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RECTANGULAR POLYGONS AND ITS SHAPE PARAMETERS

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Abstract: The author's interest in rectangular polygon shapes resulted from the observation of plans of detached houses. It is also a consequence of the author's research into roofs. This paper summarizes the basic properties of rectangular polygons and formulates three parameters characterizing the shape of a rectangular polygon: perimeter defect, area defect and span.

Keywords: rectangular polygon, perimeter defect, area defect, span of rectangular polygon, roof skeleton (straight skeleton)

1 Introduction and basic properties of rectangular polygons

Most buildings, especially detached houses, built on the plan of a polygon that has convex or concave angles [3,4]. Such polygons have interesting properties. Some of them are described in [3,5], including one that is fundamental to the figures. Let us recall it.

Figure 1: City Hall in Bialystok – example of a building on the set of rectangular polygon (author‘s photo)

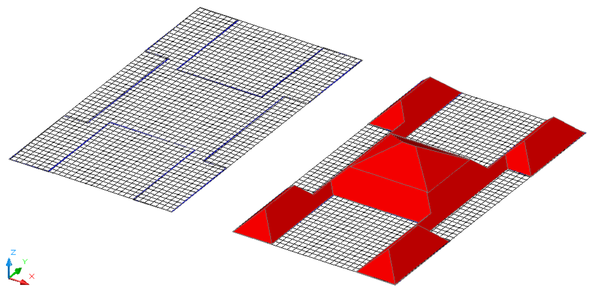

Figure 2: Plan of the Town Hall as a rectangular polygon (on the left). Study of the shape of the roof of City Hall in Białystok (on the right)

Assume that the simply connected polygon RP with n vertices and the convex angles (of 90 degrees) and the concave angles (having 270 degrees) are given (Figs. 3, 4). Such a polygon is called a *rectangular polygon* [3,4]. It turns out that the rectangular polygon must have an even number of sides. Indeed, if m denotes the number of convex angles, and k denotes the number of concave angles in any n-gon we obtain the following system of equations

$$\begin{cases} m \cdot 90^o + k \cdot 270^o = (n-2) \cdot 180^o, \\ m + k = n, \end{cases} \tag{1}$$

where m, k and n are integers. The solution of the system (1) is a pair of numbers

$$m = \frac{n}{2} + 2, k = \frac{n}{2} - 2. \tag{2}$$

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Immediately we can see that it would release (2) system (1) there where a number n must be even and (then) the difference between m and k equals 4. So, there is not any rectangular polygon with an odd number of sides. The difference between the number of convex angles and the number of concave angles is four. Every rectangle has no concave angles, a rectangular hexagon has one reflex angle, rectangular octagon has two concave corners, decagon - three, etc. [3,4].

In order not to consider degenerate polygons and have a fairly broad class figures we assume that the rectangular polygons described here are connected regions, and may have holes (Fig. 1c). Consistency of a polygon (generally determined for a planar set) means that any two points of the polygon can be connected by a broken line (a polygonal line [2]) contained within the polygon.

If we consider a rectangular l-connected n-polygon RP_n^l (i.e. with l–1-holes and n vertices, then every i-th hole is a simply connected rectangular polygon with h_i vertices (i = 1,2, ..., l–1). The sides of the rectangular simply connected polygon are mutually perpendicular or parallel. Then we have $n = n_0 + \sum_{i=1}^{l-1} h_i$, where n_0 is the number of vertices of the polygon containing a hole treated as a simply connected polygon. Then every convex angle of i-th rectangular h_i-polygon is concave, and vice versa. Using the solution (2), for the l-connected rectangular polygon with l–1 h_i-angled holes, i = 1,2, ..., l–1 the number of concave angles is equal to

$$k = \frac{n_0}{2} - 2 + \sum_{i=1}^{l-1}\left(\frac{h_i}{2} + 2\right), \tag{3}$$

and the number of convex angles is equal to

$$m = \frac{n_0}{2} + 2 + \sum_{i=1}^{l-1}\left(\frac{h_i}{2} - 2\right). \tag{4}$$

It is easy to see that $m+k=n$.

