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TRAVELLING WAVE SOLUTIONS OF THE NON-LINEAR WAVE EQUATIONS

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Abstract: This article focuses on the exact periodic solutions of nonlinear wave equations using the well-known Jacobi elliptic function expansion method. This method is more general than the hyperbolic tangent function expansion method. The periodic solutions are found using this method which contains both solitary wave and shock wave solutions. In this paper, the new results are computed using the closed-form solution including solitary or shock wave solutions which are obtained using Jacobi elliptic function method. The corresponding solitary or shock wave solutions are compared with the actual results. The results are visualised and the periodic behaviour of the solution is described in detail. The shock waves are found to break with time, whereas, solitary waves are found to be improved continuously with time.

Key words: nonlinear evolution problems, coupled equations, Jacobi elliptic function, periodic solutions

1. INTRODUCTION

In several subfields within physics and engineering, the notion of soliton is an important concept. The solitons are used in modelling of optics, hydrodynamics, nuclear physics, biomechanics, plasma physics and many other fields [1–3]. While in most cases, the solution to nonlinear governing equations is associated with a soliton, this equation describes a wave that maintains its form across time [4]. For the solution of the nonlinear differential equations, several methods are defined such as tangent hyperbolic (tanh) expansion method [5], tanh-sech expansion method [6], exponent expansion method [7], F-expansion method [8], modified simple expansion method [9], exp(-$\emptyset(\alpha)$) expansion method [10], sine-cosine method [11], expansion method [12], coth-expansion function method [13], and some more methods have also been purposed [14–16]. These methods only predict the solution of the solitary and the shock waves but are unable to predict the periodic behaviour of the solutions.

Finding travelling wave solutions of nonlinear partial differential equations is of significant interest, especially in integrable systems [17–26]. In past, the studies presented by different researchers yielded several intriguing forms of solutions, including soliton solutions, cnoidal solutions, compacton solutions and peakon solutions. Nevertheless, as the literature indicates, discovering these answers has not been simple. In recent studies [27–33], the research scholars provided a straightforward method for constructing travelling wave solutions to general nonlinear equations, which may or may not be integrable, beginning with solutions of simple equations (including even linear equations). As proved by nontrivial examples (usually involving equations of the third order), this method is particularly effective for getting travelling wave solutions of nonlinear equations. The generation of travelling wave solutions of a large class of partial differential equations beginning with a trial travelling wave and an invertible map from a trial travelling wave are the primary contributions of this paper. However, we now realise that the underlying reason for the simplicity of our earlier proposal [35] is due to some remarkable properties of travelling wave solutions.
In 1996, a well-known mathematician [36] purposed the shock and solitary wave solutions as well as the periodic solutions by using the Weierstrass elliptic function. In this article, various nonlinear wave equations are solved using the Jacobi elliptic function expansion approach, which is more general than the hyperbolic tangent function expansion method. It is demonstrated that this strategy yields periodic solutions that include certain shock wave and solitary wave solutions. In mathematics, a group of fundamental elliptic functions known as the Jacobi elliptic functions can be found easily. One can find the applications of these functions in the characterisation of the oscillation of a pendulum and in the design of electronic elliptic filters [37]. Jacobi's elliptic operations are the generalisation that refers to those other conics, the ellipse in particular, whereas trigonometry functions are specified concerning a circle. Unlike the Weierstrass elliptic functions, the Jacobi elliptic functions do not need to be explained in terms of complex analysis before they can be used in the real world.

2. METHOD EXPLANATION

The Jacobi elliptic function expansion method is summarised as follows.

Consider a given equation for nonlinear waves

$$G\left(w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial t^2}, \frac{\partial^2 \mathrm{w}}{\partial x^2}, \ldots\right) = 0, \quad (1)$$

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the travelling wave is the solution to the form

$$w(x,t) = w(\xi)\ ,\ \xi = k(x - ct), \tag{2}$$

where *k* represents the number of waves and *c* represents their speed, correspondingly. A method known as the expansion of the Jacobi elliptic function can be used to represent $w(\xi)$ as a finite series of the Jacobi elliptic function, ξ, which stands for the ansatz.

$$w(\xi) = \sum_{i=0}^{n} a_i\, sn^i\, \xi, \tag{3}$$

is produced based on the highest degree, which is

$$O(w(\xi) = m, \tag{4}$$

which represents the differential equation. From Eq. (3), we have

$$\frac{dw}{d\xi} = \sum_{i=0}^{n} i a_i\, sn^{i-1}\xi(cn\,\xi)dn\,\xi, \tag{5}$$

where all the above functions *sn* ξ, *cn* ξ and $dn\xi$ are the Jacobi elliptic functions of the third kind.

