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Reliability and risk analysis of multi-state systems with degrading components

Keywords

multi-state systems, system reliability, system risk

Abstract

Applications of multi-state approach to the reliability evaluation of systems composed of independent components are considered. The main emphasis is on multi-state systems with degrading components because of the importance of such an approach in safety analysis, assessment and prediction, and analysing the effectiveness of operation processes of real technical systems. The results concerned with multi-state series systems are applied to the reliability evaluation and risk function determination of a homogeneous bus transportation system. Results on homogeneous multi-state "*m* out of *n*" systems are applied to durability evaluation of a steel rope. A non-homogeneous series-parallel pipeline system composed of several lines of multi-state pipe segments is estimated as well. Moreover, the reliability evaluation of the model homogeneous multi-state parallel-series electrical energy distribution system is performed.

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1. Introduction

Many technical systems belong to the class of complex systems as a result of the progressive ageing of components they are built of and their complicated operating processes. Taking into account the importance of the safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis. These more general and practically important complex systems composed of multi-state components are considered among others in [1]–[36], [37] and [39]–[42]. An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is defined in the paper for systems with and degrading (ageing) in time components [5], [21], [39]-[41]. The assumption that the systems are composed of multi-state components with reliability states degrading in time without repair gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

2. Multi-state reliability analysis

In the multi-state reliability analysis to define systems with degrading components we assume that:

– E_i, $i = 1,2,...,n$, are components of a system,
– all components and a system under consideration have the state set $\{0,1,...,z\}$, $z \geq 1$,
– the state indexes are ordered, the state 0 is the worst and the state z is the best,
– $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$,
– $T(u)$ is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
– the system state degrades with time t without repair,
– $e_i(t)$ is a component E_i state at the moment t, $t \in < 0,\infty)$,
– $s(t)$ is a system state at the moment t, $t \in < 0,\infty)$.

The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse. The way in which the components and the system states change is illustrated in *Figure 1*.

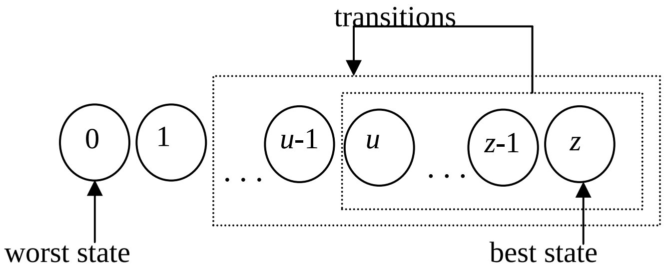

Figure 1. Illustration of states changing in system with ageing components

Definition 1. A vector

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1),..., R_i(t,z)],\ t \in <0,\infty),$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t)$$

for $t \in <0,,\infty)$, u = 0,1,...,z, i = 1,2,...,n, is the probability that the component E_i is in the state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment t = 0, is called the multi-state reliability function of a component E_i.

Definition 2. A vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],\ t \in <0,\infty),$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \qquad (1)$$

for $t \in <0,\infty)$, u = 0,1,...,z, is the probability that the system is in the state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment t = 0, is called the multi-state reliability function of a system.

Under this definition we have

$$\boldsymbol{R}_n(t,0) \geq \boldsymbol{R}_n(t,1) \geq \ldots \geq \boldsymbol{R}_n(t,z),\ t \in <0,\infty),$$

and if

$$p(t,u) = P(s(t) = u \mid s(0) = z),\ t \in <0,\infty),$$

for u = 0,1,...,z, is the probability that the system is in the state u at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment t = 0, then

$$\boldsymbol{R}_n(t,0) = 1,\ \boldsymbol{R}_n(t,z) = p(t,z),\ t \in <0,\infty), \qquad (2)$$

and

$$p(t,u) = \boldsymbol{R}_n(t,u) - \boldsymbol{R}_n(t,u+1),\ t \in <0,\infty), \qquad (3)$$

for $u = 0,1,..., z$.

