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CONVERSION OF GEODETIC COORDINATES INTO FLAT (2-DIMENSINAL) COORDINATES PL-UTM FOR THE PURPOSES OF NAVIGATION

ABSTRACT

This paper presents mathematical interdependences in converting geodetic coordinates into flat rectangular coordinates and inverse transformation recommended in the Ordinance by Prime Minister on the State System of Spatial References. It especially presents formulas contained in works by M. Hooijberg (1997) *Practical Geodesy* and R. Kadaj (2001) *Projection Formulas and Parameters of Coordinate Systems. G-1.10.*

In order to illustrate uses of the presented formulas relevant calculation examples based on the reference ellipsoids WGS 84 and GRS 80 are included.

Key words:
coordinate conversion, Gauss-Krüger projection, coordinate system PL-UTM.

INTRODUCTION

In navigation and hydrographic works carried out in the Polish maritime areas the coordinate system UTM (Universal Transverse Mercator) is used in on the WGS 84 (World Geodetic System) reference ellipsoid [9]. The fundamentals of UTM are described in textbooks on navigation, hydrography, geodesy and cartography and they are not discussed in this article. Mathematical interdependences in converting geodetic coordinates into UTM coordinates and inversely, recommended in the defense standards are presented in an article by W. Morgaś, Z. Kopacz [6].

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The Ordinance by the Council of Ministers on the state system of spatial references (2012) recommends that the flat rectangular system of coordinates PL-UTM be used during hydrographic works in maritime areas to make charts as well as in terrain-related information systems which have significance for the defense of the state [10]. Geodetic coordinate systems referred to with symbols PL-ETRF 2000 and PL-ETRF 89 (European Terrestrial Reference Frame) are a mathematical and physical realization of the European Terrestrial Reference System ETRS 89 coincidental with ITRS (International Terrestrial Reference System) at the epoch 1989.0. Differences of coordinates between the reference systems PL-ETRF 2000 and PL-ETRF 89 for the calculation examples presented in this article are $\Delta\varphi$ = 6–8 mm and $\Delta\lambda$ = 6–15 mm [5]. In accordance with the Ordinance [10], in undertakings in which the required accuracy of fixing a position does not exceed 1 m, and when geocentric reference systems (especially WGS 84, ITRS, ETRS 89) and earth-centered coordinate systems are used — coordinate transformation between these systems and reference systems PL-ETRF 2000 and PL-ETRF 89 are not used.

This article presents projection formulas for converting geodetic coordinates into coordinates PL-UTM and inversely recommended in the Ordinance, and especially the ones presented in publications:
– Marten Hooijberg (1997) *Practical Geodesy* [1];
– Roman Kadaj (2001) *G-1.10 Projection formulas and parameters of coordinate systems* [2].

The flat rectangular coordinate system PL-UTM is based on mathematical allocation of points on the reference ellipsoid GRS 80 to corresponding points on a plane in accordance with the UTM projection theory. The article presents projection interdependences on reference ellipsoids GRS 80 and WGS 84 having parameters as shown in table 1.

Tab. 1. Parameters reference ellipsoids GRS 80 and WGS 84 [7]

Parameter	Values of GRS 80	Values of WGS 84
Semi-major axis: a [m]	6 378 137,000	6 378 137,000
Flattening Factor of the Earth	298.257222101	298.257 223 563
Third flattening: n	0,001 679 220 394 63	0,001679 220 386 383 72
Semi-minor axis: b [m]	6356752.3141403558479	6356752.3142451794976
First eccentricity: e	0.081819191042815790146	0.081819190842621494335
First eccentricity square: e^2	0.0066943800229007876254	0.0066943799901413169961
Second eccentricity: e'	0,082 094 438 1519	0,082 094 437 949 696
Second eccentricity square: e'^2	0,006 739 496 775 48	0.006 739 496 742 276
Radius of a circle whose circumference is equal to the length of a meridian: R_0 [m]	6 367 449,145 771 047 5269	6 367 449,145 823 415 3093

The formulas are presented in the form easy for calculations when using electronic calculators and computer programs type MS Excel, when professional programs, e.g. TRANSPOL v.1.0 are not available.

CONVERSION OF GEODETIC COORDINATES INTO COORDINATES PL-UTM AND INVERSELY BASED ON HOOIJBERG

The article contains the results of calculations carried out with the computer software Mathcad for examples of 2 positions presented graphically in figure 1.

In accordance with the Ordinance by the Council of Ministers as of 2012 on the state system of geodetic projections the calculation formulas for the projection of a flat rectangular PL-UTM coordinate system are presented in the publication by [1], which is presented later in the article.

