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AN OPERATIONAL CALCULUS MODEL WITH THE (m, n)-SYMMETRIC DIFFERENCE

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ABSTRACT

The paper determines a non-classical Bittner operational calculus model, in which the derivative is understood as an (m, n)-symmetric difference $D_{m,n}\{x(k)\} :=\{x(k + m) - x(k - n)\}$. By considering an operation $D_{m,n,b}\{x(k)\} := \{x(k + m) - b(k)x(k - n)\}$, the formulated model has been generalized.

Key words:
operational calculus, derivative, integrals, limit conditions, (m, n)-symmetric difference.

Research article

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FOUNDATIONS OF THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [1–4] is referred to as a system

$$CO(L^0, L^1, S, T_q, s_q, Q), \tag{1}$$

where L^0 and L^1 are linear spaces (over the same field Γ of scalars) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ such that $ST_qf = f, f \in L^0$, called *integrals* and for the operations $s_q \in \mathscr{L}(L^1, L^1)$ such that $s_qx = x - T_qSx, x \in L^1$, called *limit conditions*. The kernel of S, i.e. $\operatorname{Ker} S$ is called a set of *constants* for the derivative S.

It easy to verify that the limit conditions $s_q, q \in Q$ are projections of L^1 onto the subspace $\operatorname{Ker} S$.

If we define the objects (1), then we have in mind a *representation,* or a *model* of the operational calculus. In particular, representations of the operational calculus (1) are *discrete* models in which the derivative S can be understood as:
an n^{th}*-order forward difference* [16]

$$\Delta_n x(k) := x(k+n) - x(k), \tag{2}$$

an n^{th}*-order backward difference* [15]

$$\nabla_n x(k) := x(k) - x(k-n) \tag{3}$$

or a $2n^{th}$*-order central difference* [17]

$$D_n x(k) := x(k+n) - x(k-n), \tag{4}$$

where n is a given natural number.

An (m, n)*-symmetric difference*[1]

$$D_{m,n} x(k) := x(k+m) - x(k-n), \tag{5}$$

where m, n are given non-negative integers such that $m^2 + n^2 > 0$, is a generalization of the operations (2)–(4).

In this paper, one shall determine such an operational calculus model, in which the operation (5) will be the derivative S. Next, based on this model, we shall develop a more general representation with a derivative

[1] This notion is related to the α, β-symmetric difference derivative considered in [5,6].

$$D_{m,n,b}x(k) := x(k+m) - b(k)x(k-n)^2. \tag{6}$$

Certain specific models related to (6) were considered in literature before: a model with a derivative

$$D_{1,0,b}x(k) = x(k+1) - b(k)x(k)$$

was constructed in [10], whereas a model with a derivative

$$D_{0,1,b}x(k) = x(k) - b(k)x(k-1)$$

was considered in [13].

For b being a constant sequence, representations of operational calculus with derivatives

$$D_{n,0,b}x(k) = x(k+n) - bx(k),$$
$$D_{0,n,b}x(k) = x(k) - bx(k-n),$$
$$D_{n,n,b}x(k) = x(k+n) - bx(k-n)$$

were discussed in [16], [15], [17].

(m, n)-SYMMETRIC DIFFERENCE MODEL

Let $\mathbb{Z}$ and $\mathbb{C}$ denote sets of integers and complexes, respectively. Moreover, let $C(\mathbb{Z},\mathbb{C})$ be a linear space of two-sided complex sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with a common sequences addition, and sequences multiplication by complex numbers. Further, let

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\nu-1} \tag{7}$$

be roots of unity of order $\nu := m + n$, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{\nu} + \mathrm{i}\,\sin\frac{2j\pi}{\nu}, \quad j \in \overline{0,\nu-1}\,^3,$$

where ‘i’ is the imaginary unit.

2 The sequences multiplication in (6) means a usual coordinate-wise (Hadamard) multiplication.

3 $\overline{0,\nu-1} := \{0, 1, \ldots, \nu-1\}$.

