\frac{1}{\beta}ln(1-x)\right)$ with $\beta>0$. This function generates the following disjunctive representable uninorm

$$U^d_{h_2}(x,y)=\begin{cases}1 & \text{if } (x,y)\in\{(0,1),(1,0)\},\\ 1-exp\left(-\frac{1}{\beta}ln(1-x)ln(1-y)\right) & \text{otherwise,}\end{cases}$$

with neutral element $e = 1 - exp(-\beta)$. From this $h$-generator, we obtain the following implication

$$I^{h_2,e}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ 1 - exp\left(-\beta\left(-\frac{1}{\beta}ln(1-y)\right)^{\frac{x}{1-exp(-\beta)}}\right) & \\ \quad \text{if } x > 0 \text{ and } y \leq 1 - exp(-\beta), & \\ 1 - exp\left(-\beta\left(-\frac{1}{\beta}ln(1-y)\right)^{\frac{1-exp(-\beta)}{x}}\right) & \\ \quad \text{if } x > 0 \text{ and } y > 1 - exp(-\beta). & \end{cases}$$

(iii) If we take the $h$-generator $h_3(x) = \frac{x-e}{x-x^2}$ for an $e \in (0,1)$, which is a family of additive generators of uninorms with neutral element $e$, we obtain the following family of implications $I^{h_3,e}(x,y) =$

$$= \begin{cases} 1 & \text{if } x = 0, \\ \frac{\sqrt{(xy-ex)^2+(ey-ey^2)\cdot(xy-ex)\cdot(4e-2)+(ey-ey^2)^2}}{2\cdot(xy-ex)} + & \\ + \frac{ey-ey^2-xy+ex}{-2\cdot(xy-ex)} & \text{if } x > 0 \text{ and } y \leq e, \\ \frac{\sqrt{(ey-e^2)^2+(ey-e^2)\cdot(xy-xy^2)\cdot(4e-2)+(xy-xy^2)^2}}{2\cdot(ey-e^2)} + & \\ + \frac{xy-xy^2-ey+e^2}{-2\cdot(ey-e^2)} & \text{if } x > 0 \text{ and } y > e. \end{cases}$$

Next, we prove that the $h$-generator of an $(h,e)$-implication is always unique up to a positive multiplicative constant piecewise.

***Theorem 10.*** Let $h_1, h_2 : [0,1] \to [-\infty, \infty]$ be any two $h$-generators with respect to a fixed $e \in (0,1)$. Then the following statements are equivalent:

(i) $I^{h_1,e} = I^{h_2,e}$.

(ii) There exist constants $k, c \in (0,\infty)$ such that $h_2(x) = k \cdot h_1(x)$ if $x \in [0,e)$ and $h_2(x) = c \cdot h_1(x)$ if $x \in [e,1]$.

Moreover, in this case constants $k$ and $c$ are respectively given by

$$c = h_2 \circ h_1^{-1}(1) \text{ and } k = -h_2 \circ h_1^{-1}(-1).$$

*Proof.* $(i) \Rightarrow (ii)$: Let $h_1, h_2$ be two $h$-generators of an $(h,e)$-implication with $I^{h_1,e}(x,y) = I^{h_2,e}(x,y)$ for all $x, y \in [0,1]$. For any $x \in (0,1) \setminus \{e\}$, we have that if $y \leq e$,

$$\begin{aligned} h_1^{-1}\left(\tfrac{x}{e} \cdot h_1(y)\right) &= h_2^{-1}\left(\tfrac{x}{e} \cdot h_2(y)\right) \\ \Leftrightarrow h_2 \circ h_1^{-1}\left(\tfrac{x}{e} \cdot h_1(y)\right) &= \tfrac{x}{e} \cdot h_2 \circ h_1^{-1}(h_1(y)), \end{aligned}$$

and if $y > e$, we obtain

$$\begin{aligned} h_1^{-1}\left(\tfrac{e}{x} \cdot h_1(y)\right) &= h_2^{-1}\left(\tfrac{e}{x} \cdot h_2(y)\right) \\ \Leftrightarrow h_2 \circ h_1^{-1}\left(\tfrac{e}{x} \cdot h_1(y)\right) &= \tfrac{e}{x} \cdot h_2 \circ h_1^{-1}(h_1(y)). \end{aligned}$$

By the substitution $f = h_2 \circ h_1^{-1}$ and $z = h_1(y)$ for any $y \in [0,1]$, we obtain

$$f\left(\frac{x}{e} \cdot z\right) = \frac{x}{e} \cdot f(z) \text{ for } x \in (0,1) \setminus \{e\} \text{ and } z \in [-\infty, 0], \quad (1)$$

$$f\left(\frac{e}{x} \cdot z\right) = \frac{e}{x} \cdot f(z) \text{ for } x \in (0,1) \setminus \{e\} \text{ and } z \in (0, \infty], \quad (2)$$

where $f : [-\infty,\infty] \to [-\infty,\infty]$ is a continuous strictly increasing bijection with $f(0) = 0$. Taking $z = -1$ in (1), we have $f\left(-\frac{x}{e}\right) = \frac{x}{e} \cdot f(-1)$ for any $x \in (0,1) \setminus \{e\}$. Fix arbitrarily $z \in (-\infty, 0)$. There exists $x \in (0,1) \setminus \{e\}$ such that $x \cdot z \in (-1,0) \setminus \{-e\}$. Then we get from (1) that

$$\begin{aligned} f(z) &= \tfrac{e}{x} \cdot f\left(\tfrac{x}{e} \cdot z\right) = \tfrac{e}{x} \cdot \left(-\tfrac{x}{e} \cdot z\right) \cdot f(-1) \\ &= z \cdot (-f(-1)). \end{aligned}$$