2 Rectangular polygons inscribed in rectangle. Perimeter defect and area defect

Next we will consider rectangular polygons inscribed in a rectangle (Fig. 3). The sides of the rectangular polygon l-conected RP^l are mutually perpendicular or parallel. One can imagine that when walking along the border of the rectangular polygon we are going in four directions: forward (fd), left (lt), right (rt) and back (bk). As the number of vertices of the rectangular polygon is finite, then at some point we find ourselves at the very front (a line segment ft^{-} defining a line ft ($\mathrm{ft}^{-}\subset\mathrm{ft}$) and the half-plane $\mathrm{ft}^{\rightarrow}$ containing polygon RP^l ($\mathrm{RP}^l\subset\mathrm{ft}^{\rightarrow}$)) (Fig. 3a). In another case we find ourselves in the rightmost section (line segment rt^{-} defining a line rt ($\mathrm{rt}^{-}\subset\mathrm{rt}$) and the half-plane $\mathrm{rt}^{\rightarrow}$ containing polygon RP^l (RPl$\subset\mathrm{rt}^{\rightarrow}$)) (Fig. 3a1), in still another case we are at the very back (a line segment rr^{-} defining a line rr ($\mathrm{rr}^{-}\subset\mathrm{rr}$) and the half-plane $\mathrm{rr}^{\rightarrow}$ containing polygon RP^l ($\mathrm{RP}^l\subset\mathrm{rr}^{\rightarrow}$)) (Fig. 3a2), and finally, we find ourselves in the leftmost section (a line segment lt^{-} defining a line lt ($\mathrm{lt}^{-}\subset\mathrm{lt}$) and the half-plane $\mathrm{lt}^{\rightarrow}$ containing polygon RP^l ($\mathrm{RP}^l\subset\mathrm{lt}^{\rightarrow}$)) (Fig. 3a3). These extreme lines ft, lt, rr, rt, clearly define the rectangle $\mathrm{R} = \mathrm{ft}^{\rightarrow}\cap \mathrm{lt}^{\rightarrow}\cap \mathrm{rr}^{\rightarrow}\cap \mathrm{rt}^{\rightarrow}$. Then $\mathrm{RP}^l\subset\mathrm{R}$. There is a one-to-one correspondence of this polygon with the rectangle in which it is inscribed.