Using the following properties of the sequence (7):

$$\varepsilon_j^{k+\ell\nu} = \varepsilon_j^{k}, \quad j \in \overline{0, \nu-1}, \quad k, \ell \in \mathbb{Z},$$
$$\varepsilon_0^r + \varepsilon_1^r + \ldots + \varepsilon_{\nu-1}^r = 0, \quad r \neq \ell\nu, \nu > 1, \ell, r \in \mathbb{Z}$$

we shall prove the below

Theorem 1. *A system* (1)*, where* $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{Z}, \mathbb{C}), k_0 \equiv q \in Q := \mathbb{Z}$ *and*

$$S\,x := \{x(k+m) - x(k-n)\}^4 \tag{8}$$

$$T_{k_0}x := \begin{cases} -\dfrac{1}{\nu}\displaystyle\sum_{j=0}^{\nu-1}\sum_{i=k+n}^{k_0+n-1} \varepsilon_j^{k+n-i}\, x(i) & for \quad k < k_0 \\ 0 & for \quad k = k_0 \\ \dfrac{1}{\nu}\displaystyle\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k+n-1} \varepsilon_j^{k+n-i}\, x(i) & for \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}, \tag{9}$$

$$s_{k_0}x := \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+\nu-1} \varepsilon_j^{k-i}\, x(i)\right\} \tag{10}$$

forms a discrete model of the Bittner operational calculus[5]*.*

Proof. Obviously, the operations (8)–(10) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Hence, for $k = k_0$ we obtain

$$S\{y(k)\}|_{k=k_0} = \{y(k_0+m) - y(k_0-n)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k_0+\nu-1} \varepsilon_j^{k_0-i}\, x(i) + \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k_0-i}\, x(i)\right\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+\nu-1} \varepsilon_j^{k_0-i}\, x(i)\right\}$$

$$= \left\{x(k_0) + \frac{1}{\nu}\sum_{i=k_0+1}^{k_0+\nu-1} (\,\varepsilon_0^{k_0-i} + \varepsilon_1^{k_0-i} + \cdots + \varepsilon_{\nu-1}^{k_0-i})\, x(i)\right\} = \{x(k)\}|_{k=k_0}.$$

For $k < k_0$ and $k + m = k_0$ (i.e. $k = k_0 - m$), we have in turn

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[4] $\{x(k)\}$ means a symbol of the sequence x, i.e. $x = \{x(k)\}$, whereas $x(k)$ means the k^{th}-term of the sequence $\{x(k)\}$, where $k \in \mathbb{Z}$. This notation originates from J. Mikusiński [11].

[5] Due to the definition of T_{k_0}, we assume that $\sum_{i=k_0+n}^{k_0+n-1} x(i) := 0$.

$$S\{y(k)\} = \{y(k+m) - y(k-n)\} = \{y(k_0) - y(k_0-\nu)\}$$

$$= \left\{0 + \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0-m}^{k_0+n-1} \varepsilon_j^{k_0-m-i}\, x(i)\right\}$$

$$= \left\{x(k_0-m) + \frac{1}{\nu}\sum_{i=k_0-m+1}^{k_0+n-1}\left(\varepsilon_0^{k_0-m-i} + \varepsilon_1^{k_0-m-i} + \cdots + \varepsilon_{\nu-1}^{k_0-m-i}\right) x(i)\right\} = \{x(k_0-m)\}.$$

Therefore,

$$S\{y(k)\} = \{x(k_0-m)\} \quad \text{that is} \quad S\{y(k)\} = \{x(k)\}.$$

For $k < k_0$ and $k+m < k_0$ (i.e. $k+\nu < k_0+n$), we get

$$S\{y(k)\} = \{y(k+m) - y(k-n)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i) - \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k+\nu}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i)\right\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1} \varepsilon_j^{k-i}\, x(i)\right\}$$

$$= \left\{x(k) + \frac{1}{\nu}\sum_{i=k+1}^{k+\nu-1}\left(\varepsilon_0^{k-i} + \varepsilon_1^{k-i} + \cdots + \varepsilon_{\nu-1}^{k-i}\right) x(i)\right\} = \{x(k)\}. \tag{11}$$