Now we have

$$h_2 \circ h_1^{-1}(z') = z' \cdot (-h_2 \circ h_1^{-1}(-1)).$$

So

$$h_2(x) = h_1(x) \cdot (-h_2 \circ h_1^{-1}(-1)),$$

and denoting $k = -h_2 \circ h_1^{-1}(-1) > 0$, then $h_2(x) = k \cdot h_1(x)$ for $x \in (0,e)$. Note that for $x = 0, e$ we also have $h_2(x) = k \cdot h_1(x)$ since $h_1(0) = h_2(0) = -\infty$ and $h_1(e) = h_2(e) = 0$. So, this is true for $x \in [0,e]$.
On the other hand, substituting $z = 1$ in (2), we have $f\left(\frac{e}{x}\right) = \frac{e}{x} \cdot f(1)$ for any $x \in (0,1) \setminus \{e\}$. Fix arbitrarily $z \in (0,\infty)$. There exists $x \in (0,1) \setminus \{e\}$ such that $\frac{x}{z} \in (0,1) \setminus \{e\}$. Then we get from (2) that

$$f(z) = \frac{x}{e} \cdot f\left(\frac{e}{x} \cdot z\right) = \frac{x}{e} \cdot \frac{e}{x} \cdot z \cdot f(1) = z \cdot f(1).$$

Now we have

$$h_2 \circ h_1^{-1}(z') = z' \cdot (h_2 \circ h_1^{-1}(1)).$$

Consequently

$$h_2(x) = h_1(x) \cdot (h_2 \circ h_1^{-1}(1)),$$

and denoting $c = h_2 \circ h_1^{-1}(1) > 0$, then $h_2(x) = c \cdot h_1(x)$ for $x \in (e,1)$. Finally, it is clear that for $x = 1$ we aso obtain $h_2(1) = c \cdot h_1(1) = \infty$ and so, this is true for $x \in (e,1]$.

$(ii) \Rightarrow (i)$: Let $h_1$ be an $h$-generator and $k, c \in (0,\infty)$. Define $h_2$ as follows

$$h_2(x) = \begin{cases} k \cdot h_1(x) & \text{if } x \in [0,e], \\ c \cdot h_1(x) & \text{if } x \in (e,1]. \end{cases}$$

Evidently, $h_2$ is a well-defined $h$-generator. Moreover, for any $z \in [-\infty,\infty]$,

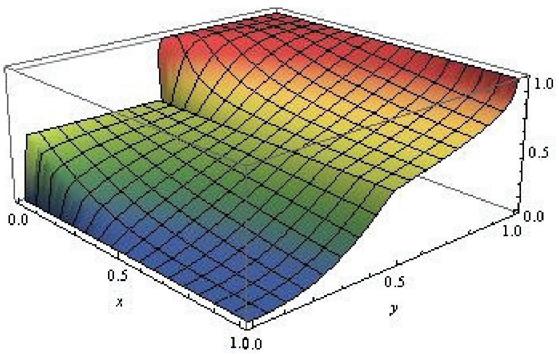
$$h_2^{-1}(z) = \begin{cases} h_1^{-1}\left(\frac{z}{k}\right) & \text{if } z \in [-\infty,0], \\ h_1^{-1}\left(\frac{z}{c}\right) & \text{if } z \in (0,\infty]. \end{cases}$$

Let us prove $I^{h_2,e}(x,y) = I^{h_1,e}(x,y)$. First, $I^{h_2,e}(0,y) = 1 = I^{h_1,e}(x,y)$ for all $y \in [0,1]$. Now, if $x > 0$ and $y \le e$,

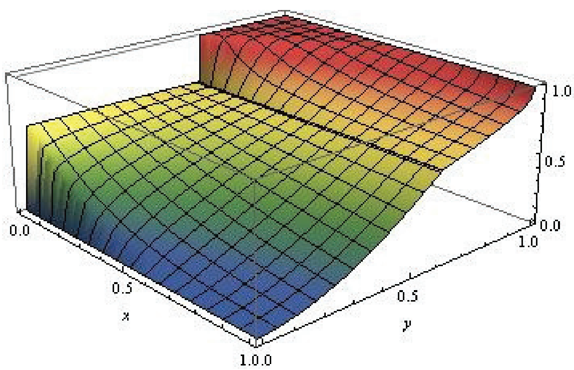
$$\begin{aligned} I^{h_2,e}(x,y) &= h_2^{-1}\left(\tfrac{x}{e} \cdot h_2(y)\right) = h_2^{-1}\left(\tfrac{x}{e} \cdot k \cdot h_1(y)\right) \\ &= h_1^{-1}\left(\tfrac{x \cdot k \cdot h_1(y)}{e \cdot k}\right) = I^{h_1,e}(x,y). \end{aligned}$$

Finally, if $x > 0$ and $y > e$,

$$\begin{aligned} I^{h_2,e}(x,y) &= h_2^{-1}\left(\tfrac{e}{x} \cdot h_2(y)\right) = h_2^{-1}\left(\tfrac{e}{x} \cdot c \cdot h_1(y)\right) \\ &= h_1^{-1}\left(\tfrac{e \cdot c \cdot h_1(y)}{x \cdot c}\right) = I^{h_1,e}(x,y). \end{aligned}$$