Moreover, if

$$\boldsymbol{R}_n(t,u) = 1 \text{ for } t \leq 0,\ u = 1,2,...,z,$$

then

$$M(u) = \int_0^\infty \boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \qquad (4)$$

is the mean lifetime of the system in the state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{N(u) - [M(u)]^2},\ u = 1,2,...,z, \qquad (5)$$

where

$$N(u) = 2\int_0^\infty t\,\boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \qquad (6)$$

is the standard deviation of the system sojourn time in the state subset $\{u, u+1,..., z\}$ and moreover

$$\overline{M}(u) = \int_0^\infty p(t,u)dt,\ u = 1,2,...,z, \qquad (7)$$

is the mean lifetime of the system in the state u while the integrals (4), (6) and (7) are convergent.
Additionally, according to (3), (4) and (7), we get the following relationship

$$\overline{M}(u) = M(u) - M(u+1),\ u = 0,1,..., z-1,$$

$$\overline{M}(z) = M(z). \qquad (8)$$

Definition 3. A probability

$$r(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \leq t),\ t \in <0,\infty),$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the state z at the moment $t = 0$ is called a risk function of the multi-state system or, in short, a risk.

Under this definition, from (1), for $t \in <0,\infty)$, we have

$$r(t) = 1 - P(s(t) \geq r \mid s(0) = z) = 1 - \boldsymbol{R}_n(t,r), \tag{9}$$

and if τ is the moment when the risk exceeds a permitted level δ, then

$$\tau = r^{-1}(\delta), \tag{10}$$

where $r^{-1}(t)$, if it exists, is the inverse function of the risk function $r(t)$.

3. Basic multi-state reliability structures

3.1. Multi-state series system

Definition 4. A multi-state system is called series if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq n}\{T_i(u)\},\ u = 1,2,...,z.$$

The above definition means that a multi-state series system is in the state subset $\{u, u+1,..., z\}$ if and only if all its components are in this subset of states.
It is easy to work out the following results.

Corollary 1. The reliability function of the multi-state series system is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1),..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = \prod_{i=1}^{n} R_i(t,u),\ t \in <0,\infty),\ u = 1,2,...,z.$$

Corollary 2. If the multi-state series system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,...,z,$$
$$i = 1,2,...,n,$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1),..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = [R(t,u)]^n \text{ for } t \in <0,\infty),\ u = 1,2,...,z.$$

Example 1 (*a bus transportation system*). The city transportation system is composed of n, $n \geq 1$, buses necessary to perform its communication tasks. We assume that the bus lifetimes are independent random variables and that the system is operating in successive cycles (days) c = 1,2,... . In each of the cycles the following three operating phases of all components are distinguished:

f_1 – components waiting for inclusion in the operation process, lasting from the moment t_0 up to the moment t_1,
f_2 – components' activation for the operation process, lasting from t_1 up to t_2,
f_3 – components operating, lasting from t_2 up to $t_3 = t_0$.

Each of the system components during the waiting phase may be damaged because of the circumstances at the stoppage place. We assume that the probability that at the end moment t_1 of the first phase the ith component is not failed is equal to $p_i^{(1)}$, where $0 \leq p_i^{(1)} \leq 1$, $i = 1,2,..., n$. Since component lifetimes are independent then the system availability at the end moment t_1 of phase f_1 is given by

$$p^{(1)} = \prod_{i=1}^{n} p_i^{(1)}. \tag{11}$$

In the activation phase f_2 system components are prepared for the operation process by the service. They are checked and small flaws are removed. Sometimes the flaws cannot be removed and particular components are not prepared to fulfill their tasks. We assume that the probability that at the end moment t_2 of the first phase the ith component is not failed is equal to $p_i^{(2)}$, where $0 \leq p_i^{(2)} \leq 1$, $i = 1,2,..., n$. Since component lifetimes are independent then the system availability at the end moment t_2 of the phase f_2 is given by

$$p^{(2)} = \prod_{i=1}^{n} p_i^{(2)}. \tag{12}$$

Thus, finally, the system availability after two phases is given by

$$p^{(1,2)} = p^{(1)} \cdot p^{(2)}, \tag{13}$$

where $p^{(1)}$ and $p^{(2)}$ are defined respectively by (11) and (12).
In the operating phase f_3, during the time $t_4 = t_3 - t_2$, each of the system components is performing one of two tasks:

z_1 – a first task (working at normal communication conditions),
z_2 – a second task (working at a communication peak),
with probabilities respectively equal to r_1 and r_2, where $0 \le r_1 \le 1$, $r_2 = 1 - r_1$.
Let

$$R^{(1)}(t,\cdot) = [1, R^{(1)}(t,1), R^{(1)}(t,2)],$$

where

$$R^{(1)}(t,u) = 1 \text{ for } t < 0,$$

$$R^{(1)}(t,u) = \exp[-\frac{1}{15-5u}t] \text{ for } t \ge 0,\ u = 1,2,$$

be the reliability function of the *i*th component during performance of task z_1 and

$$R^{(2)}(t,\cdot) = [1, R^{(2)}(t,1), R^{(2)}(t,2)],$$

where

$$R^{(2)}(t,u) = 1 \text{ for } t < 0,$$

$$R^{(2)}(t,u) = \exp[-\frac{1}{10-2u}t] \text{ for } t \ge 0,\ u = 1,2,$$

be the reliability function of the *i*th component during performance of task z_2.
Thus, by *Definition 4*, the considered transportation system is a homogeneous three-state series system and according to the formula for total probability, after applying *Corollary 2*, we conclude that

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1), \overline{\boldsymbol{R}}_n(t,2)],$$

where

$$\overline{\boldsymbol{R}}_n(t,1) = 1 \text{ for } t < 0,$$

$$\overline{\boldsymbol{R}}_n(t,1) = r_1 \exp[-\frac{n}{15-5u}t] + r_2 \exp[-\frac{n}{10-2u}t] \quad (14)$$
for $t \ge 0$, $u = 1,2$,

is the reliability function of the system performing two tasks.
The mean values of the system lifetimes $T(u)$ in the state subsets, according to (4), are:

$$M(u) = E[T(u)] = \frac{r_1(15-5u) + r_2(10-2u)}{n}$$
for $u = 1,2$.

If we assume that

$n = 30$, $r_1 = 0.8$, $r_2 = 0.2$,

then from (14), we get

$$\overline{\boldsymbol{R}}_{30}(t,\cdot) = [1,\ 0.8\exp[-3t]+0.2\exp[-3.75t],$$

$$0.8\exp[-6t]+0.2\exp[-5t]] \text{ for } t \ge 0 \quad (15)$$

and

$M(1) \cong 0.32$, $M(2) \cong 0.17$.

Thus, considering (8), the expected values of the sojourn times in the particular states are:

$\overline{M}(1) \cong 0.15$, $\overline{M}(2) \cong 0.17$.

If a critical state is $r = 1$, then according to (9), the system risk function is given by

$$\boldsymbol{r}(t) = 1 - 0.8\exp[-3t]+0.2\exp[-3.75t] \text{ for } t \ge 0.$$

The moment when the system risk exceeds a permitted level $\delta = 0.05$, according to (10), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 0.016 \text{ years} \cong 6 \text{ days}.$$

At the end moment of the system activation phase, which is simultaneously the starting moment of the system operating phase t_2 the system is able to perform its tasks with the probability $p^{(1,2)}$ defined by (13). Therefore, after applying the formula (15), we conclude that the system reliability in c cycles, $c = 1,2,\ldots$, is given by the following formula

$$G(c,\cdot) = [1,\ p^{(1,2)}\, 0.8\exp[-3ct_4]+0.2\exp[-3.75\, ct_4],$$

$$p^{(1,2)}\, 0.8\exp[-6\, ct_4]+0.2\exp[-5\, ct_4]],$$

where $t_4 = t_3 - t_2$ is the time duration of the system operating phase f_3. Further, assuming for instance

$$p^{(1,2)} = p^{(1)} p^{(2)} = 0.99 \cdot 0.99 = 0.98,$$

$t_4 = 18$ hours $= 0.002055$ years

for the number of cycles $c = 7$ days $= 1$ week, we get

$G(7,\cdot) \cong [1, 0.966, 0.902]$.

This result means that during 7 days the considered transportation system will be able to perform its tasks in state not worse than the first state with probability 0.966, whereas it will be able to perform its tasks in the second state with probability 0.902.