When $k > k_0$ and $k+m > k_0$ (i.e. $k+\nu > k_0+n$), we obtain in turn

$$S\{y(k)\} = \{y(k+m) - y(k-n)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k+\nu-1} \varepsilon_j^{k-i}\, x(i) + \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i)\right\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1} \varepsilon_j^{k-i}\, x(i)\right\},$$

so, by analogy to (11), we have $S\{y(k)\} = \{x(k)\}$.
If $k > k_0$ and $k-n = k_0$ (i.e. $k+m = k_0+\nu$), then

$$S\{y(k)\} = \{y(k+m) - y(k-n)\} = \{y(k_0+\nu) - y(k_0)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k_0+\nu+n-1} \varepsilon_j^{k_0+n-i}\, x(i) - 0\right\}$$

$$= \left\{x(k_0+n) + \frac{1}{\nu}\sum_{i=k_0+n+1}^{k_0+\nu+n-1}\left(\varepsilon_0^{k_0+n-i} + \varepsilon_1^{k_0+n-i} + \cdots + \varepsilon_{2n-1}^{k_0+n-i}\right) x(i)\right\} = \{x(k_0+n)\},$$

that is $S\{y(k)\} = \{x(k)\}$.
For $k > k_0$ and $k-n > k_0$ (i.e. $k > k_0+n$), we have

$$S\{y(k)\} = \{y(k+m) - y(k-n)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i) - \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k-1}\varepsilon_j^{k-i}\,x(i)\right\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i)\right\},$$

which, similarly to (11), also means that $S\{y(k)\} = \{x(k)\}$.
Now, if $k > k_0$ and $k - n < k_0$ (i.e. $k < k_0 + n$), then

$$S\{y(k)\} = \{y(k+m) - y(k-n)\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i) + \frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i)\right\} = \{x(k)\}.$$

Finally, we can confirm that the property $S\,T_{k_0}x = x$ is fulfilled.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+m) - x(k-n)\}$. Then, for $k < k_0$ we get

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{-\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,f(i)\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,x(i-n) - \sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,x(i+m)\,\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k}^{k_0-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=k+\nu}^{k_0+\nu-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k}^{k_0-1}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k_0}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i) - \Big(\sum_{i=k_0}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k+\nu}^{k_0+\nu-1}\varepsilon_j^{k-i}\,x(i)\,\Big)\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1}\varepsilon_j^{k-i}\,x(i)\right\} - \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+\nu-1}\varepsilon_j^{k-i}\,x(i)\right\}.$$

Hence, on the basis of (11), we eventually obtain

$$T_{k_0}S\{x(k)\} = \{x(k)\} - \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+\nu-1}\varepsilon_j^{k-i}\,x(i)\right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Similarly, if $k > k_0$, then

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0+n}^{k+n-1}\varepsilon_j^{k+n-i}f(i)\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k_0+n}^{k+n-1}\varepsilon_j^{k+n-i}x(i+m) - \sum_{i=k_0+n}^{k+n-1}\varepsilon_j^{k+n-i}x(i-n)\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k_0+\nu}^{k+\nu-1}\varepsilon_j^{k-i}x(i) - \sum_{i=k_0}^{k-1}\varepsilon_j^{k-i}x(i)\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\Big[\sum_{i=k}^{k_0+\nu-1}\varepsilon_j^{k-i}x(i) + \sum_{i=k_0+\nu}^{k+\nu-1}\varepsilon_j^{k-i}x(i) - \Big(\sum_{i=k_0}^{k-1}\varepsilon_j^{k-i}x(i) + \sum_{i=k}^{k_0+\nu-1}\varepsilon_j^{k-i}x(i)\Big)\Big]\right\}$$

$$= \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k}^{k+\nu-1}\varepsilon_j^{k-i}x(i)\right\} - \left\{\frac{1}{\nu}\sum_{j=0}^{\nu-1}\sum_{i=k_0}^{k_0+\nu-1}\varepsilon_j^{k-i}x(i)\right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Therefore, the property $T_{k_0}S\,x = x - s_{k_0}x$ is also fulfilled. □