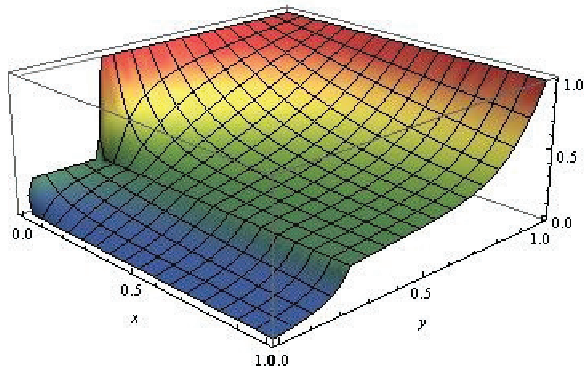


(a) $I^{h_1,e}$

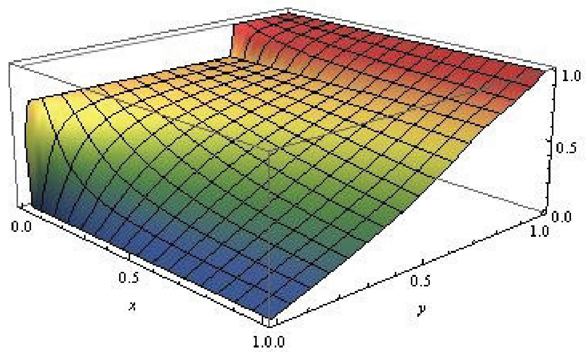


(b) $I^{h_2,e}$ with $\beta = 1$

**Figure 1**. Plots of some $(h,e)$-implications.



(a) $I^{h_3,e}$ with $e = 0.25$



(b) $I^{h_3,e}$ with $e = 0.75$

**Figure 2**. Plots of some $(h,e)$-implications with generator $h_3$

***Remark 11.*** The previous result is not so strange. While each one of $f$ and $g$-generators is a unique function satisfying some properties, an $h$-generator can be considered as a piece-wise function of two component functions with no other relation than the point where their value is zero. In other words, an $h$-generator is a piece-wise function of $h_1$ in $[0,e]$ such that $h_1(0) = -\infty$ and $h_1(e) = 0$ and $h_2$ in $[e,1]$ such that $h_2(e) = 0$ and $h_2(1) = +\infty$. So logically, $h$-generators are unique up to a multiplicative constant piecewise.

From the proof of the previous result, the following corollary is immediate.

**Corollary 12.** Let $h_1, h_2 : [0,1] \to [-\infty,\infty]$ be any two $h$-generators with respect to a fixed $e \in (0,1)$ such that $h_2 \circ h_1^{-1}(1) = -h_2 \circ h_1^{-1}(-1)$. Then the following statements are equivalent:

(i) $I^{h_1,e} = I^{h_2,e}$.

(ii) There exists a constant $c \in (0,\infty)$ such that $h_2(x) = c \cdot h_1(x)$ for all $x \in [0,1]$.

### 3.1 Basic Properties

From now on, we study which properties are satisfied by the $(h,e)$-implications. The first result deals with the natural negation associated to an $(h,e)$-implication.

**Proposition 13.** If $h$ is an $h$-generator with respect to a fixed $e \in (0,1)$, then the natural negation of $I^{h,e}$ is the Gödel negation $N_{D_1}$, that is

$$N_{D_1}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases}$$

which is not continuous.

*Proof.*

$$\begin{aligned} N_{I^{h,e}}(x) &= I^{h,e}(x,0) = \begin{cases} I^{h,e}(0,0) & \text{if } x = 0, \\ h^{-1}\left(\frac{x}{e} \cdot h(0)\right) & \text{if } x > 0, \end{cases} \\ &= \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}(-\infty) & \text{if } x > 0, \end{cases} \\ &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} = N_{D_1}(x). \end{aligned}$$

***Remark 14.*** Note that $(h,e)$-implications introduced here are completely different from $h$-generated implications introduced by J. Balasubramaniam in [6, 7] (see also Definition 2.2). Do not forget that the generator $h$ of $h$-generated implications is a multiplicative generator of a continuous Archimedean t-conorm, whereas for $(h,e)$-implications introduced here, $h$ is a generator of a representable uninorm. Thus, they do not satisfy the same properties. A first example is given by the previous proposition since the natural negation of any $h$-generated implication is always a continuous fuzzy negation, that even can be strong depending on the generator $h$ (see [6]). Other differences lie in properties $(i),(ii),(v),(vi),(vii)$ and $(x)$ in next theorem.

**Theorem 15.** Let $h$ be an $h$-generator with respect to a fixed $e \in (0,1)$. Then the following statements are true:

(i) $I^{h,e}$ does not satisfy (NP),

(ii) $I^{h,e}$ satisfies ($NP_e$),

(iii) $I^{h,e}$ satisfies (EP),

(iv) $I^{h,e}(x,x) = 1$ if and only if $x = 0$ or $x = 1$, i.e., $I^{h,e}$ does not satisfy (IP),

(v) $I^{h,e}(x,y) = 1$ if and only if $x = 0$ or $y = 1$[1], i.e., $I^{h,e}$ does not satisfy (OP),

(vi) $I^{h,e}$ does not satisfy (CP) with any fuzzy negation,

(vii) $I^{h,e}$ is continuous except at the points $(0,y)$ with $y \leq e$,

(viii) $I^{h,e}(\cdot,y)$ is one-to-one for all $y \in (0,1)\setminus\{e\}$,

(ix) $I^{h,e}(x,\cdot)$ has range $[0,1]$ for all $x \in (0,1]$.

(x) $I^{h,e}(x,y) < e$ if $x > 0$ and $y < e$,
$I^{h,e}(x,e) = e$ if $x > 0$,
$I^{h,e}(x,y) > e$ if $x > 0$ and $y > e$.

*Proof.* For points (i)-(vii) see Proposition 9 and Theorem 13 in [29]. So let us only prove points (viii)-(x).

(viii) First of all, if $x_1 = 0 < x_2$, $I^{h,e}(0,y) = 1 > I^{h,e}(x_2,y)$ for all $y \in (0,1)\setminus\{e\}$ using (v). Next let us consider $0 < y < e$ and $0 < x_1 < x_2$, then since $-\infty < h(y) < 0$ and $h^{-1}$ is strictly increasing we get

$$\begin{aligned} I^{h,e}(x_1,y) &= h^{-1}\left(\frac{x_1}{e} \cdot h(y)\right) > h^{-1}\left(\frac{x_2}{e} \cdot h(y)\right) \\ &= I^{h,e}(x_2,y). \end{aligned}$$

Now let us consider $e < y < 1$ and $0 < x_1 < x_2$, then since $0 < h(y) < \infty$ and $h^{-1}$ is strictly increasing we get

$$\begin{aligned} I^{h,e}(x_1,y) &= h^{-1}\left(\frac{e}{x_1} \cdot h(y)\right) > h^{-1}\left(\frac{e}{x_2} \cdot h(y)\right) \\ &= I^{h,e}(x_2,y). \end{aligned}$$

(ix) The vertical sections of $I^{h,e}$ are continuous for all $x \in (0,1]$ by (vii). In addition, for all $x \in (0,1]$ $I^{h,e}(x,0) = N_{I^{h,e}}(x) = 0$ and $I^{h,e}(x,1) = 1$. Therefore, vertical sections have range $[0,1]$.

(x) If $x > 0$, we have

$$I^{h,e}(x,e) = h^{-1}\left(\frac{x}{e} \cdot h(e)\right) = h^{-1}(0) = e.$$

[1]This property is studied in detail in [8] where it is shown that such property is essential for the construction of strong equality indexes.