3.2. Multi-state parallel system

Definition 5. A multi-state system is called parallel if its lifetime $T(u)$ in the state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \max_{1 \le i \le n}\{T_i(u)\}, \; u = 1,2,...,z.$$

The above definition means that the multi-state parallel system is in the state subset $\{u, u+1, ..., z\}$ if and only if at least one of its components is in this subset of states.

Corollary 3. The reliability function of the multi-state parallel system is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1 - \prod_{i=1}^{n} F_i(t,u), \; t \in <0, \infty), \; u = 1,2,...,z.$$

Corollary 4. If the multi-state parallel system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \text{ for } t \in <0,\infty), \; u = 1,2,...,z, \\ i = 1,2,...,n,$$

then its reliability function is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1 - [F(t,u)]^n \text{ for } t \in <0,\infty), \; u = 1,2,...,z.$$

3.3. Multi-state "*m* out of *n*" system

Definition 6. A multi-state system is called an "*m* out of *n*" system if its lifetime $T(u)$ in the state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = T_{(n-m+1)}(u), \; m = 1,2,...,n, \; u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the mth maximal order statistic in the sequence of the component lifetimes

$$T_1(u), T_2(u), ..., T_n(u).$$

The above definition means that the multi-state „*m* out of *n*" system is in the state subset $\{u, u+1, ..., z\}$ if and only if at least m out of its n components are in this state subset; and it is a multi-state parallel system if $m = 1$ and it is a multi-state series system if $m = n$.

Corollary 5. The reliability function of the multi-state "*m* out of *n*" system is given either by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1, \boldsymbol{R}_n^{(m)}(t,1), ..., \boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \le m-1}}^{1} [R_i(t,u)]^{r_i} [F_i(t,u)]^{1-r_i}$$

for $t \in <0,\infty)$, $u = 1,2,...,z$, or by

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,u) = \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \le \overline{m}}}^{1} [F_i(t,u)]^{r_i} [R_i(t,u)]^{1-r_i}$$

for $t \in <0,\infty)$, $\overline{m} = n - m$, $u = 1,2,...,z$.

Corollary 6. If the multi-state "*m* out of *n*" system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \text{ for } t \in <0,\infty), \; u = 1,2,...,z, \\ i = 1,2,...,n,$$

then its reliability function is given by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1, \boldsymbol{R}_n^{(m)}(t,1), ..., \boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{k=0}^{m-1} [R(t,u]^k [F(t,u)]^{n-k}$$

for $t \in <0,\infty)$, $u = 1,2,...,z$, or by

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,u) = \sum_{k=0}^{\overline{m}} [F(t,u)]^k [R(t,u)]^{n-k}$$

for $t \in <0,\infty)$, $\overline{m} = n - m$, $u = 1,2,...,z$.

Example 2 (*a three-stratum rope, durability*). Let us consider the steel rope of type *M*-80-200-10 described in [36]. It is a three-stratum rope composed of 36 strands: 18 outer strands, 12 inner strands and 6 more inner strands. All strands consist of seven still wires. The rope cross-section is presented in *Figure 2*.

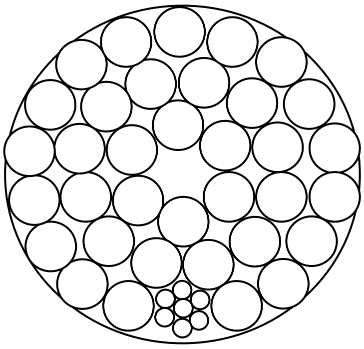

Figure 2. The steel rope *M*-80-200-10 cross-section

Considering the strands as basic components we conclude that the rope is a system composed of $n = 36$ components (strands). Due to [38] concerned with the evaluation of wear level, the following reliability states of the strands are distinguished:

state 3 – a strand is new, without any defects,

state 2 – the number of broken wires in the strand is greater than 0% and less than 25% of all its wires, or corrosion of wires is greater than 0% and less than 25%, abrasion is up to 25% and strain is up to 50%,

state 1 – the number of broken wires in the strand is greater than or equal to 25% and less than 50% of all its wires, or corrosion of wires is greater than or equal to 25% and less than 50%, abrasion is up to 50% and strain is up to 50%,

state 0 – otherwise (a strand is failed).

