

## **Selected classes of positive electrical circuits and their reachability**

Tadeusz Kaczorek

Bialystok University of Technology

15-351 Bialystok, Wiejska 45D, e-mail: kaczorek@isep.pw.edu.pl

Conditions for the positivity of linear electrical circuits composed of resistors, coils, condensators and voltage (current) sources are established. It is shown that: 1) the electrical circuit composed of resistors, coils and voltage source is positive for almost all values of their resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes, 2) the electrical circuit is not positive for any values of its resistances, capacitances and source voltages if each its branch contains resistor, capacitor and voltage source, 3) the positive  $n$ -meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if all resistances of branches belonging to two linearly independent meshes are zero 4) the electrical circuits of the structure shown on Fig.5.2 is reachable if and only if the condition (5.11) is satisfied.

### **1. Introduction**

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [2, 5]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [8, 9]. These notions are the basic concepts of the modern control theory [1, 7, 10, 11, 4]. They have been also extended to positive linear systems [2, 5]. The decomposition of the pair  $(A,B)$  and  $(A,C)$  of the positive discrete-time linear system has been addressed in [3].

The reachability of linear systems is closely related to the controllability of the systems. Specially for positive linear systems the conditions for the controllability are much stronger than for the reachability [5]. Tests for the reachability and controllability of standard and positive linear systems are given in [6, 5].

In this paper the necessary and sufficient conditions for the positivity of linear electrical circuits composed of resistors, coils, condensators and voltage (current) sources will be established. The conditions for the reachability of the positive electrical circuits will be also given.

The paper is organized as follows. In section 2 the conditions for the positivity of  $R, L, e$  type electrical circuits are established. Similar problem for  $R, C, e$  type electrical circuits is discussed in section 3. The reachability of the positive electrical circuits is analyzed in section 4. Concluding remarks are given in section 5.

The following notation will be used:  $\mathfrak{R}$  - the set of real numbers,  $\mathfrak{R}^{n \times m}$  - the set of  $n \times m$  real matrices,  $\mathfrak{R}_+^{n \times m}$  - the set of  $n \times m$  matrices with nonnegative entries and  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ,  $M_n$  - the set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries),  $I_n$  - the  $n \times n$  identity matrix.

## 2. Positive $R, L, e$ electrical circuits

Consider the linear continuous-time system described by the state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (2.1)$$

where  $x(t) \in \mathfrak{R}^n$ ,  $u(t) \in \mathfrak{R}^m$ ,  $y(t) \in \mathfrak{R}^p$  are the state, input and output vectors and  $A \in \mathfrak{R}^{n \times n}$ ,  $B \in \mathfrak{R}^{n \times m}$ ,  $C \in \mathfrak{R}^{p \times n}$ ,  $D \in \mathfrak{R}^{p \times m}$ .

**Definition 2.1.** [2, 5] The system (2.1) is called (internally) positive if for any  $x(0) = x_0 \in \mathfrak{R}_+^n$  and every  $u(t) \in \mathfrak{R}_+^m, t \geq 0$  we have  $x(t) \in \mathfrak{R}_+^n$  and  $y(t) \in \mathfrak{R}_+^p, t \geq 0$ .

**Theorem 2.1.** [2, 5] The system (1) is positive if and only if

$$A \in M_n, B \in \mathfrak{R}_+^{n \times m}, C \in \mathfrak{R}_+^{p \times n}, D \in \mathfrak{R}_+^{p \times m} \quad (2.2)$$

It is well-known [1, 4, 7, 10, 11] that any linear electrical circuit composed of resistors, coils, condensators and voltage (current) sources can be described by the state equations (2.1). Usually as the state variables  $x_1(t), \dots, x_n(t)$  (the components of the state vector  $x(t)$ ) the currents in the coils and voltages on the condensators are chosen.

**Definition 2.2.** An electrical circuit is called (internally) positive if its matrices  $A, B, C, D$  satisfy the conditions (2.2).

In this section the necessary and sufficient conditions will be established under which an electrical circuit is a positive one.

**Example 2.1.** Consider the electrical circuit shown on Figure 2.1 with given resistances  $R_k, k = 1, \dots, 8$ , inductances  $L_1, L_2$  and source voltages  $e_3, e_4, e_8$ .

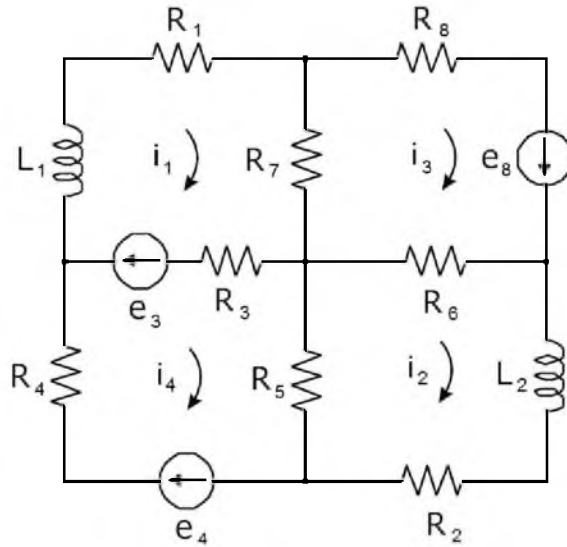


Fig. 2.1. Electrical circuit

Denote by  $i_k$ ,  $k=1, \dots, 4$  the mesh currents and by  $e_{kk}$ ,  $k=1, \dots, 4$  the mesh source voltages. Using the mesh method we obtain the following equations

$$L_1 \frac{di_1}{dt} = -R_{11}i_1 + R_{13}i_3 + R_{14}i_4 + e_{11} \quad (2.3a)$$

$$L_2 \frac{di_2}{dt} = -R_{22}i_2 + R_{23}i_3 + R_{24}i_4$$

$$0 = R_{31}i_1 + R_{32}i_2 - R_{33}i_3 + e_{33} \quad (2.3b)$$

$$0 = R_{41}i_1 + R_{42}i_2 - R_{44}i_4 + e_{44}$$

where

$$\begin{aligned} R_{11} &= R_1 + R_3 + R_7, & R_{13} &= R_{31} = R_7, & R_{14} &= R_{41} = R_3, \\ R_{22} &= R_2 + R_5 + R_6, & R_{23} &= R_{32} = R_6, & R_{24} &= R_{42} = R_5, \\ R_{33} &= R_6 + R_7 + R_8, & R_{44} &= R_3 + R_4 + R_5, \\ e_{11} &= e_3, & e_{33} &= e_8, & e_{44} &= e_4 - e_3. \end{aligned} \quad (2.3c)$$

From (2.3b) we have

$$i_3 = \frac{1}{R_{33}}(R_{31}i_1 + R_{32}i_2 + e_{33}), \quad i_4 = \frac{1}{R_{44}}(R_{41}i_1 + R_{42}i_2 + e_{44}) \quad (2.4)$$