Moreover, if $x > 0$ and $y < e$, then $h(y) < 0$ and we get

$$I^{h,e}(x,y) = h^{-1}\left(\frac{x}{e} \cdot h(y)\right) < h^{-1}(0) = e.$$

Finally, if $x > 0$ and $y > e$, then $h(y) > 0$ and we obtain

$$I^{h,e}(x,y) = h^{-1}\left(\frac{e}{x} \cdot h(y)\right) > h^{-1}(0) = e.$$

***Remark 16.*** Point (x) in the previous result shows that these implications allow to have a degree of control of the increasingness with respect to the second variable of the fuzzy implication.

### 3.2 The Law of Importation

Recently, the satisfaction of the law of importation has been studied deeply for the most common classes of fuzzy implications (see [18], [25], [26]). Even its relationship with other properties of fuzzy implications like (EP) and (WLI), a weaker version of the law of importation has been established (see [28]). Whereas Yager's implications and $h$-implications satisfy the law of importation with the product t-norm $T_P$, $(h,e)$-implications act in a total different way.

**Proposition 17.**(see Proposition 15 in [29]) Let $h$ be an $h$-generator, then $I^{h,e}$ does not satisfy (WLI) with any $F : [0,1]^2 \to [0,1]$ and consequently, it does not satisfy (LI) with any t-norm.

***Remark 18.*** In [28], there are examples of functions $F : [0,1]^2 \to [0,1]$ and of fuzzy implications $I$ that satisfy (EP) but neither (LI) nor (WLI) with the function $F$. This fact proves that (LI), and even (WLI), are stronger properties than (EP). By the previous result, we add a whole family of fuzzy implications to this divergence.

## 4 Distributivity Properties

Distributivity of fuzzy implications over different fuzzy connectives is an important topic in fuzzy logic. The peak of this interest started by Combs and Andrews in [12] where they use a distributivity classical tautology in order to reduce the complexity of fuzzy "If-Then" rules. After that, many authors have dealt with them (see [32], [33], [34]). From the four basic distributive equations involving an implication, that are tautologies in classical logic, the following generalizations in fuzzy logic are obtained:

$$I(S(x,y),z) = T(I(x,z),I(y,z)) \quad (3)$$

$$I(T(x,y),z) = S(I(x,z),I(y,z)) \quad (4)$$

$$I(x,T_1(y,z)) = T_2(I(x,y),I(x,z)) \quad (5)$$

$$I(x,S_1(y,z)) = S_2(I(x,y),I(x,z)) \quad (6)$$

for $x,y,z \in [0,1]$, where $I$ is a fuzzy implication, $T$, $T_1$, $T_2$ are t-norms and $S$, $S_1$, $S_2$ are t-conorms.

So in this section, the main goal is to study the distributivity of $(h,e)$-implications over t-norms and t-conorms, by studying the satisfaction of Equations (3)-(6).

### 4.1 On the Equation $I(S(x,y),z) = T(I(x,z),I(y,z))$

Let us begin with some necessary conditions for a t-norm and a t-conorm to satisfy Equation (3) with a binary function satisfying some minimal properties.

**Proposition 19.** Let $I : [0,1]^2 \to [0,1]$ be a function satisfying that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$, $T$ a t-norm and $S$ a t-conorm. If the triple $(I,T,S)$ satisfies (3) for all $x,y,z \in [0,1]$, then $T = T_M$.

*Proof.* Consider $x = y = 1$ in (3), we get $I(1,z) = T(I(1,z),I(1,z))$ and consequently, $I(1,z)$ must be an idempotent element of $T$ for all $z \in [0,1]$. However, since $I(1,z)$ has range $[0,1]$, $T = T_M$, the only idempotent t-norm.

**Proposition 20.** Let $I : [0,1]^2 \to [0,1]$ be a function satisfying that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$ and $I(\cdot,z_0)$ is one-to-one for some $z_0 \in [0,1)$. If the triple $(I,T,S)$, where $T$ is a t-norm and $S$ a t-conorm, satisfies (3) for all $x,y,z \in [0,1]$, then $S = S_M$.

*Proof.* The previous proposition implies that (3) is reduced to

$$I(S(x,y),z) = \min\{I(x,z),I(y,z)\}.$$

Taking $x = y$ and $z = z_0$ in this equation, we obtain $I(S(x,x),z_0) = I(x,z_0)$. Since $I(\cdot,z_0)$ is one-to-one, $S(x,x) = x$ for all $x \in [0,1]$ and then $S = S_M$.

Joining the two previous results we obtain the following corollary

**Corollary 21.** Let $I$ be a fuzzy implication that satisfies that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$ and $I(\cdot,z_0)$ is one-to-one for some $z_0 \in [0,1)$, $T$ a t-norm and $S$ a t-conorm. Then the following statements are equivalent:

(i) The triple $(I,T,S)$ satisfies (3) for all $x,y,z \in [0,1]$.

(ii) $S = S_M$ and $T = T_M$.

*Proof.* (i) implies (ii) by the two previous results. Reciprocally, the result is guaranteed by Proposition 7.2.3 in [3].

Finally, by Theorem 3.1 points $(viii)$ and $(ix)$ $(h,e)$-implications are a particular case and so we have the following result:

**Theorem 22.** Let $h$ be an $h$-generator, $T$ a t-norm and $S$ a t-conorm. Then the following statements are equivalent:

(i) The triple $(I^{h,e},T,S)$ satisfies (3) for all $x,y,z \in [0,1]$.

(ii) $S = S_M$ and $T = T_M$.

### 4.2 On the Equation $I(T(x,y),z) = S(I(x,z),I(y,z))$

This case is almost dual to the previous one and so the results follow in a similar way.

**Proposition 23.** Let $I : [0,1]^2 \to [0,1]$ be a function satisfying that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$, $T$ be a t-norm and $S$ a t-conorm. If the triple $(I,T,S)$ satisfies (4) for all $x,y,z \in [0,1]$, then $S = S_M$.