Since for $k \in \overline{k_0+1, k_0+\nu-1}$ we have

$$\sum_{i=k_0+n}^{k+n-1}\left(\varepsilon_0^{k+n-i} + \varepsilon_1^{k+n-i} + \cdots + \varepsilon_{\nu-1}^{k+n-i}\right) = 0,$$

then, from the definition (9) of the integrals T_{k_0}, we get the following

Corollary1. *If* $\{y(k)\} := T_{k_0}\{x(k)\}$*, then*

$$y(k) = 0 \quad \textit{for} \quad k \in \overline{k_0, k_0+\nu-1}.$$

Example 1. We will list terms of the sequence $\{y(k)\} := T_{k_0}\{x(k)\}$ for $k \in \overline{-17, 24}$ when $m = 3, n = 2$ and $k_0 = 4$. Using (9), we obtain[6]

[6] *Mathematica*® was used for all computations in this paper.

k	$y(k)$	k	$y(k)$
-17	$-x(-15)-x(-10)-x(-5)-x(0)-x(5)$	4	0
-16	$-x(-14)-x(-9)-x(-4)-x(1)$	5	0
-15	$-x(-13)-x(-8)-x(-3)-x(2)$	6	0
-14	$-x(-12)-x(-7)-x(-2)-x(3)$	7	0
-13	$-x(-11)-x(-6)-x(-1)-x(4)$	8	0
-12	$-x(-10)-x(-5)-x(0)-x(5)$	9	$x(6)$
-11	$-x(-9)-x(-4)-x(1)$	10	$x(7)$
-10	$-x(-8)-x(-3)-x(2)$	11	$x(8)$
-9	$-x(-7)-x(-2)-x(3)$	12	$x(9)$
-8	$-x(-6)-x(-1)-x(4)$	13	$x(10)$
-7	$-x(-5)-x(0)-x(5)$	14	$x(6)+x(11)$
-6	$-x(-4)-x(1)$	15	$x(7)+x(12)$
-5	$-x(-3)-x(2)$	16	$x(8)+x(13)$
-4	$-x(-2)-x(3)$	17	$x(9)+x(14)$
-3	$-x(-1)-x(4)$	18	$x(10)+x(15)$
-2	$-x(0)-x(5)$	19	$x(6)+x(11)+x(16)$
-1	$-x(1)$	20	$x(7)+x(12)+x(17)$
0	$-x(2)$	21	$x(8)+x(13)+x(18)$
1	$-x(3)$	22	$x(9)+x(14)+x(19)$
2	$-x(4)$	23	$x(10)+x(15)+x(20)$
3	$-x(5)$	24	$x(6)+x(11)+x(16)+x(21)$

□

Each sequence $c = \{c(k)\}$ that is a constant for the derivative (8) is a ν-periodic sequence. Therefore, from the definition (10) of limit conditions s_{k_0} there follows the below

Corollary 2. *The numbers $x(k_0), x(k_0+1), \ldots, x(k_0+\nu-1)$ form a cycle of the ν-periodic sequence $\{c(k)\} = s_{k_0}\{x(k)\}$, i.e.*

$$c(k) = c(k+\ell\nu) = x(k), \quad k \in \overline{k_0, k_0+\nu-1}, \ell \in \mathbb{Z}.$$

PARTICULAR CASES

From Theorem 1 follow difference representations of the Bittner operational calculus mentioned at the beginning of this paper. Namely, for $n = 0$ we obtain the model with the forward difference (2) described in [16]. When $m = n$, we get in turn the model with the central difference (4) constructed in [17]. In [15] there was presented a model, in which to the backward difference (3) there correspond the following limit conditions

$$s_{\nabla_n,k_0}\{x(k)\} = \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\right\} \tag{12}$$

and integrals

$$T_{\nabla_n,k_0}\{x(k)\} = \begin{cases} -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k+1}^{k_0}\varepsilon_j^{k-i}\,x(i) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0+1}^{k}\varepsilon_j^{k-i}\,x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}. \tag{13}$$