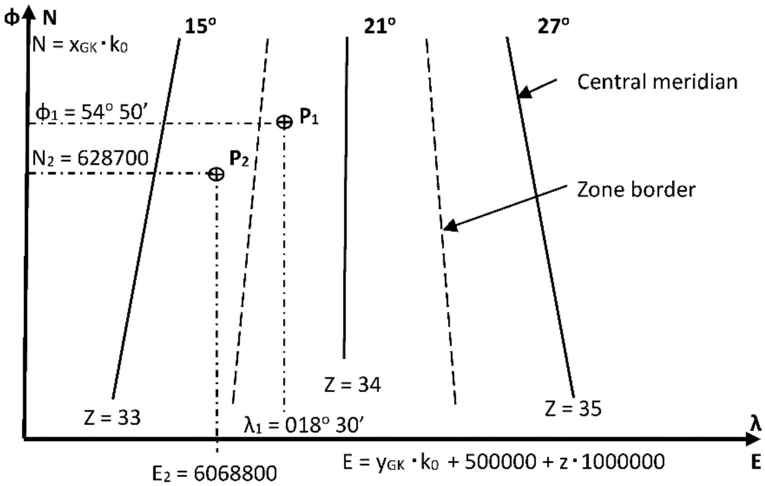

Fig. 1. Geodetic and PL-UTM coordinates for calculation examples (adopted symbols are explained later in the article) [own work]

The following mathematical formulas are presented in an established order, for converting geodetic coordinates into PL-UTM coordinates determined as a direct task.

Converting geodetic coordinates (φ, λ) into PL-UTM coordinates (E_{UTM}, N_{UTM}) — direct task

1. Length of the meridian arc (S) in coordinates PL-UTM can be calculated from the formula:

$$S = k_0 \cdot \omega, \tag{1}$$

where:

ω — length of the meridian calculated from the known formula:

$$\omega = a \cdot \left(1 - e^2\right) \cdot \int_0^{\varphi} \left(1 - e^2 \cdot \sin^2 \varphi\right)^{-\frac{3}{2}} d\varphi .$$

In the publication by [1] a solution applied to this interdependence is as follows:

$$\omega = \varphi \cdot r + \sin \varphi \cdot \cos \varphi \left(U_0 + U_2 \cdot \cos^2 \varphi + U_4 \cdot \cos^4 \varphi + U_6 \cdot \cos^6 \varphi\right), \tag{2}$$

where:

$$r = \frac{a\left(1 + n^2 / 4\right)}{1 + n}; \qquad n = \frac{f}{2 - f}; \qquad c = \frac{a}{\sqrt{1 - e^2}};$$

$$U_0 = c \cdot \left\{\left[\left(\left((-\frac{86625}{8} e'^2 + 11025)\frac{e'^2}{64} - 175\right)\frac{e'^2}{4} + 45\right)\frac{e'^2}{16} - 3\right]\frac{e'^2}{4}\right\};$$

$$U_2 = c \cdot \left\{\left[\left((-\frac{17325}{4} e'^2 + 3675)\frac{e'^2}{256} - \frac{175}{12}\right)e'^2 + 15\right]\frac{e'^4}{32}\right\}; \tag{3}$$

$$U_4 = c\left(-\frac{1495}{2} e'^2\right)\frac{e'^6}{2048}; \qquad U_6 = c\left[\left(-\frac{3465}{4} e'^2 + 315\right)\frac{e'^8}{1024}\right];$$

$$e'^2 = \frac{e^2}{1 - e^2}; \qquad f = \frac{a - b}{a},$$

where:

φ, λ — geodetic latitude and geodetic longitude;
φ_0, λ_0 — parallel of geodetic latitude origin and central meridian — CM;
a, b — length of semi-major and semi-minor axis of the ellipsoid;
e, e' — first and second eccentricity;
f, n — first flattening and second flattening of the reference ellipsoid;
c — meridional isoperimetric radius;
r — radius of the rectifying sphere;
k_0 — scale factor mandated for the central meridian;
U_0, U_2 … — coefficients in the series for meridian arc formulas are included in table 2.

The calculation results for the ellipsoid GRS 80 and WGS 84 are presented in table 2.