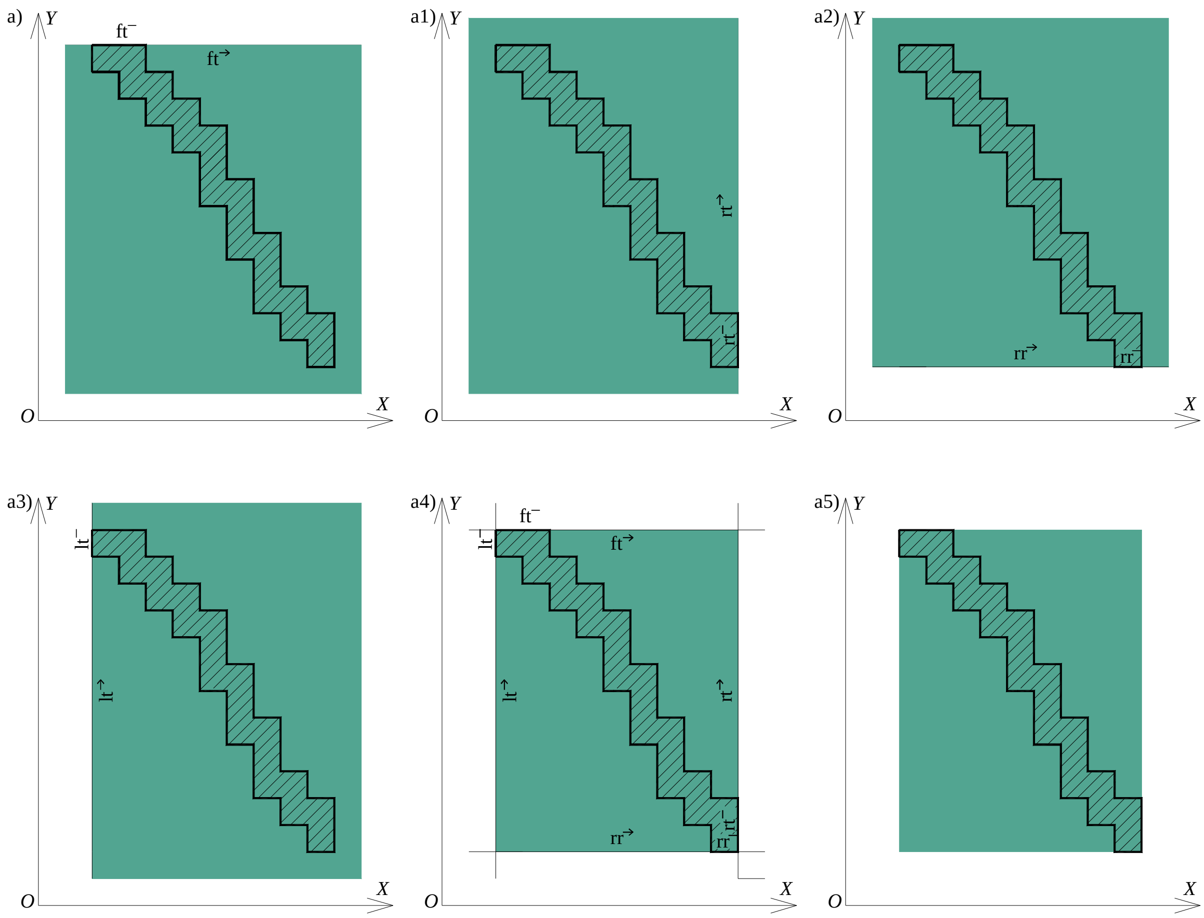

Figure 3: Creation of a rectangle circumscribed on the rectangular polygon: a) the position "at the front" a line segment ft^-, a straight line ft ($ft^- \subset ft$) and a half-plane containing $ft^{\rightarrow}$ the polygon RP^l ($RP^l \subset ft^{\rightarrow}$)); a1) a line segment rt^- , a straight line rt ($rt^- \subset rt$) and a half-plane $rt^{\rightarrow}$ containing the polygon RP^l ($RP^l \subset rt^{\rightarrow}$); a2) a line segment rr^-, a straight line rr ($rr^- \subset rr$) and a half-plane $rr^{\rightarrow}$ containing the polygon RP^l ($RP^l \subset rr^{\rightarrow}$); a3) a line segment lt^-, a straight line lt ($lt^- \subset lt$) and a half-plane $lt^{\rightarrow}$ containing the polygon RP^l ($RP^l \subset lt^{\rightarrow}$); a4) four half-planes uniquely define the rectangle circumscribed on the rectangular polygon; a5) a rectangle circumscribed on a rectangular polygon