Substitution of (2.4) into (2.3a) yields

$$\begin{aligned} L_1 \frac{di_1}{dt} &= -R'_{11} i_1 + R'_{12} i_2 + e'_{11} \\ L_2 \frac{di_2}{dt} &= R'_{21} i_1 - R'_{22} i_2 + e'_{22} \end{aligned} \quad (2.5a)$$

where

$$\begin{aligned} R'_{11} &= R_{11} - \frac{R_{13}R_{31}}{R_{33}} - \frac{R_{14}R_{41}}{R_{44}}, \quad R'_{12} = R'_{21} = \frac{R_{13}R_{32}}{R_{33}} + \frac{R_{14}R_{42}}{R_{44}}, \quad e'_{11} = e_{11} + \frac{R_{13}}{R_{33}} e_{33} + \frac{R_{14}}{R_{44}} e_{44}, \\ R'_{22} &= R_{22} - \frac{R_{23}R_{32}}{R_{33}} - \frac{R_{24}R_{42}}{R_{44}}, \quad e'_{22} = \frac{R_{23}}{R_{33}} e_{33} + \frac{R_{24}}{R_{44}} e_{44}. \end{aligned} \quad (2.5b)$$

If the mesh currents  $i_1, i_2$  are chosen as the state variables  $x_1 = i_1, x_2 = i_2$  then the equations (2.5a) can be written in the form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \begin{bmatrix} e'_{11} \\ e'_{22} \end{bmatrix} \quad (2.6a)$$

where

$$A = \begin{bmatrix} -\frac{R'_{11}}{L_1} & \frac{R'_{12}}{L_1} \\ \frac{R'_{21}}{L_2} & -\frac{R'_{22}}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \quad (2.6b)$$

From (2.6b) and (2.5b) it follows that  $A$  is the Metzler matrix. If we choose

$$y_1 = R_1 i_1, \quad y_2 = R_8 i_3 - \frac{R_8}{R_{33}} e_{33} \quad (2.7)$$

as the output then

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ \frac{R_8 R_{31}}{R_{33}} & \frac{R_8 R_{32}}{R_{33}} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad (2.8)$$

and

$$C = \begin{bmatrix} R_1 & 0 \\ \frac{R_8 R_{31}}{R_{33}} & \frac{R_8 R_{32}}{R_{33}} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.9)$$

The matrices (2.6b) and (2.9) satisfy the condition (2.2) and by Definition 2.2 the electrical circuit is positive.

In general case let us consider the  $n$ -mesh electrical circuit with some given resistances  $R_k, k = 1, \dots, q$ , inductances  $L_1, \dots, L_r$  and  $m$ -mesh source voltages

$e_{jj}$ ,  $j = 1, \dots, m$ . Let  $i_1, \dots, i_n$  be the mesh currents of the electrical circuit. Using the mesh method in a similar way as for the electrical circuit shown on Fig 2.1 we obtain the equation

$$\begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (2.10a)$$

where

$$x_1 = \begin{bmatrix} i_1 \\ \vdots \\ i_r \end{bmatrix}, \quad x_2 = \begin{bmatrix} i_{r+1} \\ \vdots \\ i_n \end{bmatrix}, \quad u = \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \quad (2.10b)$$

$$A_{11} = \begin{bmatrix} -\frac{R_{11}}{L_1} & \frac{R_{12}}{L_1} & \cdots & \frac{R_{1,r}}{L_1} \\ \frac{R_{21}}{L_2} & -\frac{R_{22}}{L_2} & \cdots & \frac{R_{2,r}}{L_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{R_{r,1}}{L_r} & \frac{R_{r,2}}{L_r} & \cdots & -\frac{R_{r,r}}{L_r} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{R_{1,r+1}}{L_1} & \cdots & \frac{R_{1,n}}{L_1} \\ \vdots & \cdots & \vdots \\ \frac{R_{r,r+1}}{L_r} & \cdots & \frac{R_{r,n}}{L_r} \end{bmatrix}, \quad (2.10b)$$

$$A_{21} = \begin{bmatrix} R_{r+1,1} & \cdots & R_{r+1,r} \\ \vdots & \cdots & \vdots \\ R_{n,1} & \cdots & R_{n,r} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -R_{r+1,r+1} & R_{r+1,r+2} & \cdots & R_{r+1,n} \\ R_{r+2,r+1} & -R_{r+2,r+2} & \cdots & R_{r+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,r+1} & R_{n,r+2} & \cdots & -R_{n,n} \end{bmatrix}$$

$$R_{ji} = R_{ij} = \begin{cases} > 0 & \text{for } i = j \\ \geq 0 & \text{for } i \neq j \end{cases} \quad (2.10c)$$

It is well-known [5] that  $(-A_{22})^{-1} \in \mathfrak{R}_+^{(n-r) \times (n-r)}$  and from (2.10a) we have

$$x_2 = (-A_{22})^{-1}(A_{21}x_1 + B_2u) \in \mathfrak{R}_+^{n-r} \text{ for } x_1 \in \mathfrak{R}_+^r \text{ and } u \in \mathfrak{R}_+^m. \quad (2.11)$$

Substituting (2.11) into

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (2.12)$$

we obtain

$$\dot{x}_1 = [A_{11} + A_{12}(-A_{22})^{-1}A_{21}]x_1 + [B_1 + A_{12}(-A_{22})^{-1}B_2]u = A'_{11}x_1 + B'_1u \quad (2.13)$$

where

$$A'_{11} = A_{11} + A_{12}(-A_{22})^{-1}A_{21} \in M_n, \quad B'_1 = B_1 + A_{12}(-A_{22})^{-1}B_2 \in \mathfrak{R}_+^{n \times m}. \quad (2.14)$$

In what follows it is assumed that by suitable choice of the outputs of the electrical circuit the matrices  $C$  and  $D$  have nonnegative entries, i.e.

$$C \in \mathfrak{R}_+^{p \times n}, \quad D \in \mathfrak{R}_+^{p \times m} \quad (2.15)$$

Therefore, the following theorem has been proved.

**Theorem 2.2.** The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if the number of coils is less or equal to the number of its linearly independent meshes and the direction of the mesh currents are consistent with the directions of the mesh source voltages.

**Example 2.2.** Consider the electrical circuit shown on Figure 2.2 with given resistances  $R_1, R_2, R_3$ , inductances  $L_1, L_2, L_3$  and source voltages  $e_1, e_2$ .

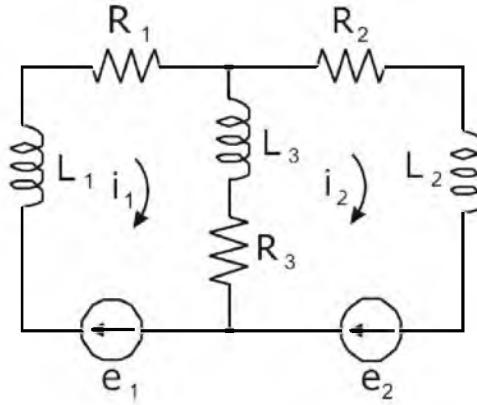


Fig. 2.2. Electrical circuit

Note that in this case the number of coils is greater than the number of linearly independent meshes. Using the mesh method we obtain the following equations

$$\begin{aligned} e_1 &= (R_1 + R_3)i_1 - R_3i_2 + (L_1 + L_3)\frac{di_1}{dt} - L_3\frac{di_2}{dt} \\ e_2 &= -R_3i_1 + (R_2 + R_3)i_2 + (L_2 + L_3)\frac{di_2}{dt} - L_3\frac{di_1}{dt} \end{aligned} \quad (2.16)$$

The equations (2.16) can be written in the form

$$\begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (2.17a)$$

where

$$\begin{aligned} R_{11} &= R_1 + R_3, & R_{12} &= R_{21} = R_3, & R_{22} &= R_2 + R_3, \\ L_{11} &= L_1 + L_3, & L_{12} &= L_{21} = L_3, & L_{22} &= L_2 + L_3. \end{aligned} \quad (2.17b)$$

Note that the inverse matrix

$$L^{-1} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{bmatrix} \quad (2.18)$$

has positive entries.