Tab. 2. Values of coefficients calculated from the formulas (2) [own work]

Parameter symbol	Values of parameter on reference elipsoid	
	GRS 80	WGS 84
c	6 399 593.625 864 023	6 399 593.625 758 493
r	6 367 449.145 771 047 5269	6 367 449.145 823 415 3093
U_0	–32 144.480 092 899 285	–32 144.479 935 001 393
U_2	135.366 899 836 125	135.366 898 504 447
U_4	–0.709 321 667 290	–0.709 321 656 818
U_6	0.003 986 101 039	0.003 986 100 961

2. Conversion of geodetic coordinates (φ, λ) into coordinates PL-UTM:

$$E = E_0 + A_1 \cdot L\{1 + L^2[A_3 + L^2(A_5 + A_7 \cdot L^2)]\};$$

$$N = S - S_0 = N_0 + A_2 \cdot L^2[1 + L^2(A_4 + A_6 \cdot L^2)], \qquad (4)$$

where:

$$A_1 = R = \frac{k_0 \cdot a}{\sqrt{1 - e^3 \cdot \sin^2\varphi}}; \qquad A_2 = \frac{1}{2}R \cdot t; \qquad A_3 = \frac{1}{6}(1 - t^2 + \eta^2);$$

$$A_4 = \frac{1}{12} \cdot [5 - t^2 + \eta^2 \cdot (9 + 4 \cdot \eta^2)]; \qquad A_5 = \frac{1}{120} \cdot [5 - 18 \cdot t^2 + t^4 + \eta^2(14 - 58t^2)]; \qquad (5)$$

$$A_6 = \frac{1}{360}[61 - 58t^2 + t^4 + \eta^2(270 - 330t^2)]; \qquad A_7 = \frac{1}{5040}(61 - 479t^2 + 179t^4 - t^6);$$

$$t = \tan\varphi; \qquad \eta^2 = e'^2 \cdot \cos^2\varphi; \qquad L = (\lambda - \lambda_0)\cos\varphi,$$

where:

N, E — northing and easting coordinates on the projection;
R — radius of curvature in the Prime Vertical;
N_0, E_0 — false northing and false easting;
S — meridional distance;
S_0 — meridional distance from the equator to φ_0, multiplied by the CM scale factor.

Example 1 of converting position geodetic coordinates into coordinates PL-UTM based on the formulas above.

Data: φ = 54°50’ N; λ = 018°30’ E; λ_o = 021°
Calculate: N and E

Tab. 3. The calculation results for example 1 based on formulas (4) and (5) [own work]

No.	Parameter symbol	Values of parameter on reference ellipsoid	
		GRS 80	WGS 84
1	L	–0.025 924 521 147 254	–0.025 130 857 277 322
2	ω	6 078 676.647 271 771	6 078 676.647 396 782
3	S	6 076 245.176 612 862	6 076 245.176 737 823
4	R	6 389 894.934 166 34	6 389 894.934 096 08
5	A_1	6 383 605.657 791 802	6 389 894.934 096 08
6	A_3	0.067 324 707 304 246	–0.168 716 340 714 015
7	A_5	–0.046 100 715 481 456	–0.228 609 862 122 027
8	A_7	–0.032 191 631 417 842	–0.036 843 923 652 902
9	A_2	24 729 021.152 005 015	4 534 725.378 084 867
10	A_4	0.369 809 260 919 236	0.250 467 274 421 501
11	A_6	0.074 577 525 401 252	0.146 297 895 349 135
12	N	6 079 109.580 583 257	6 079 109.580 708 185
13	E	339 433.587 933 6633	339 433.587 935429

As it can be seen from table 3 the differences between the calculated coordinates on the ellipsoid WGS 84 and GRS 80 are ΔN = 0.125 mm and ΔE = 0.001 765 mm, i.e. they do not exceed 1 mm.

Conversion of coordinates PL-UTM (E_{UTM}, N_{UTM}) into geodetic coordinates (φ, λ) — inverse task

1. Calculating geodetic latitude of a point having preset distance from the Equator.

Formulas for calculating geographical latitude (φ_f) of the point whose distance from the Equator is equal to N [1] are as follows:

$$\varphi_f = \omega + (\sin\omega \cdot \cos\omega) \cdot (V_0 + V_2 \cdot \cos^2\omega + V_4 \cdot \cos^4\omega + V_6 \cdot \cos^6\omega), \tag{6}$$

where:

$$\omega = \frac{N - N_0 + S_0}{k_0 \cdot r};$$

$$V_0 = \left\{\left[\left((16384 e'^2 - 11025)\frac{e'^2}{64} + 175\right)\frac{e'^2}{4} - 45\right]\frac{e'^2}{16} + 3\right\}\frac{e'^2}{4};$$

$$V_2 \left\{\left[\left(-\frac{20464721}{120}e'^2 + 19413\right)\frac{e'^2}{8} - 1477\right]\frac{e'^2}{32} + 21\right\}\frac{e'^4}{32}; \tag{7}$$

$$V_4 = \left[\left(\frac{4737141}{28}e'^2 - 17121\right)\frac{e'^2}{32} + 151\right]\frac{e'^6}{192}; \quad V_6 = \left(-\frac{427277}{35}e'^2 + 1097\right)\frac{e'^8}{1024}.$$

Example 2 of calculating geodetic latitude of a position having a preset distance from the Equator.