Thus, the considered steel rope composed of n = 36 four-state, i.e. z = 3. Let us assume that the rope strands have identical exponential reliability functions with transitions rates between the state subsets

$$\lambda(u) = 0.2u/\text{year},\ u = 1,2,3.$$

Assuming that the rope is in the state subset $\{u, u+1,..., z\}$ if at least m = 10 of its wires are in this state subset, according to *Definition 6*, we conclude the rope is a homogeneous four-state "10 out of 36" system. Thus, by *Corollary 6*, its reliability function is given by

$$\boldsymbol{R}_{36}^{(10)}(t,\cdot) = [1, \boldsymbol{R}_{36}^{(10)}(t,1), \boldsymbol{R}_{36}^{(10)}(t,2), \boldsymbol{R}_{36}^{(10)}(t,3)], \quad (16)$$

where

$\boldsymbol{R}_{36}^{(10)}(t,1) = 1$ for $t < 0$,

$\boldsymbol{R}_{36}^{(10)}(t,1) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.2t][1-\exp[-0.2t]]^{36-i}$ for $t \geq 0$,

$\boldsymbol{R}_{36}^{(10)}(t,2) = 1$ for $t < 0$,

$\boldsymbol{R}_{36}^{(10)}(t,2) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.4t][1-\exp[-0.4t]]^{36-i}$ for $t \geq 0$,

$\boldsymbol{R}_{36}^{(10)}(t,3) = 1$ for $t < 0$,

$\boldsymbol{R}_{36}^{(10)}(t,3) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.6t][1-\exp[-0.6t]]^{36-i}$ for $t \geq 0$.

By (16), the approximate mean values of the rope lifetimes $T(u)$ in the state subsets and their standard deviations in years are:

$M(1) \cong 6.66$, $M(2) \cong 3.33$, $M(3) \cong 2.22$,

$\sigma(1) \cong 1.62$, $\sigma(2) \cong 0.81$, $\sigma(3) \cong 0.54$,

whereas, the approximate mean values of the rope lifetimes in the particular reliability states are:

$\overline{M}(1) \cong 3.33$, $\overline{M}(2) \cong 1.11$, $\overline{M}(3) \cong 2.22$.

If the critical state is r = 2, then the rope risk function is approximately given by

$$\boldsymbol{r}(t) = \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.4t][1-\exp[-0.4t]]^{36-i} \text{ for } t \geq 0.$$

The moment when the risk exceeds an admissible level δ = 0.05, after applying (10), is

$\tau \cong 2.074$ years.

The behaviour of the rope system reliability function and its risk function are illustrated in *Table 1*.

Table 1. The values of the still rope multi-state reliability function and risk function

t	$\boldsymbol{R}_{36}^{(10)}(t,1)$	$\boldsymbol{R}_{36}^{(10)}(t,2)$	$\boldsymbol{R}_{36}^{(10)}(t,3)$	$\boldsymbol{r}(t)$
0.2	1.00000	1.00000	1.00000	0.00000
0.6	1.00000	0.99998	0.99979	0.00002
1.0	0.99999	0.99961	0.99425	0.00039
1.4	0.99995	0.99641	0.94590	0.00359
1.8	0.99979	0.98014	0.77675	0.01986
2.2	0.99928	0.92792	0.49332	0.07208
2.6	0.99783	0.81520	0.23107	0.18480
3.0	0.99425	0.64221	0.08058	0.35779
3.4	0.98649	0.44415	0.02168	0.55585
3.8	0.97157	0.26782	0.00469	0.73218
4.2	0.94590	0.14130	0.00085	0.85870
4.6	0.90602	0.06584	0.00013	0.93416
5.0	0.84969	0.02742	0.00002	0.97258
5.4	0.77675	0.01034	0.00000	0.98966
5.8	0.68965	0.00357	0.00000	0.99643
6.2	0.59314	0.00114	0.00000	0.99886
6.6	0.49332	0.00034	0.00000	0.99966
7.0	0.39645	0.00010	0.00000	0.99990
7.4	0.30784	0.00003	0.00000	0.99997
7.8	0.23107	0.00001	0.00000	0.99999

3.4. Multi-state series-parallel system

Other basic multi-state reliability structures with components degrading in time are series-parallel and parallel-series systems. To define them, we assume that:

- E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are components of a system,
- all components E_{ij} have the same state set as before $\{0,1,...,z\}$,
- $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are independent random variables representing the lifetimes of components E_{ij} in the state subset $\{u, u+1,..., z\}$, while they were in the state z at the moment $t = 0$,
- $e_{ij}(t)$ is a component E_{ij} state at the moment t, $t \in <0,\infty)$, while they were in the state z at the moment $t = 0$.