*Proof.* Consider $x = y = 1$ in (4), we get $I(1,z) = S(I(1,z),I(1,z))$ and consequently, $I(1,z)$ must be an idempotent element of $S$ for all $z \in [0,1]$. However, since $I(1,z)$ has range $[0,1]$, $S = S_M$, the only idempotent t-conorm.

**Proposition 24.** Let $I : [0,1]^2 \to [0,1]$ be a function satisfying that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$ and $I(\cdot,z_0)$ be one-to-one for some $z_0 \in [0,1)$. If the triple $(I,T,S)$, where $T$ is a t-norm and $S$ a t-conorm, satisfies (4) for all $x,y,z \in [0,1]$, then $T = T_M$.

*Proof.* The previous proposition implies that (4) is reduced to

$$I(T(x,y),z) = \max\{I(x,z),I(y,z)\}.$$

Taking $x = y$ and $z = z_0$ in this equation, we obtain $I(T(x,x),z_0) = I(x,z_0)$. Since $I(\cdot,z_0)$ is one-to-one, $T(x,x) = x$ for all $x \in [0,1]$ and then $T = T_M$.

Joining the two previous results we obtain the following corollary

**Corollary 25.** Let $I$ be a fuzzy implication that satisfies that $I(1,z)$ has range $[0,1]$ with $z \in [0,1]$ and $I(\cdot,z_0)$ is one-to-one for some $z_0 \in [0,1)$, $T$ a t-norm and $S$ a t-conorm. Then the following statements are equivalent:

(i) The triple $(I,T,S)$ satisfies (4) for all $x,y,z \in [0,1]$.

(ii) $S = S_M$ and $T = T_M$.

*Proof.* The proof is analogous to the one of Corollary 4.1 using in this case Proposition 7.2.12 in [3].

Finally, $(h,e)$-implications are again a particular case using Theorem 3.1-$(viii)$ and $(ix)$.

**Theorem 26.** Let $h$ be an $h$-generator, $T$ a t-norm and $S$ a t-conorm. Then the following statements are equivalent:

(i) The triple $(I^{h,e},T,S)$ satisfies (4) for all $x,y,z \in [0,1]$.

(ii) $S = S_M$ and $T = T_M$.

### 4.3 On the Equation $I(x,T_1(y,z)) = T_2(I(x,y),I(x,z))$

First of all, the next result reduces the equation to study because, under some assumptions, the involved t-norms must be the same.

**Proposition 27.** Let $T_1$, $T_2$ be t-norms. If $I : [0,1]^2 \to [0,1]$ is any function that satisfies ($NP_e$), then $T_1 = T_2$ in (5).

*Proof.* It is sufficient to take $x = e$ in (5) and apply ($NP_e$).

So, Equation (5) becomes

$$I(x,T(y,z)) = T(I(x,y),I(x,z)), \quad x,y,z \in [0,1], \tag{7}$$

taking $T = T_1 = T_2$.

In this case, we do not give a complete characterization of those $(h,e)$-implications satisfying Equation (5). Note that such a characterization is still an open problem also for $f$ and $g$-generated implications. However, there are t-norms other than $T_M$ for which an $(h,e)$-implication satisfies this equation. To see this, let us begin with a necessary condition for a t-norm to satisfy Equation (7) with an $(h,e)$-implication.

**Proposition 28.** Let $h$ be an $h$-generator and $T$ a t-norm. If $(I^{h,e},T)$ satisfies (7) then $e$ is an idempotent element of $T$, i.e., $T(e,e)=e$, or $T(e,e)=0$.

*Proof.* Take $x>0$ and $y=z=e$ in (7) then since $I^{h,e}(x,e)=e$ and $T(e,e)\leq e$ we obtain

$$\begin{array}{l} I^{h,e}(x,T(e,e))=T(e,e)\Leftrightarrow \\ h^{-1}\left(\frac{x}{e}\cdot h(T(e,e))\right)=T(e,e)\Leftrightarrow \\ \frac{x}{e}\cdot h(T(e,e))=h(T(e,e)). \end{array}$$

This implies that $h(T(e,e))=-\infty$ or $h(T(e,e))=0$ and consequently, $T(e,e)=0$ or $T(e,e)=e$, respectively.

So there are two candidate values for $T(e,e)$. The next result shows that there are solutions of Equation (7) for t-norms with $T(e,e)=0$, just taking the drastic t-norm $T_D$.

**Proposition 29.** Let $h$ be an $h$-generator. Then, the pair $(I^{h,e},T_D)$ always satisfies Equation (7).

*Proof.* If $x=0$,

$$\begin{aligned} I^{h,e}(0,T_D(y,z)) &= 1=T_D(1,1) \\ &= T_D(I^{h,e}(0,y),I^{h,e}(0,z)). \end{aligned}$$

Otherwise, we consider two cases. If $y,z<1$, using that $T_D(y,z)=0$ and by Theorem 3.1-(v), $I^{h,e}(x,y),I^{h,e}(x,z)<1$, we obtain

$$\begin{aligned} I^{h,e}(x,T_D(y,z)) &= I^{h,e}(x,0)=N_{D_1}(x)=0 \\ &= T_D(I^{h,e}(x,y),I^{h,e}(x,z)). \end{aligned}$$

Finally, if $y=1$ (or equivalently, $z=1$),

$$\begin{aligned} I^{h,e}(x,T_D(1,z)) &= I^{h,e}(x,z)=T_D(1,I^{h,e}(x,z)) \\ &= T_D(I^{h,e}(x,y),I^{h,e}(x,z)). \end{aligned}$$

On the other hand, if we consider continuous t-norms with $T(e,e)=e$, they are necessarily ordinal sums of the form $\langle(0,e,T'),(e,1,T'')\rangle$ with $T'$ and $T''$ continuous t-norms. Among them the next proposition shows three particular cases (different from minimum) that satisfy Equation (7) with a fixed $(h,e)$-implication.