Data: x_{UTM} (N_{UTM}) = 6 068 800.0 m; y_{UTM} (E_{UTM}) = 33 628 700.0 m; Z = 33
Calculate: $B(\varphi)$ and $L(\lambda)$

Tab. 4. Coefficients of formulas for calculating the geodetic latitude of a position having a preset distance from the Equator based on formulas (4) and (5) [own work]

No.	Parameter symbol	Values of parameter on reference elipsoid	
		GRS 80	WGS 84
1	ω	0,953 478 913 204 910	0,953 478 913 196 950
2	V_0	0,005 022 893 952 697	0,005 022 893 928 107
3	V_2	0,000 029 370 477 707	0,000 029 370 477 420
4	V_4	0,000 000 235 379 396	0,000 000 235 379 393
5	V_6	0,000 000 002 044 364	0,000 000 002 044 364
6	φ_f	54° 45’59,132 233 238 509”	54° 45’59,132 229 195 868”

2. Conversion of coordinates PL-UTM into geodetic coordinates.

Formulas by M. Hooijberg [1] for converting coordinates (N and E) into geodetic coordinates (φ, λ):

$$\varphi = \varphi_f + B_2 \cdot Q^2 \cdot \left[1 + Q^2 \cdot \left(B_4 + B_6 \cdot Q^2\right)\right];$$
$$L = Q \cdot \left\{1 + Q^2 \cdot \left[B_3 + Q^2 \cdot \left(B_5 + B_7 \cdot Q^3\right)\right]\right\}; \qquad (8)$$
$$\lambda = \lambda_0 + \frac{L}{\cos\varphi_f},$$

where:

$$R_f = \frac{k_0 \cdot a}{\sqrt{1 - e^2 \cdot \sin^2\varphi_f}}; \qquad Q = \frac{E'}{E_0} = \frac{E - E_0}{R_f} = \frac{E - 500000}{R_f};$$

$$B_2 = -\frac{1}{2} t_t \left(1 + n_f^2\right); \qquad B_3 = -\frac{1}{6}\left(1 + 2t_f^2 + \eta_f^2\right);$$

$$B_4 = -\frac{1}{12}\left[5 + 3t_f^2 + \eta_f^2\left(1 - 9t_f^2\right) - 4\eta_f^2\right]; \qquad B_5 = \frac{1}{120}\left[5 + 28t_f^2 + 24t_f^4 + \eta_f^2\left(6 + 8t_f^2\right)\right]; \qquad (9)$$

$$B_6 = \frac{1}{360}\left[61 + 90t_f^2 + 45t_f^4 + \eta_f^2\left(46 - 252t_f^2 - 90t_f^2\right)\right];$$

$$B_7 = -\frac{1}{5040}\left(61 + 662t_f^2 + 1320t_f^4 + 720t_f^6\right).$$

The calculation results of converting coordinates PL-UTM into geodetic coordinates of an example position 2 are presented in table 5.

Tab. 5. The calculation results for example 2 based on formulas (8) and (9) [own work]

No.	Parameter symbol	Calculation results on reference ellipsoid	
		GRS 80	WGS 84
1	φ_f	0.955 854 446 623 581	0.955 854 446 603 982
2	R_f	6 389 871.299 448 502	6 389 871.299 377 863
3	Q	0.020 141 259 733 971	0.020 141 250 734 194
4	B_2	−0.709 502 047 681 648	−0.709 502 047 644 404
5	B_4	−0.914 622 029 008 208	−0.914 622 028 982 334
6	B_6	1.167 760 117 174 568	1.167 760 117 074 864
7	B_3	−0.835 230 416 244 711	−0.835 230 416 187 307
8	B_5	1.313 471 456 092 213	1.313 471 455 917 569
9	B_7	−2.478 522 484 389 881	−2.478 522 483 905 727
10	φ	54° 44'59.786 353 317 444" N	54° 44'59.786 349 276 494" N
11	L	0.020 134 430 669 712 83	0.020 134 430 669 935 42
12	λ	016° 59'58.725 759 927 668 83" E	016° 59'58.725 759 807 483 37" E

As it can be seen from table 5 the differences between the calculated coordinates on the ellipsoid WGS 84 and GRS 80 are $\Delta\varphi$ = 0,000 004 041" and $\Delta\lambda$ = 0,000 000 12", i.e. they do not exceed 0.2 mm.