2.1. Relation between the square of the functions

$$sn^2(\xi) + cn^2(\xi) = 1, \tag{6}$$

$$dn^2(\xi) + m^2 sn^2(\xi) = 1, \quad cn^2 + m' sn^2 = dn^2, \tag{7}$$

2.2. Jacobi elliptic functions as solutions of nonlinear ordinary differential equations

The derivatives of the three basic Jacobi elliptic functions are: $(0 \prec m \prec 1)$

$$\left(\frac{d}{d\xi}\right)\left(sn(\xi)\right) = cn(\xi)dn(\xi),$$

$$\left(\frac{d}{d\xi}\right)\left(cn(\xi)\right) = -sn(\xi)dn(\xi), \tag{8}$$

$$\left(\frac{d}{d\xi}\right)\left(dn(\xi)\right) = -m^2 cn(\xi)sn(\xi),$$

the ordinary differential equation balances the highest derivative with the highest nonlinear part.

Tab. 1. When $n = 0$ or $n = 1$, the Jacobi elliptic functions are reduced to non-elliptic functions

Function	$n = 0$	$n = 1$
$sn(u,n)$	$sinu$	$tanhu$
$cn(u,n)$	$cosu$	$sechu$
$dn(u,n)$	1	$sechu$
$ns(u,n)$	$cscu$	$cothu$
$nc(u,n)$	$secu$	$coshu$
$nd(u,n)$	1	$coshu$
$sd(u,n)$	$sinu$	$sinhu$
$cd(u,n)$	$cosu$	1
$cs(u,n)$	$cotu$	$cschu$
$ds(u,n)$	$cscu$	$cschu$
$dc(u,n)$	$secu$	1
$sc(u,n)$	$tanu$	$sinhu$

After this process, we choose the order of the ordinary differential equation on its base and choose the series of the function.

$$O\left(\frac{d^q w}{d\xi^q}\right) = m + q\ ,\ q = 1,2,3,\ldots \tag{9}$$

$$O\left(w^p\,\frac{d^q w}{d\xi^q}\right) = (p+1)m + q, q = 1,2,3,\ldots \tag{10}$$

Therefore, m can be chosen in Eq. (3) to strike a good balance between the highest-order derivative term and the nonlinear component Eq. (1).

Eq. (3) in its following form is shown in Tab. 1.

$$w(\xi) = \sum_{i=0}^{n} a_i tanh^i \xi, \tag{11}$$

Therefore, the Jacobi elliptic function expansion approach is superior than the hyperbolic tangent function expansion method in terms of its universal applicability.

2.3. Applications

A number of the nonlinear models are solved using the Jacobi elliptic function method, and a lot of applications are found in different fields of real word problems.

2.3.1. Model 1

The Korteweg–De Vries (KdV) equation in mathematics is a mathematical model of waves on shallow water surfaces. It is especially significant as the classic example of a perfectly solvable model, i.e., a non-linear partial differential equation whose solutions can be precisely described. The KdV equation can be solved using the inverse scattering transform. The mathematical theory behind the KdV equation is an active area of study. The KdV equation was first presented by Boussinesq and afterwards found by Diederik Korteweg and Gustav de Vries (1895) [38]. Zabusky and Kruskal (1965) [34] discovered statistically that the solutions of the KdV equation appeared to break down at large times into a collection of 'solitons': well-separated solitary waves. Moreover, the shape of the solitons appears to be essentially unaffected by their passage through one another (though this could cause a change in their position). By demonstrating that the KdV equation represented the continuum limit of the FPUT system, they established a relationship to earlier numerical experiments by Fermi, Pasta, Ulam and Tsingou. In 1967, Gardner, Greene, Kruskal and Miura developed an analytic solution utilising the inverse scattering transform [39–41].