We shall say that RP^l is a rectangular polygon inscribed in a rectangle R if and only if $RP^l \subset R$ and each side of the rectangle R contains at least one side of the polygon RP^l. The rectangle R can be considered as circumscribed on the polygon RP^l. We agree that the sides of the rectangle R are parallel to the axis of a Cartesian coordinate system *OXY*, the axis *OX* is horizontal, the axis *OY* – vertical. Note that on every rectangular polygon (with a finite number of sides, because we are only going to deal with these), you can clearly circumscribe a rectangle. It is defined by two pairs of parallel lines, designated by parallel sides of the polygon which are the most distant from each other in *OX* and *OY* orientations. As we have already mentioned the considered rectangular polygons are coherent sets. Consistency of RPL polygon inscribed in a rectangle R means that any straight line parallel to the sides of the rectangle R containing the points inside the rectangle also includes inner points and boundary points of the polygon RP^l (Fig. 3). Then each boundary point of the rectangle R is the orthogonal projection of at least two polygon RP^l boundary points. One can use the sides of the RP^l polygon (after parallel shift to the *OX*, *OY* axes respectively to the edge of the rectangle R) as “wallpaper" for the boundary of the rectangle R. Note that if p(F) denotes the perimeter of a region F, with respect to the RP^l polygon inscribed in a rectangle R, we can write

$$p(\mathrm{RP}^l) = p(\mathrm{R}) + \Delta p(\mathrm{RP}^l), \tag{5}$$

where

$$\Delta p(\mathrm{RP}^l) = \sum_{i=0}^{n-1} 2i \cdot x_{2i+2} + \sum_{j=0}^{m-1} 2j \cdot y_{2j+2}\ , \tag{6}$$

whereas x_{2i+2} is the sum of the measurements of orthogonal projections to the sides of the rectangle R of all parts of sides of the polygon RP^l parallel to the axis OX, that intersect the straight line parallel to the axis OY at $2i$ points in total, with i=0,1,…,n–1 (Fig. 4a), y_{2j+2} is the sum of the measurements orthogonal projections to the sides of the rectangle R of all parts of sides of the polygon RP^l parallel to the axis OX(OY), which intersect the straight line parallel to the axis OY (OX) at $2j$ points in total, with j=0,1,…,m–1 (Fig. 4b). The number $2n$ ($2m$) means the maximum number of sides of the polygon RP^l parallel to the axis OX(OY), which are intersected by the line parallel to the axis OY (OX). The size Δp(RP^l), defined by the formula (5) will be called the *perimeter defect* of the polygon RP^l.

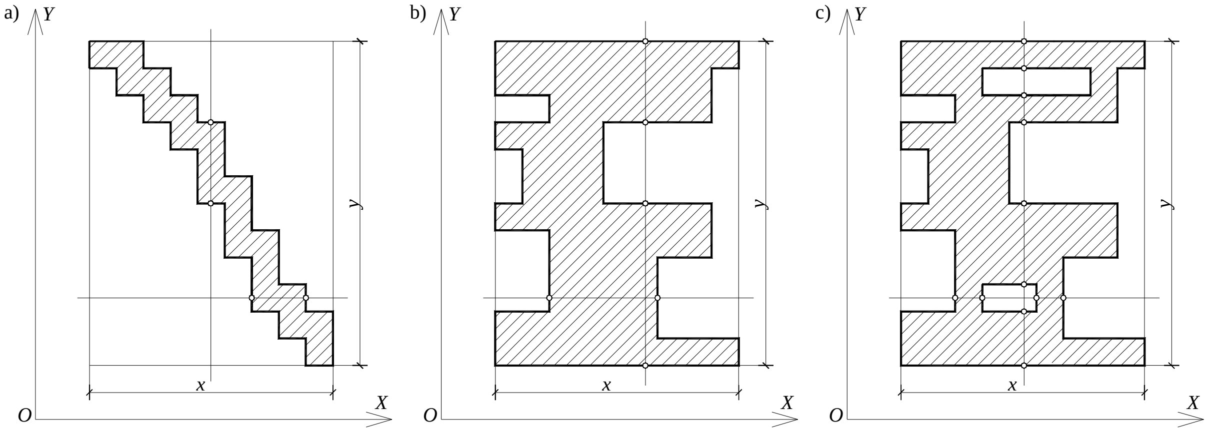

Figure 4: Rectangular polygons inscribed in a rectangle: a) a simple connected polygon, monotonic with respect to both axes, i.e. normal; b) a simple connected polygon monotonic with respect to the axis OY; but non-monotonic with respect to the axis OX; c) the polygon 3-connected non-monotonic with respect to both axes