CONVERSION OF GEODETIC COORDINATES INTO PL-UTM COORDINATES AND INVERSLY BASED ON *G-1.10* [2]

The formulas, in most cases, were taken from the technical guidelines *G-1.10 Projection formulas and parameters of coordinate systems* by prof. Roman Kadaj [2] taking into account training material *Conversions and transformations of coordinates* by L. Jaworski, R. Zdunek, A. Świątek [3].

The guidelines recommend triple (and in fact) quadruple PL-UTM projection realization based on the original formulas for projection of ellipsoid into a plane given by C. F. Gauss and modified by Krüger.

Conversion of geodetic coordinates (*B*, *L*) into coordinates PL-UTM (*N*, *E*)

Conversion of geodetic coordinates (*B*, *L*) into coordinates UTM (*N*, *E*) is carried out through combining four conversions (included in the technical guidelines), based on the following algorithms:

$$\begin{bmatrix} B \\ L \end{bmatrix} \Rightarrow (1) \Rightarrow \begin{bmatrix} \varphi \\ \lambda, (\lambda_0) \end{bmatrix} \Rightarrow (2) \Rightarrow \begin{bmatrix} x_{Merk} \\ y_{Merk} \end{bmatrix} \Rightarrow (3) \Rightarrow \begin{bmatrix} x_{GK} \\ y_{GK} \end{bmatrix} \Rightarrow (4) \Rightarrow \begin{bmatrix} N \\ E \end{bmatrix}, \quad (10)$$

where:
stage 1 — projection of ellipsoid on sphere;
stage 2 — projection of sphere on Merkator plane;
stage 3 — projection of Merkator plane on Gauss-Krüger plane;
stage 4 — UTM projection.

Projection (*Lagrange) of ellipsoid on sphere*

The projection involves converting the geodetic latitude B into the spherical latitude φ, without a change in geographical longitude, based on formulas:

$$(B, L) => \{\varphi, \lambda\}; \lambda = L; \lambda_0 = L_0$$

$$\varphi = 2 \cdot \left\{ ATN\left[k(B) \cdot \tan\left(\frac{B}{2} + \frac{\pi}{4} \right) \right] - \frac{\pi}{4} \right\}, \quad (11)$$

where:

$$k(B) = \left(\frac{1 - e \cdot \sin B}{1 + e \cdot \sin B} \right)^{\frac{e}{2}},$$

where:
$k(B)$ — auxiliary value.

Calculation results for example 1:
(B = 54° 50’00” N; L = 018° 30’00” E; L_0 =21°)

Ellipsoid:

WGS 84	GRS 80
φ = 54° 39’07.434 593 404 443 604” N	54° 39’07.434 590 199 378 59” N
$k(B)$= 0.994 534 286 397 814	0.994 534 286 371 103

Projection of sphere on Mercator plane

Transverse cylindrical projection (Mercatora*)* of meridional belt of sphere on the side surface of the tangent cylinder on latitude (λ_o) is carried out using the following formulas:

$(\varphi, \lambda/\lambda_o)$ => $(x, y)_{Mercator}$

$$x_{Merk} = R_0 \cdot ATN\left(\frac{\tan\varphi}{\cos\varphi}\right); \quad y_{Merk} = \frac{R_0}{2}\ln\left(\frac{1+\cos\varphi\cdot\sin\Delta\lambda}{1-\cos\varphi\cdot\sin\Delta\lambda}\right), \tag{12}$$

where:

$\Delta\lambda = \lambda - \lambda_0$;

$$R_0 = \frac{a}{1+n}\cdot\left(1+\frac{n^2}{4}+\frac{n^4}{64}+\frac{n^6}{256}+\frac{25\cdot n^8}{16384}+\ldots\right); \quad n = \frac{a-b}{a+b},$$

where:

R_0 — radius of sphere having the length equal to the length of the meridian (Lagrange radius);

N — third flattening of reference ellipsoid.