$$\left(\frac{\partial q}{\partial t}\right) + 12\,q\left(\frac{\partial q}{\partial x}\right) + 2\beta\left(\frac{\partial^3 q}{\partial x^3}\right) = 0 \tag{11}$$

Step 1: Using transformation for the above Eq. (11

$$q(x,t) = q(\xi), \xi = k(x - ct), \tag{12}$$

by using Eq. (12) transformation, the above partial differential equations are converted into the ordinary differential equation, and in the above transformation *k* and *c* represent the wave number and the wave speed.

$$\left(\frac{\partial q}{\partial x}\right) = k\left(\frac{\partial q}{\partial \xi}\right) \tag{13}$$

$$\left(\frac{\partial^3 q}{\partial x^3}\right) = k^3\left(\frac{\partial^3 q}{\partial \xi^3}\right) \tag{14}$$

Putting Eqs (13) and (14) in Eq. (11), the above equation becomes,

$$-2c\left(\frac{\partial q}{\partial \xi}\right)+12\mu q\left(\frac{\partial q}{\partial \xi}\right)+2\beta k^2\left(\frac{\partial^3 q}{\partial \xi^3}\right)=0, \quad (15)$$

which is the required ordinary differential equation that can be obtained by travelling wave solution.

Step 2: Now balancing the above ordinary differential equation balancing the highest derivative with the nonlinear part after this process we balanced our required ordinary differential equation balancing

$$O\left(q\left(\frac{\partial q}{\partial \xi}\right)\right) = 2m+1, \quad (16)$$

$$O\left(\left(\frac{\partial^3 q}{\partial \xi^3}\right)\right)=m+3, \quad (17)$$

Comparing Eqs (16) and (17)

$$m = 2 \quad (18)$$

The above ordinary differential equation is balanced at $\mathrm{m}=2$

Step 3: In this method, the finite series method of the Jacobi elliptic function is employed for the nonlinear equation and the values of constants are found. Here the trigonometric cosine and sine functions are used and are represented with *cn* and *sn*.

The solution to the previous equation could take the form of a travelling wave

$$q(\xi)=a_0+a_1 sn(\xi)+a_2 sn^2(\xi)+O\left(sn^3(\xi)\right), \quad (19)$$

we truncated up to two terms because the above is balanced at *m* = 2.

$$\left(\frac{\partial q}{\partial \xi}\right)= \left(a_1+2a_2 sn(\xi)\right)cn(\xi)dn(\xi), \quad (20)$$

$$q\left(\frac{\partial q}{\partial \xi}\right)= [a_0a_1+(a_1^2+2a_0a_2)sn(\xi)+ 3a_1a_2sn^2(\xi)+2a_2^2sn^3(\xi)]cn(\xi)dn(\xi), \quad (21)$$

$$\left(\frac{\partial^2 q}{\partial \xi^2}\right)= \left.\begin{matrix}2a_2-(1+m^2)a_1sn(\xi)-4a_2(1+m^2)sn^2(\xi)\\ +2m^2a_1sn^3(\xi)+6m^2a_2sn^4(\xi)\end{matrix}\right\}, \quad (22)$$

$$\left(\frac{\partial^3 q}{\partial \xi^3}\right)= \left.\begin{matrix}[-(1+m^2)a_1-8(1+m^2)a_2sn(\xi)]cn(\xi)dn(\xi)\\ +[6m^2a_1sn^2(\xi)+24m^2a_2sn^3(\xi)]cn(\xi)dn(\xi)\end{matrix}\right\}, \quad (23)$$

Substituting Eqs (20–23) in Eq. (15)

$$\left.\begin{matrix}-2c\left[\left(a_1+2a_2sn(\xi)\right)cn(\xi)dn(\xi)\right]\\ +12\mu\left[\begin{bmatrix}a_0a_1+(a_1^2+2a_0a_2)sn(\xi)\\ +3a_1a_2sn^2(\xi)+2a_2^2sn^3(\xi)\end{bmatrix}cn(\xi)dn(\xi)\right]\\ +2\beta k^2\left[\begin{bmatrix}-(1+m^2)a_1-8(1+m^2)a_2sn(\xi)\\ +6m^2a_1sn^2(\xi)+24m^2a_2sn^3(\xi)\end{bmatrix}cn(\xi)dn(\xi)\right]\end{matrix}\right\}=0, \quad (24)$$