This model is different to the one obtained from Theorem 1 for $m = 0$. The limit conditions (10) are uniquely determined by the initial conditions

$$x(k_0), x(k_0 + 1), \ldots, x(k_0 + n - 1),$$

while the limit conditions (12) depend on

$$x(k_0 - n + 1), x(k_0 - n + 2), \ldots, x(k_0).$$

Hence, the integrals (9) and (13) are also different, so

$$T_{k_0}T_{\nabla_n,k_0} \neq T_{\nabla_n,k_0}T_{k_0}.$$

GENERALIZATIONS

Let us consider the following generalization of the symmetric difference (8):

$$S_b\{x(k)\} := \{x(k + m) - b(k)x(k - n)\}, \tag{14}$$

where $b := \{b(k)\}$ is a real sequence satisfying the condition

$$\bigwedge_{k\in\mathbb{Z}} b(k) > 0.$$

In the spirit of the method described in [18], we shall define the integrals T_{b,k_0} and the limit conditions s_{b,k_0} corresponding to the derivative (14), called an *(m, n)-symmetric difference with the base $b = \{b(k)\}$*. In order to do this, we shall also use the following auxiliary theorems:

Lemma 1 (Th. 3 [4]). *An abstract differential equation*

$$S\,x = f, \quad f \in L^0, x \in L^1$$

with a limit condition

$$s_q x = x_{0,q}, \quad x_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = x_{0,q} + T_q f.$$

Lemma 2 (Th. 4 [4]). *With the given derivative* $S \in \mathscr{L}(L^1, L^0)$*, the projection* $s_q \in \mathscr{L}(L^1, \operatorname{Ker} S)$ *determines an integral* $T_q \in \mathscr{L}(L^0, L^1)$ *from the condition*

$$x = T_q f \quad \textit{if and only if} \quad S\,x = f, s_q x = 0.$$

Moreover, the projection s_q *is a limit condition corresponding to the integral* T_q*.*

First, we will determine a positive sequence $d = \{d(k)\}$ such that $d \in \operatorname{Ker} S_b$ and

$$d(0) = d(1) = \ldots = d(\nu - 1) = 1.$$

Thus, we have

$$\ln(d(k+m)) - \ln(d(k-n)) = \ln(b(k)), \quad k \in \mathbb{Z},$$

so

$$h(k+m) - h(k-n) = u(k), \quad k \in \mathbb{Z} \tag{15}$$

and

$$h(0) = h(1) = \ldots = h(\nu - 1) = 0, \tag{16}$$

where

$$h = \{h(k)\} := \{\ln(d(k))\}, \quad u = \{u(k)\} := \{\ln(b(k))\}.$$

The initial value problem (15), (16) can be presented in the form of

$$S\,h = u, \quad s_0 h = 0,$$

where S is the symmetric difference (8), whereas s_0 is the limit condition (10) for $k_0 := 0$. A solution to this problem, based on Lemma 1, is of the following form:

$$h = T_0 u,$$

where T_0 is the integral (9) for $k_0 := 0$.

Finally, the sequence

$$d = \exp(T_0 u) \tag{17}$$

is the desired constant for the derivative (14), i.e.

$$d(k+m) = b(k)d(k-n), \quad k \in \mathbb{Z}.$$

Example 2. If b is a constant sequence and $m := 2, n := 1$ or $m := 1, n := 2$, then from (17) we obtain

$$\{d(k)\}=\{\ldots, b^{-3}, b^{-3}, b^{-3}, b^{-2}, b^{-2}, b^{-2}, b^{-1}, b^{-1}, b^{-1}, \underset{k=0}{1}, 1, 1, b, b, b, b^2, b^2, b^2, b^3, b^3, b^3, \ldots\}.$$

Generally, we have

$$d(k) = b^{\lfloor k/\nu \rfloor}, \quad k \in \mathbb{Z}^7. \qquad \square$$

Let us consider a difference equation

$$S_b\{x(k)\} = \{f(k)\},$$

i.e.

$$x(k+m) - b(k)x(k-n) = f(k), \quad k \in \mathbb{Z}. \tag{18}$$

Hence, we have

$$\frac{x(k+m)}{d(k+m)} - \frac{x(k-n)}{d(k-n)} = \frac{f(k)}{d(k+m)}, \quad k \in \mathbb{Z},$$

so

$$y(k+m) - y(k-n) = g(k), \quad k \in \mathbb{Z}, \tag{19}$$

where

$$y(k) := \frac{x(k)}{d(k)}, \quad g(k) := \frac{f(k)}{d(k+m)}, \quad k \in \mathbb{Z}. \tag{20}$$

The equation (19) can be presented in the form of

$$S\{y(k)\} = \{g(k)\}, \tag{21}$$

where $S \equiv D_{m,n}$ is the operation (8).