The calculation results for example 1 (continued):

	WGS 84	GRS 80
x_{Merk} =	6 076 506,109 517 529	6 076 506,109 368 645
y_{Merk} =	–160 720,331 498 260 55	–160 720,331 500 461 27

Conversion of Mercator projection plane [$(x, y)_{Merk}$] into Gauss-Kruger plane [$(x, y)_{GK}$]

The conversion of Mercator coordinates into Gauss-Krüger coordinates based on two trigonometric polynomials [2, 3] is presented by the following interdependencies:

$(x, y)_{Merk}$ => $(x, y)_{GK}$

$$\begin{aligned} x_{GK} &= x_{Merk} + R_0\cdot[a_2\cdot\sin(2\alpha)\cdot\cosh(2\beta) + a_4\cdot\sin(4\alpha)\cdot\cosh(4\beta) + \\ &\quad + a_6\cdot\sin(6\alpha)\cdot\cosh(6\beta) + a_8\cdot\sin(8\alpha)\cdot\cosh(8\beta) + \ldots] \\ y_{GK} &= y_{Merk} + R_0\cdot[a_2\cdot\cos(2\alpha)\cdot\sinh(2\beta) + a_4\cdot\cos(4\alpha)\cdot\sinh(4\beta) + \\ &\quad + a_6\cdot\cos(6\alpha)\cdot\sinh(6\beta) + a_8\cdot\cos(8\alpha)\cdot\sinh(8\beta) + \ldots], \end{aligned} \tag{13}$$

where: $\alpha = \dfrac{x_{Merk}}{R_0}$; $\beta = \dfrac{y_{Merk}}{R_0}$;

$$\sinh\beta = \frac{e^{\beta} - e^{-\beta}}{2}; \qquad \cosh\beta = \frac{e^{\beta} + e^{-\beta}}{2};$$

$$a_2 = \frac{n}{2} - \frac{2n^2}{3} + \frac{5n^3}{16} + \frac{41n^4}{180} + \ldots; \qquad a_4 = \frac{13n^2}{48} - \frac{3n^3}{5} + \frac{557n^4}{1440} + \ldots;$$
$$a_6 = \frac{61n^3}{240} - \frac{103n^4}{140} + \ldots; \qquad a_8 = \frac{49561n^4}{161280} + \ldots \tag{14}$$

The calculation results based on the formulas above from example 1 (continued):

	WGS 84	GRS 80
x_{GK} =	6 081 542.197 603 935 m	6 081 542.197 455 065
y_{GK} =	–160 630.664 329 579 36 m	–160 630.664 331 7867
α =	0.954 307 756 584 6405	0.954 307 756 569 1066
β =	–0.025 240 928 952 480 35	–0.025 240 928 953 033 55

Tab. 6. Coefficients of trigonometric polynomians [*own work], [**2]

No.	Coefficient	Values of coefficients on reference ellipsoid	
		WGS 84*	GRS 80**
1	a_2	0,837 731 820 628 509 · 10^{-3}	0.837 731 824 7344 · 10^{-3}
2	a_4	0,760 852 771 421 549 · 10^{-6}	0,760 852 778 8826 · 10^{-6}
3	a_6	0,119 763 800 155 261 · 10^{-8}	0,119 763 801 9173 · 10^{-8}
4	a_8	0,244 337 619 450 050 · 10^{-11}	0,244 337 624 2510 · 10^{-11}

Conversion of Gauss-Krüger coordinates into coordinates PL-UTM

$$(x, y)_{GK} \Rightarrow (N, E)_{UTM}$$

The flat rectangular coordinates in the projection PL-UTM (N, E) can be determined on the basis of coordinates in Gauss-Krüger projection (x_{GK}, y_{GK}) from the following interdependence:

$$x_{UTM}\ (N_{UTM}) = k_0 \cdot x_{GK}: \qquad S_{UTM} = k_0 \cdot x_{GK} + 10\,000\,000 \text{ m};$$
$$y_{UTM}\ (E_{UTM}) = k_0 \cdot y_{GK} + 500\,000 \text{m} + z \cdot 1\,000\,000 \text{ m}, \tag{15}$$

where:

k_0 — scale factor in the central meridian:

$k_0 = 0.9996$;

Z — number of projection zone:

$Z = (180 + L)/6$ $Z = 33$ for central meridian 15° E,

$Z = 34$ for central meridian 21° E,

$Z = 35$ for central meridian 27° E.