Comparing coefficients on both side

$$[-ca_1+a_0a_1-k^2\beta(1+m^2)a_1]= 0, \quad (25)$$

$$[-2a^2c+a_1^2+2a_0a_2-8k^2\beta(1+m^2)a_2]= 0, \quad (26)$$

$$3a_1a_2+6m^2\beta k^2= 0, \quad (27)$$

$$2a_2^2+24m^2\beta k^2a_2= 0, \quad (28)$$

By solving the above-mentioned equations using Mathematica Bulit in Software for finding the values of the constants a_0, a_1 and a_2.

$$a_0= c+4\beta k^2+4\beta k^2m^2, \quad (29)$$

$$a_1= 0, \quad (30)$$

$$a_2= -12k^2m^2\beta, \quad (31)$$

Putting the above values of the constant in the above-mentioned equation

$$q(\xi)=c+4\beta k^2+4\beta k^2m^2- 12k^2m^2\beta sn^2(\xi)+ \mathrm{O}\,(sn^2(\xi)), \quad (32)$$

This is the periodic solution precise to the model Eq. (1). In common usage, this solution is known as the cnoidal wave solution to the model equation that came before it. When $\mathrm{m}=1$, the expression Eq. (22) is simplified as

$$q(\xi)=c-4\beta k^2+12k^2\beta sech^2(\xi), \quad (33)$$

Which is the model equation's solitary wave solution particularly when c = 1.

$$q(\xi)=3c\left(sech^2(x-ct)\right)\sqrt{\frac{c}{4\beta}}(x-ct), \quad (34)$$

2.3.2. Model 2

This model represents the coupled system of partial differential equations [37].

$$q\left(\frac{\partial q}{\partial t}\right)+12q\left(\frac{\partial q}{\partial x}\right)+\left(\frac{\partial z}{\partial x}\right)+\alpha\left(\frac{\partial^3 q}{\partial t\partial x^2}\right)= 0, \quad (35)$$

$$\left(\frac{\partial q}{\partial t}\right)+\left(\frac{\partial (qz)}{\partial x}\right)+6\beta\left(\frac{\partial^3 q}{\partial x^3}\right)= 0, \quad (36)$$

These are known as couple equations, so we solve these equations by using a travelling wave solution.

Step 1: Here we suppose the possible travelling wave solution for the above-coupled equations

$$\begin{matrix}q(x,t) = q(\xi), \xi = k(x-ct),\\ z(x,t) = z(\xi), \xi = k(x-ct),\end{matrix} \quad (37)$$

By using Eq. (37), we transform the above coupled partial differential equation into an ordinary differential equation. In the above transformation, k and c represent the wave number and the wave speed, respectively.

$$\begin{matrix}(q)= -ck\left(\frac{\partial q}{\partial \xi}\right),\\ \left(\frac{\partial q}{\partial x}\right)= k\left(\frac{\partial q}{\partial \xi}\right), \left(\frac{\partial z}{\partial x}\right)= k\left(\frac{\partial z}{\partial \xi}\right),\\ \left(\frac{\partial^2 q}{\partial x^2}\right)= k^2\left(\frac{\partial^2 q}{\partial \xi^2}\right),\\ \left(\frac{\partial^3 q}{\partial x^3}\right)= k^3\left(\frac{\partial^3 q}{\partial \xi^3}\right),\end{matrix} \quad (38)$$

Putting Eq. (38) in Eq. (35), we have the following equation:

$$-cq\left(\frac{\partial q}{\partial \xi}\right)+12q\left(\frac{\partial q}{\partial \xi}\right)+\left(\frac{\partial z}{\partial \xi}\right)-ck^2\alpha\left(\frac{\partial^3 q}{\partial \xi^3}\right)=0, \quad (39)$$

which is the required ordinary differential equation that can be obtained using travelling wave solution.