From Lemma 1 it follows that the below sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (9) and (10), is the solution to (21).

[7] $\lfloor r \rfloor$ means the floor (the integer part) of a real number r.

From (20) we obtain $x(k) = d(k)y(k), k \in \mathbb{Z}$. Finally,

$$\{x(k)\} = \{d(k)\}s_{k_0}\left\{\frac{x(k)}{d(k)}\right\} + \{d(k)\}T_{k_0}\left\{\frac{f(k)}{d(k+m)}\right\} \tag{22}$$

is the solution to (18).

If we assume that

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{d(k)}\right\},$$

then $\{\widetilde{c}(k)\} \in \operatorname{Ker} S$, so

$$\widetilde{c}(k+m) = \widetilde{c}(k-n), \quad k \in \mathbb{Z}.$$

Let

$$s_{b,k_0}\{x(k)\} := \{d(k)\}s_{k_0}\left\{\frac{x(k)}{d(k)}\right\}, \quad k_0 \in Q := \mathbb{Z}, \{x(k)\} \in L^1. \tag{23}$$

Thus, for each $k \in \mathbb{Z}$ we get

$$\begin{aligned} S_b s_{b,k_0} x(k) &= d(k+m)\widetilde{c}(k+m) - b(k)d(k-n)\widetilde{c}(k-n) \\ &= d(k+m)(\widetilde{c}(k+m) - \widetilde{c}(k-n)) = d(k+m) \cdot 0 = 0, \end{aligned}$$

that is $s_{b,k_0} \in \mathscr{L}(L^1, \operatorname{Ker} S_b)$. Moreover, since $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$, for each $k \in \mathbb{Z}$ we have

$$\begin{aligned} s_{b,k_0}^2 x(k) &= s_{b,k_0}[d(k)\widetilde{c}(k)] = e(k)s_{k_0}\left[\frac{d(k)\widetilde{c}(k)}{d(k)}\right] \\ &= d(k)s_{k_0}\widetilde{c}(k) = d(k)\widetilde{c}(k) = s_{b,k_0}x(k). \end{aligned}$$

Eventually, s_{b,k_0} is a projection of L^1 onto $\operatorname{Ker} S_b$ for each $k_0 \in \mathbb{Z}$. From Lemma 2 it follows that the projection s_{b,k_0} determines an *integral* T_{b,k_0} from (22). Namely,

$$T_{b,k_0}\{f(k)\} := \{d(k)\}T_{k_0}\left\{\frac{f(k)}{d(k+m)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \tag{24}$$

What is more, s_{b,k_0} is a *limit condition* corresponding to the integral (24).

Finally, we arrive at the

Corollary 3. *The system* (14), (23), (24) *forms a discrete model of the Bittner operational calculus.*

Example 3. Let us consider an initial value problem

$$x(k+1) - e^k x(k-2) = 0, \quad k \in \mathbb{Z} \tag{25}$$

$$x(0) = 3, x(1) = x(2) = 0. \tag{26}$$

The homogeneous equation (25) takes the form of $S_b x = 0$, where S_b is the forward difference (14) for $m := 1, n := 2, b := \{e^k\}$. From (22) we get the solution to (25) expressed by the limit condition (23), i.e.