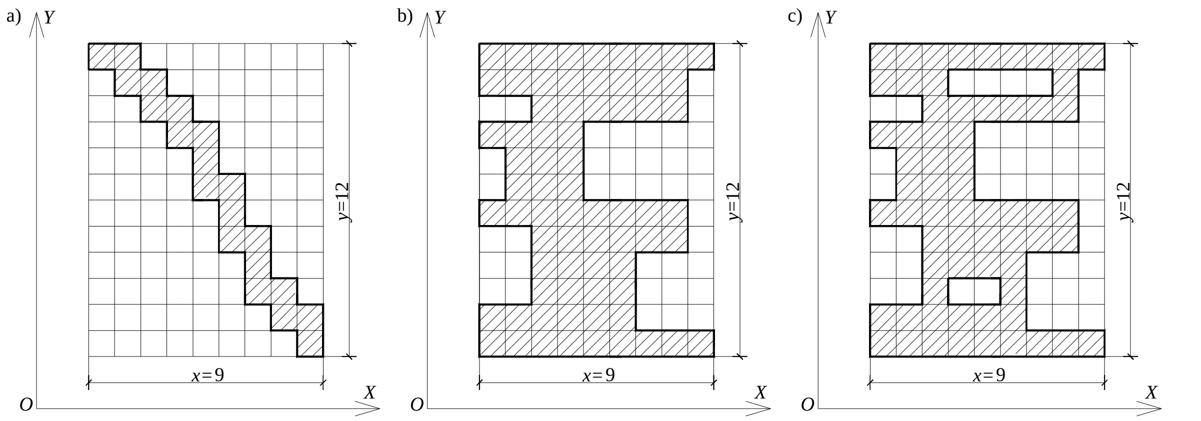

Figure 5: Rectangular polygons inscribed in a rectangle with dimensions of $x \times y$, x=9u, y=12u: a) with a big area defect (Δa=88u^2), rda=0,81 (81%), and zero's perimeter defect, rdp = 0 (0%); b) with a positive perimeter defect (Δp=24u, rdp=0,57 (57%)), with area defect (Δa=38u^2, rda=0,35 (35%)); c) with a positive perimeter defect (Δp=40u, rdp=0,95 (95%)), with area defect (Δa=44u^2, rda=0,41 (41%))

If all straight lines parallel to the axis OY (OX) intersect only two sides of the polygon RP^l parallel to the axis OX(OY), then n=1 and i=0 (m=1 and j=0) and then due to (5) we have $\Delta p(RP^l) = 0$, which is a defect of the rectangular polygon inscribed in a rectangle which equals zero (Fig. 4a). Then, according to (3) $p(RP^l) = p(R)$, i.e. a perimeter of a polygon inscribed in a rectangle is equal to the perimeter of the rectangle. Such a rectangular polygon will be called a *normal* rectangular polygon inscribed in a rectangle (Fig. 1a). A normal rectangular polygon is a monotonic polygon with respect to any straight line not parallel to the sides of the bounding rectangle (a polygon is called *monotonic* with respect to the line l, if for any line l' perpendicular to l the intersection of the polygon with l' is connected) [2]. In other words, a normal rectangular polygon with respect to the rectangle has the property that any straight line parallel to the sides of the rectangle, to the axis OY(OX) intersecting the rectangle in inner points intersects exactly two sides of the rectangular polygon RP^l parallely to the axis OX (OY).

The second parameter that characterizes the geometry of the rectangular polygon RP^l is its area $a(RP^l)$ (general area $a(F)$ of a region F). A area of a rectangular polygon RP^l is expressed

$$a(RP^l)= a(R) - \Delta a(RP^l), \tag{6}$$

where $\Delta a(RP^l)$ will be called the *area defect* of the polygon RP^l.