From (2.17) we have

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (2.19)$$

where

$$A = L^{-1} \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} -L_2(R_1 + R_3) - L_3R_1 & L_2R_3 - L_3R_2 \\ L_1R_3 - L_3R_1 & -L_1(R_2 + R_3) - L_3R_2 \end{bmatrix}$$

$$B = L^{-1} \in \mathfrak{R}_+^{2 \times 2}. \quad (2.20)$$

Note that the matrix  $A \in M_2$  if and only if

$$L_2R_3 \geq L_3R_2 \text{ and } L_1R_3 \geq L_3R_1. \quad (2.21)$$

Therefore, the electrical circuit is positive if and only if  $A \in M_2$  i.e. the condition (2.21) is met.

In general case let us consider the  $n$ -mesh electrical circuit with given resistances  $R_k$ ,  $k = 1, \dots, q$ , inductances  $L_1, \dots, L_r$  for  $r \geq n$  and  $m \leq n$  mesh source voltages  $e_{jj}$ ,  $j = 1, \dots, m$ . Denote by  $i_1, \dots, i_n$  the mesh currents. In a similar way as for the electrical circuit shown on Fig 2.2 using the mesh method we obtain the equation

$$L \frac{d}{dt} \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A' \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix} \quad (2.22a)$$

where

$$L = \begin{bmatrix} L_{11} & -L_{12} & \dots & -L_{1,n} \\ -L_{21} & L_{22} & \dots & -L_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{n,1} & -L_{n,2} & \dots & L_{nn} \end{bmatrix}, \quad A' = \begin{bmatrix} -R_{11} & R_{12} & \dots & R_{1,n} \\ R_{21} & -R_{22} & \dots & R_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,1} & R_{n,2} & \dots & -R_{nn} \end{bmatrix}. \quad (2.22b)$$

Note that  $-L \in M_n$ ,  $A' \in M_n$  and  $L^{-1} \in \mathfrak{R}_+^{n \times n}$ .

Premultiplying (2.22a) by  $L^{-1}$  we obtain

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + B \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix} \quad (2.23a)$$

where

$$A = L^{-1}A', \quad B = L^{-1} \in \mathfrak{R}_+^{n \times n}. \quad (2.23b)$$

The electrical circuit is positive if and only if the matrix  $L^{-1}A'$  is a Metzler matrix, i.e.

$$L^{-1}A' \in M_n. \quad (2.24)$$

By suitable choice of the outputs of the electrical circuit we have (2.15). Therefore, the following theorem has been proved.

**Theorem 2.3.** The linear electrical circuit composed of resistors, coils and voltage sources is positive for  $r > n$  if its resistances and inductances satisfy the condition (2.24).

**Remark 2.1.** In the case  $r = n$  if it is possible to choose the  $n$  linearly independent meshes so that to each mesh belongs only one coil. Then the matrix  $L = \text{diag}[L_1 \ \dots \ L_n]$  and the condition (2.24) is met for almost all values of the resistances and inductances of the electrical circuit.

**Remark 2.2.** Note that it is impossible to choose the  $n$  linearly independent meshes so that to each mesh belongs only one coil if all branches belonging to the same node contain the coils. In this case we can eliminate one of the branch currents using the fact that the sum of the currents in the coils is equal to zero.

From Theorem 2.2 and 2.3 and Remark 2.1 we have the following important theorem.

**Theorem 2.4.** The linear electrical circuit composed of resistors, coils and voltage sources is positive for almost all values of the resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes and the directions of the mesh currents are consistent with the directions of the mesh source voltages.

### 3. Positive $R, C, e$ electrical circuits

Now let us consider the linear electrical circuits composed of resistors, condensators and source voltages.



**Example 3.1.** Consider the electrical circuit shown on Figure 3.1 with given conductances  $G_k$ ,  $k=1,2,3$ , capacitances  $C_1, C_2, C_3$  and source voltages  $e_1, e_2, e_3$ .

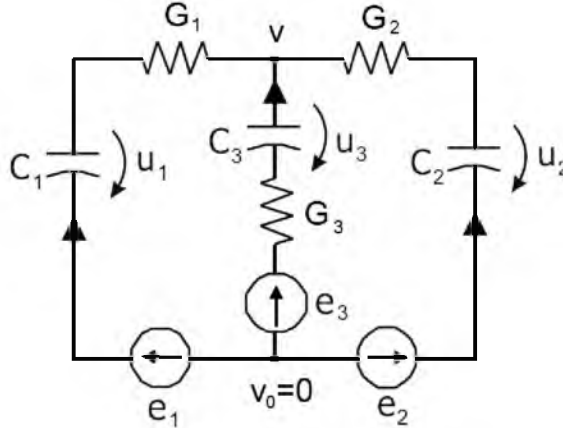


Fig. 3.1. Electrical circuit

Using the Kirchhoff's laws we may write the equations

$$\begin{aligned} C_1 \frac{du_1}{dt} &= G_1(e_1 - u_1 - v) \\ C_2 \frac{du_2}{dt} &= G_2(e_2 - u_2 - v) \\ C_3 \frac{du_3}{dt} &= G_3(e_3 - u_3 - v) \end{aligned} \quad (3.1a)$$

and

$$G_3(e_3 - u_3 - v) + G_1(e_1 - u_1 - v) + G_2(e_2 - u_2 - v) = 0 \quad (3.1b)$$

From (3.1b) we have

$$v = \frac{G_1(e_1 - u_1) + G_2(e_2 - u_2) + G_3(e_3 - u_3)}{G_1 + G_2 + G_3} \quad (3.2)$$