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Step 2: Now balancing the above ordinary differential equation balancing the highest derivative with the nonlinear part after this process we balanced our required ordinary differential equation balancing

$$O\left(q\left(\frac{\partial q}{\partial \xi}\right)\right) = 2m+1, \quad (40)$$

$$O\left(\left(\frac{\partial^3 q}{\partial \xi^3}\right)\right) = m+3, \quad (41)$$

Comparing Eqs (40) and (41)

$$m = 2 \quad (41)$$

The above ordinary differential equation is balanced at $\mathrm{m} = 2$.

Step 3: In this method, we apply a finite series method of the Jacobi elliptic function on nonlinear equation and find the values of constants. Here we use cosine and sine functions which are represented by *cn* and *sn*.

The solution to the previous equation could take the form of a travelling wave.

$$\begin{aligned} q(\xi) &= a_0 + a_1 sn(\xi) + a_2 sn^2(\xi) + O\left(sn^3(\xi)\right), \\ z(\xi) &= b_0 + b_1 sn(\xi) + b_2 sn^2(\xi) + O\left(sn^3(\xi)\right), \end{aligned} \quad (42)$$

We truncated up to two terms because the above is balanced at $m = 2$.

$$\left(\frac{\partial q}{\partial \xi}\right) = (a_1 + 2a_2 sn\xi)cn(\xi)dn(\xi), \quad (43)$$

$$q\left(\frac{\partial q}{\partial \xi}\right) = [a_0a_1 + (a_1^2 + 2a_0a_2)sn(\xi) + 3a_1a_2sn^2(\xi) + 2a_2^2sn^3\xi]cn(\xi)dn(\xi), \quad (44)$$

$$\left(\frac{\partial z}{\partial \xi}\right) = (b_1 + 2b_2 sn\xi)cn(\xi)dn(\xi), \quad (45)$$

$$\left(\frac{\partial^3 q}{\partial \xi^3}\right) = \left.\begin{matrix}[-(1+m^2)a_1 - 8(1+m^2)a_2sn(\xi)]cn(\xi)dn(\xi) \\ +[6m^2a_1sn^2(\xi) + 24m^2a_2sn^3(\xi)]cn(\xi)dn(\xi)\end{matrix}\right\}, \quad (46)$$

By using the defining values of the above equations, we have,

$$\left.\begin{matrix} -cq[(a_1+2a_2sn\xi)cn(\xi)dn(\xi)] + [a_0a_1 + (a_1^2+2a_0a_2)sn(\xi)]cn(\xi)dn(\xi) \\ +12[3a_1a_2sn^2(\xi) + 2a_2^2sn^3\xi]cn(\xi)dn(\xi) + [(b_1+2b_2sn\xi)cn(\xi)dn(\xi)] \\ -ck^2\alpha\begin{bmatrix}[-(1+m^2)a_1 - 8(1+m^2)a_2sn(\xi)]cn(\xi)dn(\xi) \\ -ck^2\alpha[+6m^2a_1sn^2(\xi) + 24m^2a_2sn^3(\xi)]cn(\xi)dn(\xi)\end{bmatrix}\end{matrix}\right\}, = 0 \quad (47)$$

Comparing coefficients on both sides

$$[-ca_1 + a_0a_1 + b_1 + ck^2\alpha(1+m^2)a_1] = 0, \quad (48)$$

$$[-2a_2c + a_1^2 + 2a_0a_2 + 2b_2 + 8ck^2\alpha(1+m^2)a^2] = 0, \quad (49)$$

$$3a_1a_2 + 6m^2 = 0, \quad (50)$$

$$2a_2^2 - 24m^2a_2 = 0, \quad (51)$$

Step 2: Again, we repeated Step 2 for the coupled partial differential equation.

Now balancing the above ordinary differential equation balancing the highest derivative with the nonlinear part after this process we balanced our required ordinary differential equation balancing

$$O\left(z\left(\frac{\partial q}{\partial \xi}\right)\right) = O\left(q\left(\frac{\partial z}{\partial \xi}\right)\right) = 2m+1, \quad (52)$$

$$O\left(\left(\frac{\partial^3 q}{\partial \xi^3}\right)\right) = m+3, \quad (53)$$

Comparing Eqs (52) and (53)

$$m = 2 \quad (54)$$

The above ordinary differential equation is balanced at $m = 2$

Step 3: In this method, we apply a finite series method of the Jacobi elliptic Function on a nonlinear equation and find the values of constants. Here we use cosine and sine functions which are represented by *cn* and *sn*.