Perimeter defect and area defect of the polygon defined in absolute terms do not reflect the size of the measurement deviations from perimeter and area of a rectangle. Besides, in practical applications these will depend on the accepted measurement units of length and area. Therefore, it is desirable to describe these measurement deviations (from the ideal figure – a rectangle) in a relative manner. Let us introduce therefore, two concepts: the *relative perimeter defect* of the rectangular polygon

$$rdp(RP^l)= \frac{\Delta p}{p(R)}, \tag{7}$$

and the *relative area defect* of the rectangular polygon

$$rda(RP^l) = \frac{\Delta a}{a(R)}. \tag{8}$$

Relative defect of area, with the defect of perimeter equal to zero, shows the degree of "imperfections" of the border line. The same length of perimeter takes up roughly $\frac{\Delta a}{a(R)} \cdot 100\%$ smaller area; so there is loss in the area at the same perimeter. Because the larger the perimeter defect the larger perimeter, area defect with increased perimeter results in even greater losses of the area.

3 Span of rectangular polygon

An important structure parameter of a building is its span. This insight allows you to apply the concept of span to a rectangular polygon. In the case of a building this will be the maximum distance between two parallel walls. The question remains which of the walls, if there are a number of such parallel walls? It turns out that in the case of a building it is good to use the hipped-roof ends, but prior to that the roof needs to be solved. In the case of a rectangular polygon we shall act in a similar manner by constructing a straight skeleton (clearly defined for the polygon) [1,3]. Treating the edge of the rectangular polygon as a line of eaves we construct the skeleton of the roof (Fig. 6b). By the span $s(RP^l)$ of a rectangular polygon we mean the largest projection height of all roof surfaces measured relative to the eaves of the roof (Fig. 6). For a given orthographic skeleton of a roof stretched over the edge of the rectangular polygon it is found in the following way:

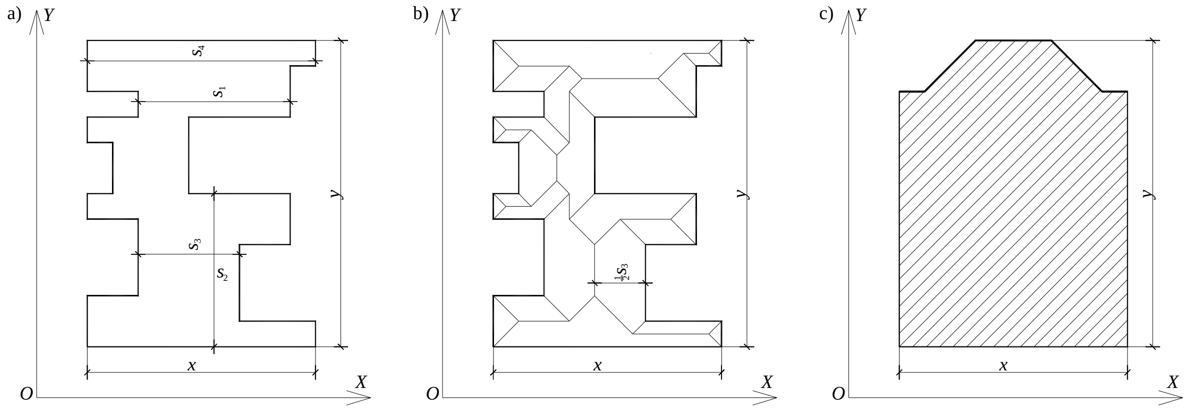

Figure 6: Determining the span of a rectangular polygon: a) after the first overview the span value of a polygon is not directly visible: s2 or s3 or perhaps s4?; b) after constructing a simple skeleton the algorithm for determining the span is easy to formulate; c) a shape close to rectangular polygon

1: From the corner points skeleton of the roof we construct line segments s_{ij} of s_{ij} lengths descended into the eaves (i, j) (the base of the corresponding polygon of (i, j)-th hipped roof end) [3]. The two-index hipped roof end indicators are derived from [3], where for the generalized l-connected polygons i represents the number of a polygon (i = 1) or sub-polygon-hole (i = 2,3,..., l), j - eaves number (hipped roof end number) of the polygon (sub-polygon) (i = 1,2,3,..., l) (Fig. 7).