**Proposition 30.** Let $h$ be an $h$-generator and $T$ a t-norm. Then $I^{h,e}$ satisfies (7) if $T=T_M$ or $T$ is one of the following t-norms

$$T_1(x,y)=\begin{cases} h^{-1}(h(x)+h(y)) & \text{if } (x,y)\in[0,e]^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

$$T_2(x,y)=\begin{cases} h^{-1}\left(\frac{h(x)\cdot h(y)}{h(x)+h(y)}\right) & \text{if } (x,y)\in[e,1]^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

$$T_3(x,y)=\begin{cases} h^{-1}(h(x)+h(y)) & \text{if } (x,y)\in[0,e]^2, \\ h^{-1}\left(\frac{h(x)\cdot h(y)}{h(x)+h(y)}\right) & \text{if } (x,y)\in[e,1]^2, \\ \min\{x,y\} & \text{otherwise.} \end{cases}.$$

Note that the three last t-norms are ordinal sums of $T'$ and $T_M$, $T_M$ and $T''$, $T'$ and $T''$ respectively, where $T'$ and $T''$ are the Archimedean t-norms with additive generators $f(x)=-h(e\cdot x)$ and $g(x)=\frac{1}{h(e+(1-e)\cdot x)}$ respectively.

*Proof.* First of all, $I^{h,e}$ satisfies (7) with $T=T_M$ by Proposition 7.2.15 in [3]. Next let us prove that $I^{h,e}$ satisfies (7) with $T=T_1$. If $x=0$, the equation holds trivially. Otherwise, note that if $y>e$ (or equivalently $z>e$) then $I^{h,e}(x,y)>e$ and we have that

$$\begin{aligned} I^{h,e}(x,T_1(y,z)) &= I^{h,e}(x,T_M(y,z)), \\ T_1(I^{h,e}(x,y),I^{h,e}(x,z)) &= T_M(I^{h,e}(x,y),I^{h,e}(x,z)) \end{aligned}$$

and the result follows. Now if $y,z\leq e$, since $T_1(y,z)\leq e$ and $I^{h,e}(x,y),I^{h,e}(x,z)\leq e$ we have that

$$\begin{array}{l} I^{h,e}(x,T_1(y,z))=I^{h,e}(x,h^{-1}(h(y)+h(z)))= \\ =h^{-1}\left(\frac{x}{e}\cdot h(h^{-1}(h(y)+h(z)))\right) \\ =h^{-1}\left(\frac{x}{e}\cdot(h(y)+h(z))\right) \\ =h^{-1}\left(h\left(h^{-1}\left(\frac{x}{e}\cdot h(y)\right)\right)+h\left(h^{-1}\left(\frac{x}{e}\cdot h(z)\right)\right)\right) \\ =h^{-1}(h(I^{h,e}(x,y))+h(I^{h,e}(x,z))) \\ =T_1(I^{h,e}(x,y),I^{h,e}(x,z)). \end{array}$$

Consider now $T=T_2$. Again if $x=0$, the equation holds trivially. Otherwise, note that if $y\leq e$ (or equivalently $z\leq e$) then $I^{h,e}(x,y)\leq e$ and we have that

$$\begin{aligned} I^{h,e}(x,T_2(y,z)) &= I^{h,e}(x,T_M(y,z)), \\ T_2(I^{h,e}(x,y),I^{h,e}(x,z)) &= T_M(I^{h,e}(x,y),I^{h,e}(x,z)) \end{aligned}$$

and the result follows. Now if $y,z>e$, since $T_2(y,z)>e$ because $\frac{h(y)\cdot h(z)}{h(y)+h(z)}>0$, and $I^{h,e}(x,y),I^{h,e}(x,z)>e$ we have that

$$\begin{array}{l} I^{h,e}(x,T_2(y,z))=I^{h,e}\left(x,h^{-1}\left(\frac{h(y)\cdot h(z)}{h(y)+h(z)}\right)\right)= \\ =h^{-1}\left(\frac{e}{x}\cdot h\left(h^{-1}\left(\frac{h(y)\cdot h(z)}{h(y)+h(z)}\right)\right)\right) \\ =h^{-1}\left(\frac{h\left(h^{-1}\left(\frac{e}{x}\cdot h(y)\right)\right)\cdot h\left(h^{-1}\left(\frac{e}{x}\cdot h(z)\right)\right)}{h\left(h^{-1}\left(\frac{e}{x}\cdot h(y)\right)\right)+h\left(h^{-1}\left(\frac{e}{x}\cdot h(z)\right)\right)}\right) \\ =h^{-1}\left(\frac{h(I^{h,e}(x,y))\cdot h(I^{h,e}(x,z))}{h(I^{h,e}(x,y))+h(I^{h,e}(x,z))}\right) \\ =T_2(I^{h,e}(x,y),I^{h,e}(x,z)). \end{array}$$

Finally, let us prove that $I^{h,e}$ satisfies (7) with $T = T_3$. If $x = 0$, the equation holds trivially. Otherwise, if $y,z \leq e$, $T_3(y,z) = T_1(y,z)$ and since $I^{h,e}(x,y), I^{h,e}(x,z) \leq e$, we get

$$T_3(I^{h,e}(x,y), I^{h,e}(x,z)) = T_1(I^{h,e}(x,y), I^{h,e}(x,z))$$

and the result is straightforward. Now if $y,z > e$, $T_3(y,z) = T_2(y,z)$ and since $I^{h,e}(x,y), I^{h,e}(x,z) > e$, we obtain

$$T_3(I^{h,e}(x,y), I^{h,e}(x,z)) = T_2(I^{h,e}(x,y), I^{h,e}(x,z))$$

and the result is clear. Otherwise, suppose $y \leq e < z$ (equivalently $z \leq e < y$) $T_3(y,z) = T_M(y,z)$ and since $I^{h,e}(x,y) \leq e < I^{h,e}(x,z)$ we have

$$T_3(I^{h,e}(x,y), I^{h,e}(x,z)) = T_M(I^{h,e}(x,y), I^{h,e}(x,z))$$

and the result follows.
Let us prove now that the three last t-norms are the ordinal sums given in the statement. First, we are going to prove that $f(x) = -h(e \cdot x)$ and $g(x) = \frac{1}{h(e+(1-e)\cdot x)}$ are additive generators of Archimedean t-norms. It is clear that they are continuous and strictly decreasing functions since $h$ is continuous and strictly increasing. Now

$$\begin{aligned} f(0) &= -h(0) = \infty \\ g(0) &= \tfrac{1}{h(e)} = \tfrac{1}{0} = \infty \\ f(1) &= -h(e) = 0 \\ g(1) &= \tfrac{1}{h(1)} = \tfrac{1}{\infty} = 0. \end{aligned}$$