The calculation results for example 1 (continued):

	WGS 84	GRS 80
x_{UTM} (N_{UTM}) =	6 079 109.580 724 894 m	6 079 109.580 576 084
y_{UTM} (E_{UTM}) =	34 339 433.587 936 15 m	34 339 433.587 933 946
($Z = 34$)		

Conversion of coordinates PL-UTM into geodetic coordinates — inverse task

The conversion (N_{UTM}, E_{UTM}) recommended in the technical guide (*G-1.10*) is carried out with the method of four stages in the order inverse to (10):

$$\begin{bmatrix} N_{UTM} \\ E_{UTM} \end{bmatrix} \Rightarrow (4) \Rightarrow \begin{bmatrix} x_{GK} \\ y_{GK} \end{bmatrix} \Rightarrow (3) \Rightarrow \begin{bmatrix} x_{Merk} \\ y_{Merk} \end{bmatrix} \Rightarrow (2) \Rightarrow \begin{bmatrix} \varphi \\ \lambda, (\lambda_0) \end{bmatrix} \Rightarrow (1) \Rightarrow \begin{bmatrix} B \\ L \end{bmatrix}. \quad (16)$$

Conversion of coordinates PL-UTM (N_{UTM}, E_{UTM}) into Gauss-Krüger coordinates (x_{GK}, y_{GK})

The conversion of coordinates (N_{UTM}, E_{UTM}) into coordinates (x_{GK}, y_{GK}) is based on the following interdependencies:

$$x_{GK} = N_{UTM}/k_0;$$
$$y_{GK} = (E_{UTM} - 500\,000 - z{*}1\,000\,000)/k_0. \quad (17)$$

Solution to example 2:

(N_{UTM} = 6 068 800,0 m; E_{UTM} = 33 628 700.0 m; $Z = 33$)

	WGS 84	GRS 80
x_{GK} =	6 071 228.491 396 5585	6 071 228.491 396 5585
y_{GK} =	128 751.500 600 240 09	128 751.500 600 240 09

Conversion of Gauss-Krüger coordinates (x_{GK}, y_{GK}) into Mercator coordinates (x_{Merk}, y_{Merk})

As in the case of formulas (13) the complex formula is distributed into the following two actual formulas[2, 3]:

$$x_{Merk} = x_{GK} + R_o \cdot [b_2 \cdot \sin(2\alpha) \cdot \cosh(2\beta) + b_4 \cdot \sin(4\alpha) \cdot \cosh(4\beta) + \\ + b_6 \cdot \sin(6\alpha) \cdot \cosh(6\beta) + b_8 \sin(8\alpha) \cdot \cosh(8\beta)]$$
$$y_{Merk} = y_{GK} + R_o \cdot [b_2 \cdot \cos(2\alpha) \cdot \sinh(2\beta) + b_4 \cdot \cos(4\alpha) \cdot \sinh(4\beta) + \\ + b_6 \cdot \cos(6\alpha) \cdot \sinh(6\beta) + b_8 \cos(8\alpha) \cdot \sinh(8\beta)], \quad (18)$$

where:

b_2, b_4, b_6, b_8 — coefficients of trigonometric series in inverse operation:

s => φ

$$b_2 = -\frac{n}{2} + \frac{2 \cdot n^2}{3} - \frac{37 \cdot n^3}{96} + \frac{n^4}{360} n^4 + \ldots \quad b_4 = -\frac{n^2}{48} - \frac{n^3}{15} + \frac{437 \cdot n^4}{1440} + \ldots$$
$$b_6 = -\frac{17 \cdot n^3}{480} + \frac{37 \cdot n^4}{840} + \ldots \quad b_8 = -\frac{4397 \cdot n^4}{161280} + \ldots \quad (19)$$

Tab. 7. Coefficients of trigonometric polynomians [*own work], [**2]

No.	Coefficient	Values of coefficients on reference elipsoid	
		WGS 84*	GRS 80**
1	b_2	$-0.837\,732\,164\,058\,213 \cdot 10^{-3}$	$-0.837\,732\,168\,1641 \cdot 10^{-3}$
2	b_4	$-0.590\,586\,956\,790\,786 \cdot 10^{-7}$	$-0.590\,586\,962\,6083 \cdot 10^{-7}$
3	b_6	$-0.167\,348\,888\,034\,378 \cdot 10^{-9}$	$-0.167\,348\,890\,4988 \cdot 10^{-9}$
4	b_8	$-0.216\,773\,776\,300\,291 \cdot 10^{-12}$	$-0.216\,773\,780\,5597 \cdot 10^{-12}$

The calculation results of example 2:

	WGS 84	GRS 80
x_{Merk} =	6 066 188.959 480 454	6 066 503.959 480 523
y_{Merk} =	128 822.701 898 837 72	128 822.701 898 84091

Conversion of Mercator coordinates into Lagrange coordinates

The conversion of Mercator coordinates (x_{Merk}, y_{Merk}) into Lagrange coordinates (φ, λ):

$$\varphi = ATN\left(\frac{\cos\beta\cdot\sin\alpha}{\sqrt{1-\cos^2\beta\cdot\sin^2\alpha}}\right); \qquad \Delta\lambda = ATN\left(\frac{\tan\beta}{\cos\alpha}\right), \tag{20}$$

where:

$$\alpha = \frac{x_{Merk}}{R_0}; \qquad \beta = 2\cdot ATN\left(e^{\frac{y_{Merk}}{R_0}}\right)-\frac{\pi}{2}.$$