Definition 7. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1),..., R_{ij}(t,z)] \text{ for } t \in <0,\infty),$$
$$i = 1,2,...,k,\ j = 1,2,...,l_i,$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t)$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, is the probability that the component E_{ij} is in the state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

Definition 8. A multi-state system is called series-parallel if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \max_{1\leq i\leq k}\{\min_{1\leq j\leq l_i}\{T_{ij}(u)\}\},\ u = 1,2,...,z.$$

Corollary 7. The reliability function of the multi-state series-parallel system is given by

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,1),..., \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,u) = 1 - \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} R_{ij}(t,u)] \text{ for } t \in <0,\infty),$$
$$u = 1,2,...,z,$$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Corollary 8. If the multi-state series-parallel system is homogeneous, i.e.

$$R_{ij}(t,u) = R(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,...,z,$$
$$i = 1,2,...,k,\ j = 1,2,...,l_i,$$

then its reliability function is given by

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,1),..., \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,u) = 1 - \prod_{i=1}^{k}[1 - [R(t,u)]^{l_i}] \text{ for } t \in <0,\infty),$$
$$u = 1,2,...,z,$$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Corollary 9. If the multi-state series-parallel system is homogeneous, i.e.

$R_{ij}(t,u) = R(t,u)$ for $t \in <0,\infty)$, $u = 1,2,...,z$, $i = 1,2,...,k$, $j = 1,2,...,l_i$,

and regular, i.e.

$l_1 = l_2 = \ldots = l_k = l$, $l \in N$.

then its reliability function is given by

$$\boldsymbol{R}_{k,l}(t,\cdot) = [1, \boldsymbol{R}_{k,l}(t,1), ..., \boldsymbol{R}_{k,l}(t,z)],$$

and

$$\boldsymbol{R}_{k,l}(t,u) = 1 - [1 - [R(t,u)]^l]^k \text{ for } t \in <0,\infty),$$
$u = 1,2,...,z$,

where k is the number of series subsystems linked in parallel and l is the number of components in the series subsystems.

Example 3 (*a pipeline system*). Let us consider the pipeline system composed of k = 3 lines of pipe segments linked in parallel, each of them composed of l = 100 five-state identical segments linked in series. The scheme of the considered system is shown in *Figure 3*.

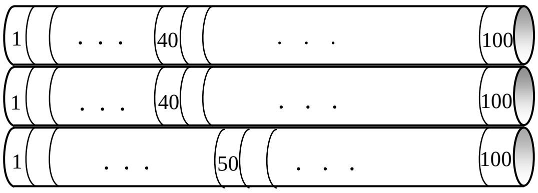

Figure 3. The model of a regular series-parallel pipeline system

Considering pipe segments as basic components of the pipeline system, according to *Definition 8*, we conclude that it is a homogeneous regular five-state series-parallel system. Therefore, by *Corollary 9*, the pipeline system reliability function is given by

$$\boldsymbol{R}_{3,100}(t,\cdot) = [1, \boldsymbol{R}_{3.1000}(t,1), \boldsymbol{R}_{3,100}(t,2), \boldsymbol{R}_{3,100}(t,3),$$
$$\boldsymbol{R}_{3,100}(t,4)],$$

where

$$\boldsymbol{R}_{3,100}(t,u) = 1 - [1 - [R(t,u)]^{100}]^3$$

for $t \in (-\infty,\infty)$, $u = 1,2,3,4$.