Substitution of (3.2) into (3.1a) yields

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + B_3 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad (3.3a)$$

where

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad (3.3b)$$

and

$$\begin{aligned} a_{11} &= -\frac{G_1(G_2 + G_3)}{C_1(G_1 + G_2 + G_3)}, & a_{12} &= \frac{G_1G_2}{C_1(G_1 + G_2 + G_3)}, & a_{13} &= \frac{G_1G_3}{C_1(G_1 + G_2 + G_3)}, \\ a_{21} &= \frac{G_1G_2}{C_2(G_1 + G_2 + G_3)}, & a_{22} &= -\frac{G_2(G_1 + G_3)}{C_2(G_1 + G_2 + G_3)}, & a_{23} &= \frac{G_2G_3}{C_2(G_1 + G_2 + G_3)}, \\ a_{31} &= \frac{G_1G_3}{C_3(G_1 + G_2 + G_3)}, & a_{32} &= \frac{G_2G_3}{C_3(G_1 + G_2 + G_3)}, & a_{33} &= -\frac{G_3(G_1 + G_2)}{C_3(G_1 + G_2 + G_3)}, \end{aligned} \quad (3.4a)$$

$$\begin{aligned} b_{11} &= \frac{G_1(G_2 + G_3)}{C_1(G_1 + G_2 + G_3)}, & b_{12} &= -\frac{G_1G_2}{C_1(G_1 + G_2 + G_3)}, & b_{13} &= -\frac{G_1G_3}{C_1(G_1 + G_2 + G_3)}, \\ b_{21} &= -\frac{G_1G_2}{C_2(G_1 + G_2 + G_3)}, & b_{22} &= \frac{G_2(G_1 + G_3)}{C_2(G_1 + G_2 + G_3)}, & b_{23} &= -\frac{G_2G_3}{C_2(G_1 + G_2 + G_3)}, \\ b_{31} &= -\frac{G_1G_3}{C_3(G_1 + G_2 + G_3)}, & b_{32} &= -\frac{G_2G_3}{C_3(G_1 + G_2 + G_3)}, & b_{33} &= \frac{G_3(G_1 + G_2)}{C_3(G_1 + G_2 + G_3)}, \end{aligned} \quad (3.4b)$$

**Remark 3.1.** From (3.4) we have

$$1_3^T (\text{diag}[C_1 \ C_2 \ C_3])A_3 = 0 \quad \text{and} \quad 1_3^T (\text{diag}[C_1 \ C_2 \ C_3])B_3 = 0 \quad (3.5)$$

where  $1_3^T = [1 \ 1 \ 1]$ . The equations (3.5) follow from the first Kirchhoff's law

$$C_1 \frac{du_1}{dt} + C_2 \frac{du_2}{dt} + C_3 \frac{du_3}{dt} = 0. \quad (3.6)$$

From (3.3) and (3.4) it follows that the electrical circuit shown on Fig. 3.1 is not positive for any nonzero its parameters  $C_k, G_k, k=1,2,3$  and source voltages  $e_k, k=1,2,3$ .

The electrical circuit is positive if  $C_3 = 0$  and  $e_1 = e_2 = 0$ . In this case we have

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_2 e_3 \quad (3.7a)$$

where

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2 \text{ and } B_2 = \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \quad (3.7b)$$

In general case for the linear electrical circuits with  $n$  branches we have the following theorem.

**Theorem 3.1.** The linear electrical circuit is not positive for almost all values of its resistances, capacitances and source voltages if each its branch contains resistor, condensator and voltage source .

**Proof.** From the first Kirchhoff's law we have (Remark 3.1)

$$I_n^T (\text{diag}[C_1 \dots C_n])B = 0 \quad (3.8)$$

where  $C_i$  is the capacitance of  $i$ th ( $i=1, \dots, n$ ) branch. The equality (3.8) implies that some entries of the matrix  $B$  are negative. Therefore, the electrical circuit is not positive.  $\square$

Consider the electrical circuit shown on Figure 3.2 with given conductances  $G_k$ ,  $k=0,1,\dots,n$ ; capacitances  $C_j$ ,  $j=1,\dots,n$  and source voltage  $e$ .

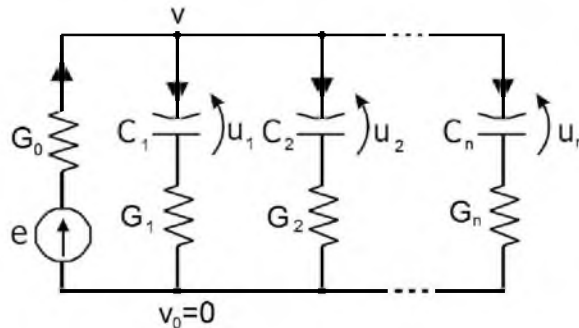


Fig. 3.2. Electrical circuit.

Using the Kirchhoff's laws we may write the equations

$$G_0(e - v) = \sum_{j=1}^n G_j(v - u_j) \quad (3.9a)$$

and

$$C_k \frac{du_k}{dt} = G_k(v - u_k), \quad k = 1, \dots, n \quad (3.9b)$$

From (3.9a) we have

$$v = \frac{1}{G} \left( G_0 e + \sum_{j=1}^n G_j u_j \right), \quad G = \sum_{i=0}^n G_i \quad (3.10)$$

Substitution of (3.10) into (3.9b) yields

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + B e \quad (3.11a)$$

where

$$A = \begin{bmatrix} -\frac{G_1 G - G_1^2}{C_1 G} & \frac{G_1 G_2}{C_1 G} & \cdots & \frac{G_1 G_n}{C_1 G} \\ \frac{G_2 G_1}{C_2 G} & -\frac{G_2 G - G_2^2}{C_2 G} & \cdots & \frac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n G_1}{C_n G} & \frac{G_n G_2}{C_n G} & \cdots & -\frac{G_n G - G_n^2}{C_n G} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{G_0 G_1}{C_1 G} \\ \vdots \\ \frac{G_0 G_n}{C_n G} \end{bmatrix} \quad (3.11b)$$

From (3.11) it follows that  $A \in M_n$  and  $B \in \mathfrak{R}_+^n$ . Therefore, the electrical circuit is positive for all values of the conductances  $G_k$ ,  $k = 0, 1, \dots, n$ ; all values of capacitances  $C_j$ ,  $j = 1, \dots, n$  and source voltage  $e$ . The following theorem has been proved.

**Theorem 3.2.** The electrical circuit shown on Fig. 3.2 is positive for any values of the conductances  $G_k$ ,  $k = 0, 1, \dots, n$ ; capacitances  $C_j$ ,  $j = 1, \dots, n$  and source voltage  $e$ .

**Example 3.2.** Consider the electrical circuit shown on Figure 3.3 with given conductances  $G_k$ ,  $k = 1, \dots, 6$ , capacitances  $C_1, C_2$  and source voltages  $e_2, e_4$ .

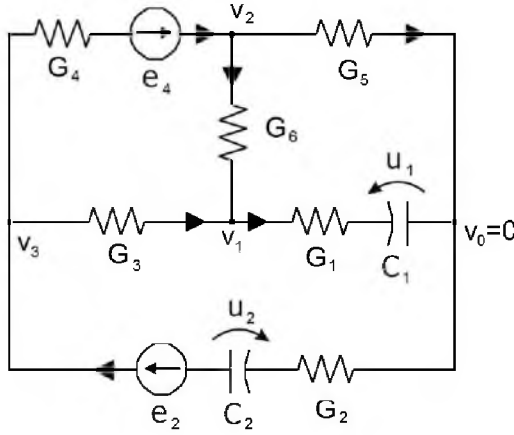


Fig. 3.3. Electrical circuit.