If $(x,y) \in [0,e]^2$ we have

$$\begin{aligned} T_1(x,y) &= h^{-1}(h(x)+h(y)) \\ &= e \cdot f^{-1}(-h(x)-h(y)) \\ &= e \cdot f^{-1}\left(-h\left(e \cdot \frac{x}{e}\right) - h\left(e \cdot \frac{y}{e}\right)\right) \\ &= e \cdot f^{-1}\left(f\left(\frac{x}{e}\right) + f\left(\frac{y}{e}\right)\right) \\ &= e \cdot T'\left(\frac{x}{e}, \frac{y}{e}\right) \end{aligned}$$

and if $(x,y) \in [e,1]^2$ we get

$$\begin{aligned} T_2(x,y) &= h^{-1}\left(\frac{h(x)\cdot h(y)}{h(x)+h(y)}\right) \\ &= e + (1-e) \cdot \frac{h^{-1}\left(\frac{1}{\frac{1}{h(x)} + \frac{1}{h(y)}}\right) - e}{1-e} \\ &= e + (1-e) \cdot g^{-1}\left(\frac{1}{h(x)} + \frac{1}{h(y)}\right) \\ &= e + (1-e) \cdot g^{-1}\left(g\left(\frac{x-e}{1-e}\right) + g\left(\frac{y-e}{1-e}\right)\right) \\ &= e + (1-e) \cdot T''\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right). \end{aligned}$$

The result for $T_3$ is now straightforward.

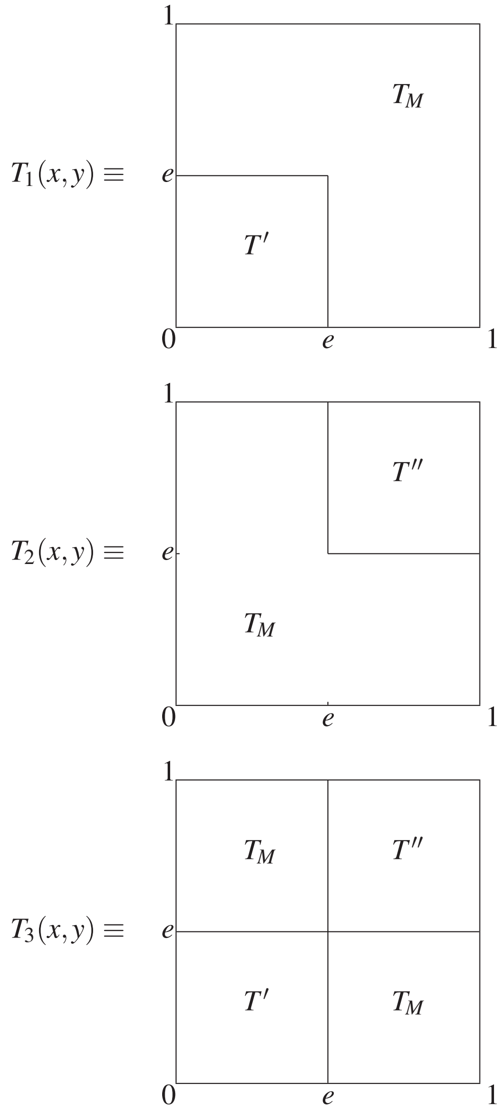



**Figure 3**. t-norms $T_1$, $T_2$ and $T_3$ which are solutions of Equation (7) for a fixed $(h,e)$-implication $I^{h,e}$. $T'$ and $T''$ are the Archimedean t-norms with additive generators $f(x) = -h(e \cdot x)$ and $g(x) = \frac{1}{h(e+(1-e)\cdot x)}$ respectively

The three t-norms $T_1$-$T_3$ given in Proposition 4.3 can be viewed in Figure 3.

### 4.4 On the Equation $I(x,S_1(y,z)) = S_2(I(x,y),I(x,z))$

The behaviour of this case is again dual of the previous one and thus all results follow similarly.

**Proposition 31.** Let $S_1$, $S_2$ be t-conorms. If $I : [0,1]^2 \to [0,1]$ is any function that satisfies ($NP_e$), then $S_1 = S_2$ in (6).

So, Equation (6) becomes taking $S = S_1 = S_2$

$$I(x,S(y,z)) = S(I(x,y),I(x,z)), \;\; x,y,z \in [0,1]. \quad (8)$$

In this case also, there is not a complete characterization of those $(h,e)$-implications satisfying Equation (6), similarly as for $f$ and $g$ implications. However, there are t-conorms other than $S_M$ for which an $(h,e)$-implication satisfies this equation. Again we have the following necessary condition.

**Proposition 32.** Let $h$ be an $h$-generator and $S$ a t-norm. If $(I^{h,e},S)$ satisfies (8) then $e$ is an idempotent element of $S$, i.e., $S(e,e) = e$, or $S(e,e) = 1$.

*Proof.* Take $x > 0$ and $y = z = e$ in (8) then since $I^{h,e}(x,e) = e$ we obtain

$$I^{h,e}(x,S(e,e)) = S(e,e).$$

Now if $S(e,e) = e$, the equality holds. Otherwise, $S(e,e) > e$ and in this case,

$$h^{-1}\left(\tfrac{e}{x}\cdot h(S(e,e))\right) = S(e,e) \Leftrightarrow \\ \tfrac{e}{x}\cdot h(S(e,e)) = h(S(e,e)).$$

This implies that $h(S(e,e)) = \infty$ and consequently, $S(e,e) = 1$.

So there are two candidate values for $S(e,e)$. The next result shows that there are solutions of Equation (8) for t-conorms with $S(e,e) = 1$, just taking the drastic t-conorm $S_D$.