Taking into account pipe segment reliability data given in their technical certificates and expert opinions we assume that they have Weibull reliability functions

$$R(t,\cdot) = [1, R(t,1), R(t,2), R(t,3), R(t,4)],$$

where

$$R(t,u) = 1 \text{ for } t < 0,$$

$$R(t,u) = \exp[-\beta(u)t^{\alpha(u)}] \text{ for } t \ge 0, u = 1,2,3,4,$$

with the following parameters:

$$\alpha(1) = 3, \ \beta(1) = 0.00001,$$

$$\alpha(2) = 2.5, \ \beta(2) = 0.0001,$$

$$\alpha(3) = 2, \ \beta(3) = 0.0016,$$

$$\alpha(4) = 1, \ \beta(4) = 0.05.$$

Hence it follows that the pipeline system exact reliability function is given by

$$\boldsymbol{R}_{3,100}(t,\cdot) = [1, 1 - [1 - \exp[-0.001t^3]]^3,$$
$$1 - [1 - \exp[-0.01t^{5/2}]]^3, 1 - [1 - \exp[-0.16t^2]]^3,$$
$$1 - [1 - \exp[-5t]]^3] \text{ for } t \ge 0. \quad (17)$$

By (17), the expected values $M(u)$, u = 1,2,3,4, of the system sojourn times in the state subsets in years, calculated on the basis of the approximate formula are:

$$M(1)$$
$$= \Gamma(4/3)[3(0.001)^{-1/3} - 3(0.002)^{-1/3} + (0.003)^{-1/3}]$$
$$\cong 11.72,$$

$$M(2)$$
$$= \Gamma(7/5)[3(0.01)^{-2/5} - 3(0.02)^{-2/5} + (0.03)^{-2/5}]$$
$$\cong 7.67,$$

$$M(3)$$
$$= \Gamma(3/2)[3(0.16)^{-1/2} - 3(0.32)^{-1/2} + (0.48)^{-1/2}]$$
$$\cong 3.23,$$

$M(4) = \Gamma(2)[3(5)^{-1} - 3(10)^{-1} + (15)^{-1}] \cong 0.37.$

Hence, the system mean lifetimes $\overline{M}(u)$ in particular states are:

$\overline{M}(1) \cong 4.05,\ \overline{M}(2) \cong 4.44,\ \overline{M}(3) \cong 2.86,$

$\overline{M}(4) \cong 0.37.$

If the critical state is $r = 2$, then the system risk function, according (9), is given by

$\boldsymbol{r}(t) = [1 - \exp[-0.01t^{5/2}]]^3.$

The moment when the system risk exceeds an admissible level $\delta = 0.05$, from (10), is

$\tau = \boldsymbol{r}^{-1}(\delta) = [-100\log(1-\sqrt[3]{\delta})]^{2/5} \cong 4.62.$

The behaviour of the system risk function is presented in *Table 2* and *Figure 4*.

Table 2. The values of the piping system risk function

t	$\boldsymbol{r}(t)$
0.0	0.000
1.5	0.000
3.0	0.003
4.5	0.043
6.0	0.201
7.5	0.485
9.0	0.758
10.5	0.918
12.0	0.980
13.5	0.996
15.0	1.000

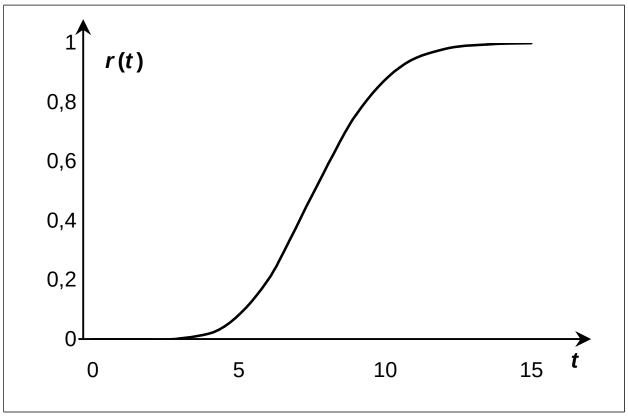

Figure 4. The graph of the piping system risk function

3.5. Multi-state parallel-series system

Definition 9. A multi-state system is called parallel-series if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1\le i\le k}\{\max_{1\le j\le l_i}\{T_{ij}(u)\}, \ u = 1,2,...,z.$$

Corollary 10. The reliability function of the multi-state parallel-series system is given by

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,1),..., \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,u) = \prod_{i=1}^{k}[1-\prod_{j=1}^{l_i}F_{ij}(t,u)] \ \text{for} \ t \in <0,\infty),$$
$u = 1,2,...,z,$

where k is the number of its parallel subsystems linked in series and l_i are the numbers of components in the parallel subsystems.