Using the Kirchhoff's laws we may write the equations

$$\begin{aligned} C_1 \frac{du_1}{dt} &= G_1(v_1 - u_1) \\ C_2 \frac{du_2}{dt} &= G_2(e_2 - u_2 - v_3) \end{aligned} \quad (3.12a)$$

and

$$\begin{aligned} G_1 u_1 - (G_1 + G_3 + G_6) v_1 + G_6 v_2 + G_3 v_3 &= 0 \\ G_6 v_1 - (G_4 + G_5 + G_6) v_2 + G_4 v_3 + G_4 e_4 &= 0 \\ -G_2 u_2 + G_3 v_1 + G_4 v_2 - (G_2 + G_3 + G_4) v_3 + G_2 e_2 - G_4 e_4 &= 0 \end{aligned} \quad (3.12b)$$

The equations (3.12) can be rewritten in the form

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{G_1}{C_1} & 0 \\ 0 & -\frac{G_2}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \frac{G_1}{C_1} & 0 & 0 \\ 0 & 0 & -\frac{G_2}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{G_2}{C_2} & 0 \end{bmatrix} \begin{bmatrix} e_2 \\ e_4 \end{bmatrix} \quad (3.13a)$$

and

$$G \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = - \begin{bmatrix} G_1 & 0 \\ 0 & 0 \\ 0 & -G_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & G_4 \\ G_2 & -G_4 \end{bmatrix} \begin{bmatrix} e_2 \\ e_4 \end{bmatrix} \quad (3.13b)$$

where the matrix

$$G = \begin{bmatrix} -(G_1 + G_3 + G_6) & G_6 & G_3 \\ G_6 & -(G_4 + G_5 + G_6) & G_4 \\ G_3 & G_4 & -(G_2 + G_3 + G_4) \end{bmatrix} \quad (3.14)$$

is a Metzler matrix and  $-G^{-1} \in \mathfrak{R}_+^{3 \times 3}$ . From (3.13b) we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -G^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & 0 \\ 0 & -G_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - G^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G_4 \\ G_2 & -G_4 \end{bmatrix} \begin{bmatrix} e_2 \\ e_4 \end{bmatrix} \quad (3.15)$$

Substituting (3.15) into (3.13a) we obtain

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} e_2 \\ e_4 \end{bmatrix} \quad (3.16)$$

where

$$A = \begin{bmatrix} -\frac{G_1}{C_1} & 0 \\ 0 & -\frac{G_2}{C_2} \end{bmatrix} - \begin{bmatrix} \frac{G_1}{C_1} & 0 & 0 \\ 0 & 0 & -\frac{G_2}{C_2} \end{bmatrix} G^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & 0 \\ 0 & -G_2 \end{bmatrix}, \quad (3.17)$$

$$B = \begin{bmatrix} 0 & 0 \\ \frac{G_2}{C_2} & 0 \end{bmatrix} - \begin{bmatrix} \frac{G_1}{C_1} & 0 & 0 \\ 0 & 0 & -\frac{G_2}{C_2} \end{bmatrix} G^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G_4 \\ G_2 & -G_4 \end{bmatrix}$$

The electrical circuit is positive if and only if the matrix  $A$  is the Metzler matrix and the matrix  $B$  has nonnegative entries.

In general case let us consider the electrical circuit composed of  $q$  conductances  $G_k$ ,  $k=1, \dots, q$ ,  $r$  capacitances  $C_i$ ,  $i=1, \dots, r$  and  $m$  source voltages  $e_j$ ,  $j=1, \dots, m$ . Let  $n$  be the number of linearly independent nodes of the electrical circuit and  $n > r$ .

In a similar way as in Example 3.2 we may write the equation

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A_r \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + A_n \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + B_m \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \quad (3.18)$$

where  $u_i$  is the  $i$ -th ( $i=1, \dots, r$ ) voltage on the condenser,  $v_j$  is the voltage of the  $j$ -th node ( $j=1, \dots, n$ ),  $A_r \in \mathfrak{R}^{r \times r}$  is the diagonal Metzler matrix,  $A_n \in \mathfrak{R}^{r \times n}$  and  $B_m \in \mathfrak{R}^{r \times m}$ .

Using the node method we obtain

$$G \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -F \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - H \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \quad (3.19)$$

where  $G \in \mathfrak{R}^{n \times n}$  is a Metzler matrix  $F \in \mathfrak{R}^{n \times r}$  and  $H \in \mathfrak{R}^{n \times m}$ .

Taking into account that  $-G^{-1} \in \mathfrak{R}_+^{n \times n}$  from (3.19) we obtain

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G^{-1}F \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - G^{-1}H \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}. \quad (3.20)$$

Substitution of (3.20) into (3.18) yields

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + B \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \quad (3.21)$$

where

$$A = A_r - A_n G^{-1} F, \quad B = B_m - A_n G^{-1} H. \quad (3.22)$$

The electrical circuit described by the equation (3.21) is positive if and only if (under the assumption  $C \in \mathfrak{R}_+^{p \times r}$ ,  $D \in \mathfrak{R}_+^{p \times m}$ ) the matrix  $A$  is a Metzler matrix and the matrix  $B$  has nonnegative entries. Therefore, the following theorem has been proved.

**Theorem 3.3.** The linear electrical circuit composed of  $q$  resistors,  $r$  condensators and  $m$  voltage sources is positive if and only if  $r < n$  and

$$A_r - A_n G^{-1} F \in M_r, \quad B_m - A_n G^{-1} H \in \mathfrak{R}_+^{r \times m}. \quad (3.23)$$

#### 4. R, L, C type electrical circuits

Consider the electrical circuit shown on Figure 4.1 with given resistance  $R$ , inductance  $L$ , capacitance  $C$  and source voltage  $e$ .

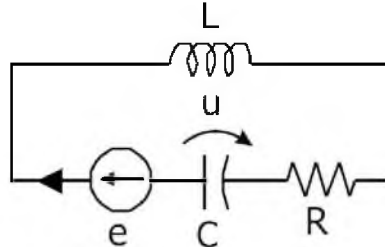


Fig. 4.1. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$\begin{aligned} i &= C \frac{du}{dt} \\ e &= Ri + L \frac{di}{dt} + u \end{aligned} \tag{4.1}$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be \tag{4.2}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{4.3}$$

The matrix  $A$  has negative off-diagonal entry  $(-1/L)$  and it is not a Metzler matrix for any values of  $R, L, C$ . Therefore, the electrical circuit is not positive one for any values of the resistances  $R$ , inductance  $L$ , capacitance  $C$ .

In general case we have the following theorem.

**Theorem 4.1.** The electrical circuits of  $R, L, C$  type are not positive for any values of its resistances, inductances, capacitances and source voltages if at least one its branch contains coil and condensator.

**Proof.** It is well-known that the linear independent meshes of the electrical circuits can be chosen so that the branch containing the inductance  $L$  and capacitance  $C$  belongs to the first one. The equation for the first mesh contains the following term

$$e_{11} = L \frac{di_1}{dt} + u_1 + \dots \tag{4.4}$$

where  $e_{11}$  and  $i_1$  are the source voltage and current of the first mesh and  $u_1$  is the voltage on the capacitance  $C$ . From (4.4) and  $i_1 = C \frac{du_1}{dt}$  it follows that the matrix

$A$  of the electrical circuit has at least one negative off-diagonal entry. Therefore the matrix  $A$  is not a Metzler matrix and the electrical circuit is not positive one.  $\square$

Consider the electrical circuit shown on Figure 4.2 with given resistances  $R_1, R_2$ , inductance  $L$ , capacitance  $C$  and source voltages  $e_1, e_2$ .