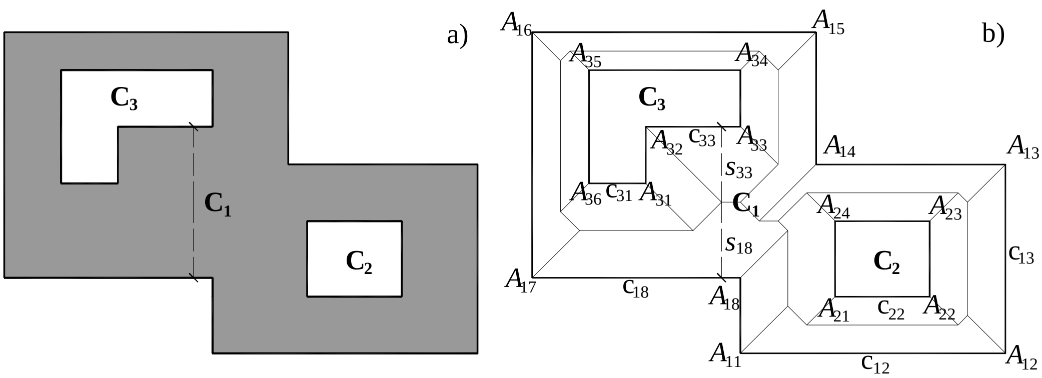

Figure 7: Determining the span of a 3-connected rectangular polygon RP^3; a) a 3-connected rectangular polygon indicating the polygon C_1 and sub-polygons (holes) C_2, C_3 of the line segment defined in Figure 7b e defining a span; b) the process of determining the span $s(RP^3)$ by solving the roof: $s(RP^3) = 2{\cdot}s_{18}$, $s(RP^3) = 2{\cdot}s_{33}$, as well as the sum of $s(RP^3) = s_{18} + s_{33}$ heights of hipped roof end adjacent to each other along the ridge

2: The length of the maximum hipped roof end height multiplied by 2 is the span of a polygon, ie.

$$s(RP^l) = 2\cdot \max_{ij}\{s_{ij}\}. \tag{10}$$

This is another interesting application of geometry of the roof (straight skeletons), this time to facilitate the formulation and determination of the rectangular polygon span, as a result of a fairly simple geometric process involving viewing the height of slope following the eaves. The span of a polygon indicates its "slender" qualities. Note that the span of a rectangle is equal to the length of its shorter side.

It is worth noting that the concept of span can be generalized to any polygon, not necessarily rectangular and that if eg. span of a triangle is equal to the double distance of the point of intersection of the angle bisectors from any side. The concept of span can be generalized to any planar region in which a polygon can be inscribed.

4 Conclusions

The properties of rectangular polygons and their parameters can be used to describe and analyze the shape of buildings (especially detached houses) and their optimization. These properties, in practice, for example, applied to optimize the shape of the building, can also be used for a number of shapes similar to rectangular polygons (Fig. 6c). Naturally, this requires further study and discussion to check the possibility to override the lack of precise shape of a rectangular polygon.

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WIELOKĄTY PROSTOKĄTNE I PARAMETRY ICH KSZTAŁTU

Zainteresowanie wielokątami prostokątnymi wynikło z obserwacji kształtu planów domów jednorodzinnych. Jest też konsekwencją badań autora dotyczących dachów. W pracy podano podstawowe własności wielokątów prostokątnych i sformułowano trzy parametry charakteryzujące kształt wielokąta prostokątnego: defekt obwodu, defekt pola i rozpiętość.