The calculation results of example 2:

WGS 84	GRS 80
α = 0,952 687 460 952 779	0,952 687 460 960 625
β =0,020 231 445 740 456	0,020 231 445 749 623
φ = 54° 34’06,549 823 148 092 173”	54° 34’06,549 824 764 437 631”
$\Delta\lambda$ = 1° 59’58,725 758 687 077 416”	1° 59’58,725 758 835 749 64”

Conversion of spherical coordinates (φ, λ) into geodetic coordinates (B, L)

$$L = \Delta\lambda + L_0; \qquad B = \varphi + c_2\cdot\sin(2\varphi) + c_4\sin(4\varphi) + c_6\cdot\sin(6\varphi) + c_8\sin(8\varphi) + \ldots$$

where:

$$c_2 = 2\cdot n - \frac{2\cdot n^2}{3} - 2\cdot n^3 + \frac{116\cdot n^4}{45} + \ldots; \qquad c_4 = \frac{7\cdot n^2}{3} - \frac{8\cdot n^3}{5} - \frac{227\cdot n^4}{45} + \ldots$$

$$c_6 = \frac{56\cdot n^3}{15} - \frac{136\cdot n^4}{35} = \ldots; \qquad c_8 = \frac{4279\cdot n^4}{630} + \ldots \tag{21}$$

Tab. 8. Coefficients of trigonometric polynomial [*own work], [**2]

No.	Coefficient	Values of coefficients on reference ellipsoid	
		WGS 84*	GRS 80**
1	c_2	$0.335\,655\,146\,911\,7811\cdot10^{-2}$	$0{,}335\,655\,148\,5597\cdot10^{-2}$
2	c_4	$0.657\,187\,308\,393\,1152\cdot10^{-5}$	$0.657\,187\,314\,8459\cdot10^{-5}$
3	c_6	$0.176\,465\,640\,046\,4177\cdot10^{-7}$	$0.176\,465\,642\,6454\cdot10^{-7}$
4	c_8	$0.540\,048\,208\,164\,5885\cdot10^{-10}$	$0.540\,048\,218\,7760\cdot10^{10}$

The calculation results of example 2:

Ellipsoid	WGS 84	GRS 80
	B = 54° 44’59.786 353 057 660 904”	54° 44’59.786 354 670 399 646”
	L = 016° 59’58.725 758 687 074 006”	016° 59’58.725 758 825 740 684”

CONCLUSIONS

Standards adopted in marine navigation and hydrography recommend using use of World Geodetic System of 1984 with the reference ellipsoid WGS 84 (5), (6), and the state system of spatial references obligates using ETRS 89 with the reference ellipsoid GRS 80 (7).

The conceptualization of projection formulas presented by M. Hooijberg (1997) differs from technical guidelines *G-1.10* (2001) presented above, nevertheless the results of calculation examples are comparable, i.e. ΔN = 0,007 mm and ΔE = 0,0003 mm as well as $\Delta\varphi$ = 0,04 mm and $\Delta\lambda$ = 0,002 mm.

There exists a lot of computer software for converting geodetic coordinates into UTM coordinates (usually available commercially). The presented formulas allow for verifying the available software and using it to develop own or complex algorithms in which converting geodetic coordinates into UTM coordinates will be one of the modules.

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PRZELICZANIE WSPÓŁRZĘDNYCH GEODEZYJNYCH NA WSPÓŁRZĘDNE PŁASKIE PL-UTM DLA POTRZEB NAWIGACJI

STRESZCZENIE

W artykule przestawiono zależności matematyczne przekształcenia współrzędnych geodezyjnych na współrzędne płaskie prostokątne PL-UTM i zamiany odwrotnej zalecane postanowieniami rozporządzenia Rady Ministrów w sprawie państwowego systemu odniesień przestrzennych. Przede wszystkim przedstawiono formuły zaczerpnięte z prac M. Hooijberga (1997) *Practical Geodesy* oraz R. Kadaja (2001) *Formuły odwzorowawcze i parametry układów współrzędnych. G-1.10*.

Dla zilustrowania przedstawionych wzorów zamieszczono stosowne przykłady obliczeniowe na elipsoidzie odniesienia WGS 84 i GRS 80.

Słowa kluczowe:

przeliczanie współrzędnych, odwzorowanie Gaussa-Krügera, układ współrzędnych PL-UTM.