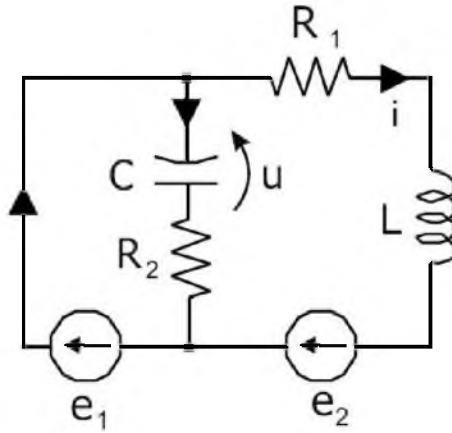


Fig. 4.2. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_2 C \frac{du}{dt} + u \quad (4.5)$$

$$e_1 + e_2 = R_1 i + L \frac{di}{dt}$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (4.6)$$

where

$$A = \begin{bmatrix} -\frac{1}{R_2 C} & 0 \\ 0 & -\frac{R_1}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_2 C} & 0 \\ \frac{1}{L} & \frac{1}{L} \end{bmatrix}. \quad (4.7)$$

From (4.7) it follows that the matrix  $A$  is a Metzler matrix and the matrix  $B$  has nonnegative entries. Therefore, the  $R, L, C$  type electrical circuit is positive for any values of the resistances  $R_1, R_2$ , inductance  $L$  and capacitance  $C$ .

Consider the electrical circuit shown on Figure 4.3 with given resistances  $R_k, k = 1, \dots, 8$ , inductances  $L_2, L_4, L_6, L_8$ , capacitances  $C_1, C_3, C_5, C_7$  and source voltages  $e_1, e_2, e_4, e_6, e_8$ .

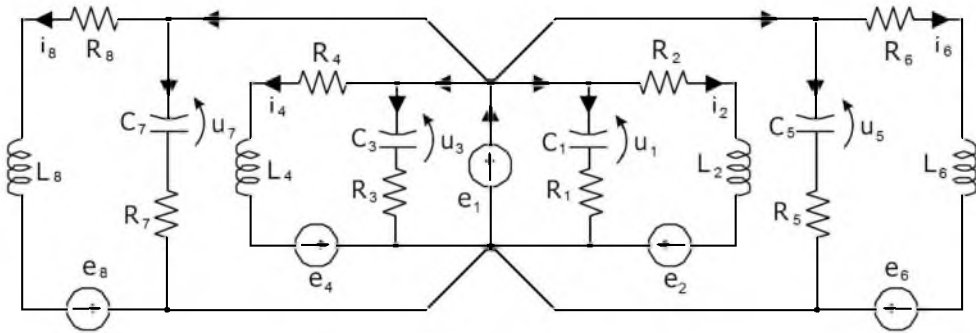


Fig. 4.3. Electrical circuit.

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_k C_k \frac{du_k}{dt} + u_k \quad \text{for } k = 1, 3, 5, 7 \quad (4.8)$$

$$e_1 + e_j = L_j \frac{di_j}{dt} + R_j i_j \quad \text{for } j = 2, 4, 6, 8 \quad (4.9)$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ i_2 \\ i_4 \\ i_6 \\ i_8 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ i_2 \\ i_4 \\ i_6 \\ i_8 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ e_6 \\ e_8 \end{bmatrix} \quad (4.10)$$

where

$$A = \text{diag}\left[-\frac{1}{R_1 C_1}, -\frac{1}{R_3 C_3}, -\frac{1}{R_5 C_5}, -\frac{1}{R_7 C_7}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, -\frac{R_6}{L_6}, -\frac{R_8}{L_8}\right],$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{R_1 C_1} & 0 & 0 & 0 & 0 \\ \frac{1}{R_3 C_3} & 0 & 0 & 0 & 0 \\ \frac{1}{R_5 C_5} & 0 & 0 & 0 & 0 \\ \frac{1}{R_7 C_7} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & 0 & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & 0 & 0 \\ \frac{1}{L_6} & 0 & 0 & \frac{1}{L_6} & 0 \\ \frac{1}{L_8} & 0 & 0 & 0 & \frac{1}{L_8} \end{bmatrix} \quad (4.11)$$

From (4.11) it follows that the matrix  $A$  is a Metzler matrix and the matrix  $B$  has nonnegative entries. Therefore, the  $R, L, C$  type electrical circuit is positive for any values of the resistances, inductances and capacitances.

In general case the equation (4.10) has the form

$$\frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + B e \quad (4.12)$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ \vdots \\ e_n \end{bmatrix} \quad (4.13a)$$

and

$$A = \text{diag}\left[-\frac{1}{R_1 C_1}, -\frac{1}{R_3 C_3}, \dots, -\frac{1}{R_{n_1} C_{n_1}}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, \dots, -\frac{R_{n_2}}{L_{n_2}}\right] \in M_n \quad (4.13b)$$

$n_1$  and  $n_2$  are odd and even numbers,

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathfrak{R}_+^{n \times \left(\frac{n_2+1}{2}\right)}, \quad B_1 = \begin{bmatrix} \frac{1}{R_1 C_1} & 0 & 0 & 0 & 0 \\ \frac{1}{R_3 C_3} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{R_{n_1} C_{n_1}} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \dots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \dots & \frac{1}{L_{n_2}} \end{bmatrix} \quad (4.13c)$$

The electrical circuit described by the equation (4.12) is positive for all value of the resistances  $R_k, k = 1, \dots, n$ , inductances  $L_k, k = 2, 4, \dots, n_2$ , capacitances  $C_k, k = 1, 3, \dots, n_1$ . Therefore, the following theorem has been proved.

**Theorem 4.2.** The linear electrical circuit of the structure shown on Fig. 4.3 for  $n_1 = n_2 = 4$  is positive for any values of its resistances, inductances and capacitances.

Consider the electrical circuit shown on Figure 4.4 with given resistances  $R_k, k = 1, \dots, 5$ , inductances  $L_1, L_2$ , capacitance  $C$  and source voltage  $e$ .

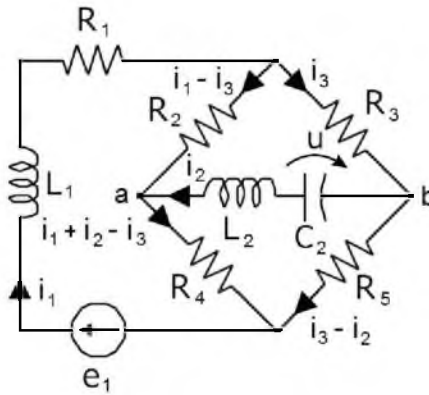


Fig. 4.4. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$\begin{aligned}
 e_1 &= R_1 i_1 + L_1 \frac{di_1}{dt} - R_5 i_2 + (R_3 + R_5) i_3 \\
 L_2 \frac{di_2}{dt} + u + (R_2 + R_3) i_3 - R_2 i_1 &= 0 \\
 C \frac{du}{dt} &= i_2
 \end{aligned}
 \tag{4.14a}$$

and

$$(R_2 + R_4) i_1 + (R_4 + R_5) i_2 - (R_2 + R_3 + R_4 + R_5) i_3 = 0
 \tag{4.14b}$$