Corollary 11. If the multi-state parallel-series system is homogeneous, i.e.

$R_{ij}(t,u) = R(t,u)$ for $t \in <0,\infty)$, $u = 1,2,...,z,$
$i = 1,2,...,k,\ j = 1,2,...,l_i,$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,1),..., \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,u) = \prod_{i=1}^{k}[1-[F(t,u)]^{l_i}] \ \text{for} \ t \in <0,\infty),$$
$u = 1,2,...,z,$

where k is the number of its parallel subsystems linked in series and l_i are the numbers of components in the parallel subsystems.

Corollary 12. If the multi-state parallel-series system is homogeneous, i.e.

$R_{ij}(t,u) = R(t,u)$ for $t \in <0,\infty)$, $u = 1,2,...,z,$
$i = 1,2,...,k,\ j = 1,2,...,l_i,$

and regular, i.e.

$l_1 = l_2 = ... = l_k = l,\ l \in N.$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,l}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k,l}(t,1), ..., \overline{\boldsymbol{R}}_{k,l}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l}(t,u) = [1-[F(t,u]^l]^k \quad \text{for } t \in <0,\infty),$$
$$u = 1,2,...,z,$$

where k is the number of its parallel subsystems linked in series and l is the number of components in the parallel subsystems.

Example 4 (*an electrical energy distribution system*). Let us consider a model energetic network stretched between two poles and composed of three energetic cables, six insulators and two bearers and analyze the reliability of all cables only. Each cable consists of 36 identical wires. Assuming that the cable is able to conduct the current if at least one of its wires is not failed we conclude that it is a homogeneous parallel-series system composed of k = 3 parallel subsystems linked in series, each of them consisting of l = 36 basic components. Further, assuming that the wires are four-state components, i.e. z = 3, having Weibull reliability functions with parameters

$$\alpha(u) = 2,\ \beta(u) = (7.07)^{2u-8},\ u = 1,2,3.$$

According to Corollary 12, we obtain the following form of the system multi-state reliability function

$$\overline{\boldsymbol{R}}_{3,36}(t,\cdot) \cong [1, [1-[1-\exp[-0.000008007t^2]]^{36}]^3,$$

$$[1-[1-\exp[-0.000400242t^2]]^{36}]^3,$$

$$[1-[1-\exp[-0.20006042t^2]]^{36}]^3] \text{ for } t \in <0,\infty).(18)$$

By (18), the values of the system sojourn times $T(u)$ in the state subsystems in months, after applying (4), are given by

$$E[T(u)] \cong \int_0^\infty [1-[1-\exp[-(7.07)^{2u-8}t^2]]^{36}]^3\, dt$$

for u = 1,2,3, and particularly

$$M(1) \cong 650,\ M(2) \cong 100,\ M(3) \cong 15.$$

Hence, from (8), the system mean lifetimes in particular states are:

$$\overline{M}(1) \cong 550,\ \overline{M}(2) \cong 85,\ \overline{M}(3) \cong 15.$$

If the critical reliability state of the system is r = 2, then its risk function, according to (9), is given by

$$\boldsymbol{r}(t) \cong 1-[1-[1-\exp[-0.000400242t^2]]^{36}]^3.$$

The moment when the system risk exceeds an admissible level δ = 0.05, calculated due to (10), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 76 \text{ months}.$$

4. Conclusion

In the paper the multi-state approach to the reliability evaluation of systems with degrading components have been considered. Theoretical results presented in have been illustrated by examples of their application in reliability evaluation of technical systems. These evaluations, despite not being precise may be a very useful, simple and quick tool in approximate reliability evaluation, especially during the design of large systems, and when planning and improving their safety and effectiveness operation processes.

The results presented in the paper suggest that it seems reasonable to continue the investigations focusing on:

- methods of improving reliability for multi-state systems,
- methods of reliability optimisation for multi-state systems related to costs and safety of the system operation processes,
- availability and maintenance of multi-state systems.

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