The solution to the previous equation could take the form of a travelling wave.

$$\left(\frac{\partial z}{\partial \xi}\right) = (b_1 + 2b_2 sn\xi)cn(\xi)dn(\xi), \quad (55)$$

$$q\left(\frac{\partial z}{\partial \xi}\right) = \left.\begin{matrix}[a_0b_1 + (a_1b_1 + 2a_0b_2)sn(\xi)]cn(\xi)dn(\xi) \\ +[(2a_1b_2 + a_2b_1)sn^2(\xi) + 2a_2b_2sn^3\xi]cn(\xi)dn(\xi)\end{matrix}\right\}, \quad (56)$$

$$z\left(\frac{\partial q}{\partial \xi}\right) = \left.\begin{matrix}[b_0a_1 + (a_1b_1 + 2b_0a_2)sn(\xi)]cn(\xi)dn(\xi) \\ +[(2b_1a_2 + a_2b_1)sn^2(\xi) + 2b_2a_2sn^3\xi]cn(\xi)dn(\xi)\end{matrix}\right\}, \quad (57)$$

$$\left(\frac{\partial^3 q}{\partial \xi^3}\right) = \left.\begin{matrix}[-(1+m^2)a_1 - 8(1+m^2)a_2sn(\xi)]cn(\xi)dn(\xi) \\ +[6m^2a_1sn^2(\xi) + 24m^2a_2sn^3(\xi)]cn(\xi)dn(\xi)\end{matrix}\right\}, \quad (58)$$

By substituting the values of Eqs (55)–(58) in Eq. (36), we have the following equation:

$$\left.\begin{matrix} -c[(b_1+2b_2sn\xi)cn(\xi)dn(\xi)] + [a_0b_1 + (a_1b_1+2a_0b_2)sn(\xi)]cn(\xi)dn(\xi) \\ +[(2a_1b_2 + a_2b_1)sn^2(\xi) + 2a_2b_2sn^3\xi]cn(\xi)dn(\xi) \\ +[b_0a_1 + (a_1b_1 + 2b_0a_2)sn(\xi)]cn(\xi)dn(\xi) \\ +[(2b_1a_2 + a_2b_1)sn^2(\xi) + 2b^2a_2sn^3\xi]cn(\xi)dn(\xi) \\ +\beta k^2[-(1+m^2)a_1 - 8(1+m^2)a_2sn(\xi)]cn(\xi)dn(\xi) \\ +\beta k^2[6m^2a_1sn^2(\xi) + 24m^2a_2sn^3(\xi)]cn(\xi)dn(\xi)\end{matrix}\right\}, = 0 \quad (59)$$

By comparing the coefficients on both sides, we have,

$$[-cb_1 + a_0b_1 + a_1b_0 - k^2\beta(1+m^2)a_1] = 0, \tag{60}$$

$$[-2b_2c + 2a_0b_2 + 2a_1b_1 + 2b_0a_2 - 8k^2\beta(1+m^2)a_2] = 0, \tag{61}$$

$$3a_1b_2 + 3a_2b_1 + 6\beta k^2m^2a_1 = 0, \tag{62}$$

$$4a_2b_2 + 24\beta k^2m^2a_2 = 0, \tag{63}$$

By solving the above-mentioned equations by using Mathematica Bulit in Software for finding the values of the constants a_0, a_1, a_2, b_0, b_1 and b_2.

$$a_0 = c + \left(\frac{\beta}{2\alpha c}\right) - 4ck^2\alpha(1+m^2), \tag{64}$$

$$a_1 = 0, \tag{65}$$

$$a_2 = 12k^2m^2\alpha c, \tag{66}$$

$$b_0 = \left(\frac{\beta^2}{4\alpha^2c^2}\right) + 2\beta(1+m^2)k^2, \tag{67}$$

$$b_1 = 0, \tag{68}$$

$$b_2 = -6c\beta^2m^2, \tag{69}$$

Hence the series solution of the sn for the coupled Eqs (35) and (36) written as

$$q(\xi) = c + \left(\frac{\beta}{2\alpha c}\right) - 4ck^2\alpha(1+m^2) + [12k^2m^2\alpha c]sn^2(\xi) + O\left(sn^3(\xi)\right), \tag{70}$$

$$z(\xi) = \left(\frac{\beta^2}{4\alpha^2c^2}\right) + 2\beta(1+m^2)k^2 + [-6c\beta^2m^2]sn^2(\xi) + O\left(sn^3(\xi)\right), \tag{71}$$