From (4.14b) we have

$$i_3 = \frac{(R_2 + R_4) i_1 + (R_4 + R_5) i_2}{R_2 + R_3 + R_4 + R_5}
 \tag{4.15}$$

Substituting (4.15) into (4.14a) we obtain

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + Be \quad (4.16)$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} - \frac{(R_2 + R_4)(R_3 + R_5)}{L_1(R_2 + R_3 + R_4 + R_5)} & \frac{R_2R_5 - R_3R_4}{L_1(R_2 + R_3 + R_4 + R_5)} & 0 \\ \frac{R_2R_5 - R_3R_4}{L_2(R_2 + R_3 + R_4 + R_5)} & -\frac{(R_2 + R_3)(R_4 + R_5)}{L_2(R_2 + R_3 + R_4 + R_5)} & -\frac{1}{L_2} \\ 0 & \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}. \quad (4.17)$$

From (4.17) it follows that the matrix  $A$  is not a Metzler matrix if

$$R_2R_5 = R_3R_4 \quad (4.18)$$

If the condition (4.18) is met then the voltage between the points  $a, b$  is equal to zero and  $u_2 = 0$ ,  $L_2 \frac{di_2}{dt} = 0$ ,  $i_2 = 0$ . In this case the equation (4.16) takes the form

$$\frac{di_1}{dt} = \left( -\frac{R_1}{L_1} - \frac{(R_2 + R_4)(R_3 + R_5)}{L_1(R_2 + R_3 + R_4 + R_5)} \right) i_1 + \frac{1}{L_1} e_1 \quad (4.19)$$

The system described by the equation (4.19) is positive. Therefore, we have the following corollary.

**Corollary 4.1.** If the resistances of the electrical circuit satisfy the condition (4.18) then the electrical circuit is positive.

In general case we have.

**Corollary 4.2.** Nonpositive electrical circuit for some special choice of the parameters (resistances) can be positive one.

## 5. Reachability of standard and positive electrical circuits

Consider the positive electrical circuit described by the equations (2.1).

**Definition 5.1.** The electrical circuit described by the equations (2.1) (or the pair  $(A, B)$ ) is called reachable in time  $t_f$  if for any given final state  $x_f \in \mathfrak{R}^n$  there exists

an input  $u(t) \in \mathfrak{R}^m$ , for  $t \in [0, t_f]$  that steers the state of the circuit from zero initial state  $x(0) = x_0 = 0$  to the state  $x_f$ , i.e.  $x(t_f) = x_f$ .

**Theorem 5.1.** The electrical circuit (2.1) (or the pair  $(A, B)$ ) is reachable if and only if one of the following conditions is satisfied:

- i)  $\text{rank}[B \ AB \ \dots \ A^{n-1}B] = n$ ,
- ii)  $\text{rank}[I_n s - A \ B] = n$  for  $s \in \delta_A = \{s_1, \dots, s_n\}$

where  $s_1, \dots, s_n$  are the eigenvalues (not necessary distinct) of the matrix  $A$ .

Proof is given in [1, 4, 7, 9].

**Definition 5.2.** The positive electrical circuit (2.1) (or the positive pair  $(A, B)$ ) is called reachable in time  $t_f$  if for any given final state  $x_f \in \mathfrak{R}_+^n$  there exists an input  $u(t) \in \mathfrak{R}_+^m$ , for  $t \in [0, t_f]$  that steers the state of the circuit from zero initial state  $x(0) = 0$  to the state  $x_f$ , i.e.  $x(t_f) = x_f$ .

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

**Theorem 5.2.** The positive electrical circuit (2.1) is reachable if the matrix

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau, \quad t_f > 0 \quad (5.1)$$

is monomial. The input that steers the state of the electrical circuit in time  $t_f$  from  $x(0) = x_0 = 0$  to the state  $x_f$  is given by the formula

$$u(t) = B^T e^{A^T(t_f-t)} R_f^{-1} x_f \quad \text{for } t \in [0, t_f]. \quad (5.2)$$

The proof is given in [5].

**Example 5.1.** Consider the electrical circuit shown on Figure 5.1 with given resistances  $R_1, R_2, R_3$ , inductances  $L_1, L_2$  and source voltages  $e_1, e_2$ .

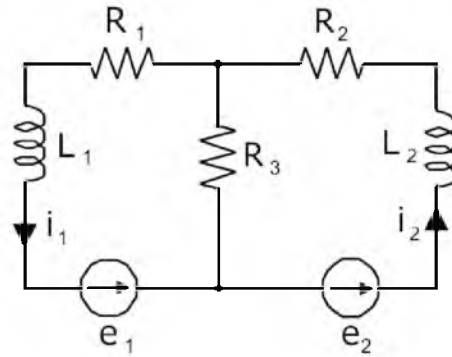


Fig. 5.1. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{di_1}{dt} \tag{5.3}$$

$$e_2 = R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{di_2}{dt}$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{5.4a}$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{5.4b}$$

The electrical circuit is positive since the matrix  $A$  is Metzler matrix and the matrix  $B$  has nonnegative entries. Note that by Theorem 5.1 the standard pair (5.4b) is reachable since  $\det B \neq 0$ .

We shall show that the positive electrical circuit is reachable if  $R_3 = 0$ . In this case

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} \tag{5.5}$$

and

$$e^{A\tau} = \begin{bmatrix} e^{-\frac{R_1}{L_1}\tau} & 0 \\ 0 & e^{-\frac{R_2}{L_2}\tau} \end{bmatrix} \quad (5.6)$$

from (5.1) we obtain

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{-A^T \tau} d\tau = \int_0^{t_f} \begin{bmatrix} \frac{1}{L_1^2} e^{-\frac{2R_1}{L_1}\tau} & 0 \\ 0 & \frac{1}{L_2^2} e^{-\frac{2R_2}{L_2}\tau} \end{bmatrix} d\tau. \quad (5.7)$$

The matrix (5.7) is monomial and by Theorem 5.2 the positive electrical circuit is reachable if  $R_3 = 0$ .

Now let us consider the  $n$ -mesh electrical circuit with given resistances  $R_k$ ,  $k = 1, \dots, q$ , inductances  $L_i$ ,  $i = 1, \dots, n$  and  $m$ -mesh source voltages  $e_{jj}$ ,  $j = 1, \dots, m$ . It is assumed that to each linearly independent mesh belongs only one inductance. In this case the matrix  $L$  defined by (2.22b) is diagonal one and the condition (2.24) is met. By Theorem 5.1 the standard electrical circuit is reachable since  $\det B \neq 0$ .

**Theorem 5.3.** The positive  $n$ -meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if

$$R_{ij} = 0 \text{ for } i \neq j, \quad i, j = 1, \dots, n \quad (5.8)$$

where  $R_{ij}$  are entries of the matrix  $A'$  defined by (2.22b).