**Proposition 33.** Let $h$ be an $h$-generator. Then, the pair $(I^{h,e},S_D)$ always satisfies Equation (8).

*Proof.* If $x = 0$,

$$\begin{aligned} I^{h,e}(0,S_D(y,z)) &= 1 = S_D(1,1) \\ &= S_D(I^{h,e}(0,y),I^{h,e}(0,z)). \end{aligned}$$

Otherwise, we consider two cases. If $y,z > 0$, using that $S_D(y,z) = 1$ and since $I^{h,e}(x,y), I^{h,e}(x,z) > 0$ in this case, we obtain

$$\begin{aligned} I^{h,e}(x,S_D(y,z)) &= I^{h,e}(x,1) = 1 \\ &= S_D(1,1) = S_D(I^{h,e}(x,y),I^{h,e}(x,z)). \end{aligned}$$

Finally, if $y = 0$ (or equivalently, $z = 0$),

$$\begin{aligned} I^{h,e}(x,S_D(0,z)) &= I^{h,e}(x,z) = S_D(0,I^{h,e}(x,z)) \\ &= S_D(I^{h,e}(x,y),I^{h,e}(x,z)). \end{aligned}$$

On the other hand, if $S(e,e) = e$ we have the following result:

**Proposition 34.** Let $h$ be an $h$-generator and $S$ a t-conorm. Then $I^{h,e}$ satisfies (8) if $S = S_M$ or $S$ is one of the following t-conorms

$$S_1(x,y) = \begin{cases} h^{-1}(h(x)+h(y)) & \text{if } (x,y) \in [e,1]^2, \\ \max\{x,y\} & \text{otherwise}, \end{cases}$$

$$S_2(x,y) = \begin{cases} h^{-1}\left(\frac{h(x)\cdot h(y)}{h(x)+h(y)}\right) & \text{if } (x,y) \in [0,e]^2, \\ \max\{x,y\} & \text{otherwise}, \end{cases}$$

$$S_3(x,y) = \begin{cases} h^{-1}\left(\frac{h(x)\cdot h(y)}{h(x)+h(y)}\right) & \text{if } (x,y) \in [0,e]^2, \\ h^{-1}(h(x)+h(y)) & \text{if } (x,y) \in [e,1]^2, \\ \max\{x,y\} & \text{otherwise}. \end{cases}$$

Note that the three last t-conorms are ordinal sums of $S_M$ and $S'$, $S''$ and $S_M$, $S'$ and $S''$ respectively, where $S'$ and $S''$ are the Archimedean t-conorms with additive generators $f(x) = h(e+(1-e)\cdot x)$ and $g(x) = \frac{-1}{h(e\cdot x)}$ respectively.

*Proof.* The proof is quite similar to the one of Proposition 4.3.

## 5 Intersections with other Families

When a class of fuzzy implications is presented, it is worth to study their intersections with the existing classes of fuzzy implications. Yager's implications have not been an exception and their intersections among themselves and with the most used classes of implications were stated in [2] and more recently in [30]. Whereas $f$- and $g$-implications have non-trivial intersections with $(S,N)$- and $R$-implications, Balasubramaniam's $h$-generated implications (see Definition 2.2) are contained in the family of all $(S,N)$-implications obtained from continuous negations (see [2]). With respect to $QL$-implications, $g$-implications and $f$-implications with $f(0) = \infty$ do not intersect with this class, however the intersection of $f$-implications generated from a bounded generator with $QL$-implications is non-trivial. From the properties studied above we will see that $(h,e)$-implications are a new class of fuzzy implications literally since they do not intersect with any of the most used classes of implications.

First of all, note that $(h,e)$-implications do not satisfy (NP) and consequently, they do not intersect with $(S,N)$, *R*, *QL*, *D*-implications or Yager's implications because all these families satisfy this property (see [3]).

**Proposition 35.** If $h$ is an $h$-generator, then $I^{h,e}$ is neither an $(S,N)$-, *R*-, *QL*-, *D*- nor a Yager's implication.

The previous result is coherent since $(h,e)$-implications are more related to implications derived from uninorms as they satisfy ($NP_e$). However, they do not intersect $(U,N)$-implications with $N$ a strict negation and *RU*-implications because these two types of implications satisfy the law of importation with some uninorm (see [26]) whereas $(h,e)$-implications do not satisfy even (WLI). In addition, there is also no intersection with $(U,N)$-implications with $N$ just continuous because they satisfy (WLI) with some function $F$ (see [28]). Furthermore, there are no intersections neither with *QLU* nor *DU*-implications because the first ones only satisfy the exchange principle when they are in fact *QL*-implications and the second ones satisfy (NP) (see [24]). Finally, they do not intersect $e$- and pseudo-$e$-implications (see [20]) because $(h,e)$-implications do not satisfy $I(x,x)=e$ for all $x \in (0,1)$, an axiom of those classes of implications.

**Proposition 36.** If $h$ is an $h$-generator, then $I^{h,e}$ is neither an $(U,N)$-implication with $N$ continuous, *RU*, *QLU*-, *DU*-, $e$ nor pseudo-$e$-implication.

# 6 Conclusion

We have presented and studied in detail a new class of implications, called $(h,e)$-implications, in a similar way to Yager's and Balasubramaniam's implications. They are generated from an additive generator of a representable uninorm, i.e., from a continuous and strictly increasing function $h: [0,1]^2 \to [-\infty,\infty]$ with $h(0)=-\infty$, $h(1)=\infty$ and $h(e)=0$ for a fixed $e \in (0,1)$. We studied the most common properties that can be satisfied by a class of fuzzy implications. The most interesting ones are the exchange principle (EP) and some distributive properties with respect to t-norms and t-conorms, that are useful in order to avoid combinatorial rule explosion in fuzzy systems (see [12, 34]). Moreover, we have proved that they do not satisfy either the contrapositive symmetry with any negation $N$, or the law of importation with any t-norm (even the weak law of importation is not satisfied with any function $F$). So, due to the fact that they satisfy (EP), they constitute a whole class of implications supporting the non-equivalence between (EP) and (LI) (see [22] and [28]). In addition, the intersections with the most usual classes of implications have been studied. Thus, we proved that $(h,e)$-implications are a new class of implications since they do not intersect with any of the existing classes. However, some questions remain open:

- Is there any relation between $(h,e)$-implications and Yager's implications that could lead us to characterize this new class of implications?
- Characterize all the t-norms and t-conorms that are solutions of Equation (7) and Equation (8) for a fixed $(h,e)$-implication. Recall that this is also an open problem for Yager's implications.

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