It is the solution for cnoidal waves and the precise periodic solution of Eqs (70) and (71), while the corresponding solitary wave solution for them is

$$q(\xi) = c + \left(\frac{\beta}{2\alpha c}\right) + 4ck^2\alpha - [12k^2\alpha c]sech^2(\xi), \tag{72}$$

$$z(\xi) = -\left(\frac{\beta^2}{4\alpha^2c^2}\right) - 2\beta k^2 - [6c\beta^2]sech^2(\xi), \tag{73}$$

2.3.3. Model 3

Exact solutions of nonlinear evolution equations (NLEEs) are very important to figure out how complex physical phenomena work on the inside. In this work, the new generalised Jacobi elliptic function expansion method is used to look at the exact travelling wave solutions of the Boussinesq equation [42]. With this method, one can get a lot of travelling wave solutions with any parameters, and the wave solutions are written in terms of elliptic functions. It is shown that the new generalised Jacobi elliptic function expansion method is a powerful and clear way to solve nonlinear partial differential equations in mathematical physics and engineering [37].

$$\left(\frac{\partial^2 q}{\partial t^2}\right) - 12c_0^2\left(\frac{\partial^2 q}{\partial x^2}\right) - \alpha q\left(\frac{\partial^4 q}{\partial x^4}\right) - 6\beta\left(\frac{\partial^2 q^2}{\partial x^2}\right) = 0, \tag{75}$$

The solution to Eq. (75) is

$$c^2\left(a_2 - (1+m^2)a_1sn(\xi) - 4a_2(1+m^2)sn^2(\xi) + 2m^2a_1sn^3(\xi) + 6m^2a_2sn^4(\xi)\right) \tag{76}$$

By comparing the coefficients on both sides of the Eq. (76), we find the values of the constants given below:

$$a_0 = \left(\frac{c^2-c_0^2}{2\beta}\right) + \left(\frac{2\alpha k^2}{\beta}\right) + \left(\frac{2m^2\alpha k^2}{\beta}\right), \tag{77}$$

$$a_1 = 0, \tag{78}$$

$$a_2 = -\left(\frac{6m^2\alpha k^2}{\beta}\right), \tag{79}$$

The solution of Model 3 is in the form of the periodic solitary wave by using Eqs (77–79) in Eq. (19) we have,

$$q(\xi) = \left(\frac{c^2-c_0^2}{2\beta}\right) + \left(\frac{2\alpha k^2}{\beta}\right) + \left(\frac{2m^2\alpha k^2}{\beta}\right) - \left(\frac{6m^2\alpha k^2}{\beta}\right)sn^2(\xi) + O\left(sn^3(\xi)\right), \tag{80}$$

The solitary wave solution that corresponds to this one is

$$q(\xi) = \left(\frac{c^2-c_0^2}{2\beta}\right) - \left(\frac{2\alpha k^2}{\beta}\right) + \left(\frac{6\alpha k^2}{\beta}\right)sech^2(\xi), \tag{81}$$

3. CONCLUDING REMARKS

The Jacobi elliptic function expansion approach is presented and applied to nonlinear wave equations in this study. This method is more general than the hyperbolic tangent function expansion method, as demonstrated. In addition, the shock wave and solitary wave solutions are included in the periodic wave solutions derived from the Jacobi elliptic function expansion approach. In the applications, it is demonstrated that the Jacobi elliptic function expansion approach applies to both single and coupled equations. In fact, this method can be used to solve more nonlinear wave equations so long as the odd-order and even-order derivative terms do not overlap in the nonlinear wave equations. It is found that the shock waves break with time, whereas, solitary waves are improved continuously with time.

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