**Proof.** If the condition (5.8) is met then the Metzler matrix  $A'$  is diagonal. The matrix  $L$  defined by (2.22b) is also diagonal since by assumption only one inductance belongs to each linearly independent mesh. In this case the matrix  $A = L^{-1}A'$  is diagonal Metzler matrix and  $B = L^{-1} \in \mathfrak{R}_+^{n \times n}$  is also diagonal. For diagonal Metzler matrix  $A$  and diagonal  $B$  the matrix  $e^{A\tau} B$  is also diagonal and the matrix  $R_f$  defined by (5.1) is monomial. By Theorem 5.1 the positive electrical circuit is reachable.  $\square$

**Remark 5.1.** The condition (5.8) is met if the resistance of the branch belonging to two linearly independent meshes is zero. This result is consistent with the one obtained in Example 5.1.



Consider the electrical circuit shown on Figure 5.2 with given conductances  $G_k, G'_k, G_{kj}$   $k, j=1, \dots, n$ ; capacitances  $C_k$ ,  $k=1, \dots, n$  and source voltages  $e_k$ ,  $k=1, \dots, n$ .

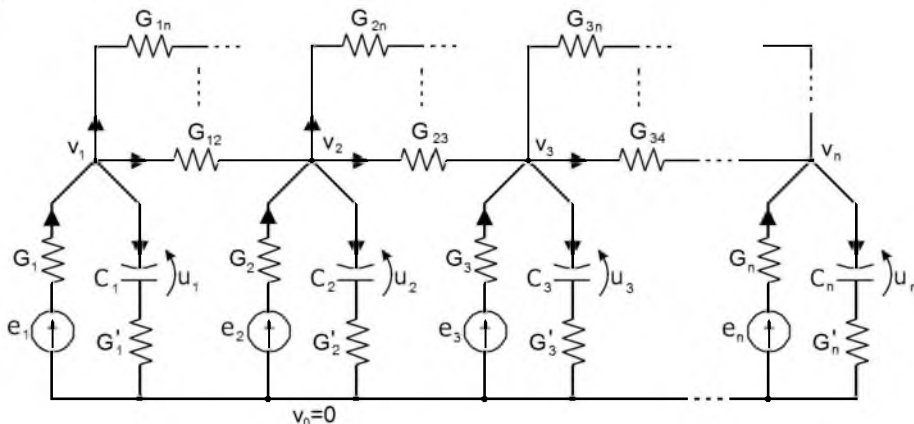


Fig. 5.2. Electrical circuit.

**Theorem 5.4.** The electrical circuit shown on Fig. 5.2 is positive for all values of the conductances, capacitances and source voltages.

**Proof.** Using the Kirchhoff's laws and the node method for the electrical circuit we may write the equations

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -C^{-1}G' \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + C^{-1}G \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad (5.9a)$$

and

$$\overline{G} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G' \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} - G \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \quad (5.9b)$$

where

$$C^{-1} = \text{diag}[C_1^{-1}, \dots, C_n^{-1}], \quad G' = \text{diag}[G'_1, \dots, G'_n], \quad G = \text{diag}[G_1, \dots, G_n],$$

$$\overline{G} = \begin{bmatrix} -G_{11} & G_{12} & \dots & G_{1,n} \\ G_{12} & -G_{22} & \dots & G_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ G_{1,n} & G_{2,n} & \dots & -G_{n,n} \end{bmatrix} \quad (5.9c)$$

$G_{ii}$  is the sum of conductances of all branches belonging to the  $i$ th node,  $i = 1, \dots, n$ . The matrix  $\bar{G} \in M_n$  and its inverse has nonnegative entries. Substituting (5.9b) into (5.9a) we have

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + B \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \quad (5.10a)$$

where

$$A = -C^{-1}G'[I_n + \bar{G}^{-1}G'] \in M_n \quad (5.10b)$$

and

$$B = -C^{-1}G'\bar{G}^{-1}G \in \mathfrak{R}_+^{n \times n} \quad (5.10c)$$

since the matrices  $C^{-1}$ ,  $G'$ ,  $G$  and  $-\bar{G}^{-1}$  have nonnegative entries. Therefore, the electrical circuit is positive.  $\square$

Note that by Theorem 5.1 the standard electrical circuit is reachable for all nonzero values of the conductances and capacitances since  $\det B \neq 0$ .

**Theorem 5.5.** The electrical circuit shown on Fig. 5.2 is reachable if and only if

$$G_{k,j} = 0 \quad \text{for } k \neq j \quad \text{and } k, j = 1, \dots, n. \quad (5.11)$$

**Proof.** The matrix  $\bar{G}$  defined by (5.9c) is diagonal if and only if the condition (5.11) is met. In this case the matrices  $\bar{G}^{-1}G'$ ,  $A$  and  $B$  are also diagonal and from (5.10) we obtain

$$\frac{du_k}{dt} = \frac{1}{C_k} (G'_k G_{k,k}^{-1} G'_k - G_k) u_k + \frac{1}{C_k} G_k G_{k,k}^{-1} G_k e_k, \quad k = 1, \dots, n \quad (5.12)$$

where

$$G_{k,k} = G_k + G'_k, \quad k = 1, \dots, n. \quad (5.13)$$

Note that the subsystem (5.12) is reachable. Therefore, the positive electrical circuit is reachable if and only if the condition (5.11) is satisfied.  $\square$

Consider the electrical circuit shown on Fig 4.4 which is reachable only if the condition (4.18) is not satisfied. In this case using (4.17) we obtain

$$\text{rank}[B \quad AB \quad A^2B] = \begin{cases} 3 & \text{for } R_2R_5 \neq R_3R_4 \\ 1 & \text{for } R_2R_5 = R_3R_4 \end{cases}$$

If (4.18) holds then the first order electrical circuit (4.19) is positive and reachable.

## 6. Concluding remarks

The conditions for the positivity of linear electrical circuits composed of resistors, coils, condensators and voltage (current) sources have been established. It has been shown that:

- 1) The electrical circuits composed of resistors, coils and voltage source (shortly called  $R, L, e$  type) are positive for any values of their resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes (Theorem 2.4).
- 2) The electrical circuits composed of resistors, condensators and voltage source (shortly called  $R, C, e$  type) are not positive for almost all values of its resistances, capacitances and voltage sources if each their branch contains resistor, condensator and voltage source (Theorem 3.1).
- 3) The electrical circuit shown on Fig. 3.2 is positive for any values of the conductances, capacitances and source voltage  $e$  (Theorem 3.2).
- 4) The electrical circuit of  $R, L, C, e$  type is not positive for any values of its resistances, inductances, capacitances and source voltages if at least one its branch contains a coil and condensator (Theorem 4.1).
- 5) The electrical circuit of the structure shown on Fig. 4.3 is positive for any values of its resistances, inductances and capacitances (Theorem 4.2)
- 6) The positive  $n$ -meshes electrical circuits with only one coil in each linearly independent mesh are reachable if all resistances of branches belonging to two linearly independent meshes are zero (Theorem 5.3).
- 7) The positive electrical circuit shown on Fig. 5.2 is reachable if and only if the conductances between their nodes are zero (Theorem 5.5 - condition (5.11)).

The considerations have been illustrated by examples of linear electrical circuits.

Some of these results can be also extended for the controllability and observability of the linear electrical circuit. Open problems are extension of these considerations for the following classes of the systems:

- 1) disturbed parameters linear systems,
- 2) nonlinear electrical circuits.

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