$$\{x(k)\} = s_{\{e^k\},0}\{x(k)\} = \{d(k)\}s_0\left\{\frac{x(k)}{d(k)}\right\},$$

which depends of the sequence

$$\{d(k)\} = \exp(T_0\{k\})$$

and initial conditions (26).
Therefore,

$$x(k) = \begin{cases} \left(1 + 2\cos\left(\frac{2k\pi}{3}\right)\right)\exp\left(-\frac{1}{3}\sum_{j=0}^{2}\sum_{i=k+2}^{1} i\,\varepsilon_j^{k+2-i}\right) & \text{for} \quad k < 0 \\ \left(1 + 2\cos\left(\frac{2k\pi}{3}\right)\right)\exp\left(\frac{1}{3}\sum_{j=0}^{2}\sum_{i=2}^{k+1} i\,\varepsilon_j^{k+2-i}\right) & \text{for} \quad k \geqslant 0 \end{cases}, \quad k \in \mathbb{Z}, \quad (27)$$

i.e.

$$\{x(k)\} = \{\ldots, 0, 0, 3e^{12}, 0, 0, 3e^5, 0, 0, 3e, 0, 0, \underset{k=0}{3}, 0, 0, 3e^2, 0, 0, 3e^7, 0, 0, 3e^{15}, 0, 0, \ldots\}.$$

□

Example 4. Referring to the previous example, let us consider the Cauchy problem

$$x(k+1) - e^k x(k-2) = 1, \quad k \in \mathbb{Z} \quad (28)$$

$$x(0) = 3, x(1) = x(2) = 0. \quad (29)$$

From (24) and (22) it follows that the sequence

$$\{x(k)\} = T_{\{e^k\},0}\{1\} = \{d(k)\}T_0\left\{\frac{1}{d(k+1)}\right\}$$

is a solution to the nonhomogeneous equation (28) with the conditions $x(0) = x(1) = x(2) = 0$. Thus,

$$x(k) = \begin{cases} -\frac{1}{3}\exp\left(-\frac{1}{3}\sum_{j=0}^{2}\sum_{i=k+2}^{1} i\,\varepsilon_j^{k+2-i}\right)\sum_{j=0}^{2}\sum_{i=k+2}^{1}\exp\left(-\frac{1}{2}(i+1)(i+4)\right)\varepsilon_j^{k+2-i} & \text{for} \quad k < 0 \\ \frac{1}{3}\exp\left(\frac{1}{3}\sum_{j=0}^{2}\sum_{i=2}^{k+1} i\,\varepsilon_j^{k+2-i}\right)\sum_{j=0}^{2}\sum_{i=2}^{k+1}\exp\left(-\frac{1}{2}(i+1)(i+4)\right)\varepsilon_j^{k+2-i} & \text{for} \quad k \geqslant 0 \end{cases},$$

$$k \in \mathbb{Z}. \quad (30)$$

Eventually, basing on (22), the sum of the sequences (27) and (30) is a solution to the problem (28),(29), the values of which (for $k \in \overline{-17, 16}$) are presented in the table below:

k	$x(k)$	k	$x(k)$
-17	$-e^{15}-e^{27}-e^{36}-e^{42}-2e^{45}$	0	3
-16	$-e^{14}-e^{25}-e^{33}-e^{38}-e^{39}-e^{40}$	1	0
-15	$-e^{13}-e^{23}-e^{30}-e^{34}-2e^{35}$	2	0
-14	$-e^{12}-e^{21}-e^{27}-2e^{30}$	3	$1+3e^{2}$
-13	$-e^{11}-e^{19}-e^{24}-e^{25}-e^{26}$	4	1
-12	$-e^{10}-e^{17}-e^{21}-2e^{22}$	5	1
-11	$-e^{9}-e^{15}-2e^{18}$	6	$1+e^{5}+3e^{7}$
-10	$-e^{8}-e^{13}-e^{14}-e^{15}$	7	$1+e^{6}$
-9	$-e^{7}-e^{11}-2e^{12}$	8	$1+e^{7}$
-8	$-e^{6}-2e^{9}$	9	$1+e^{8}+e^{13}+3e^{15}$
-7	$-e^{5}-e^{6}-e^{7}$	10	$1+e^{9}+e^{15}$
-6	$-e^{4}+2e^{5}$	11	$1+e^{10}+e^{17}$
-5	$-2e^{3}$	12	$1+e^{11}+e^{19}+e^{24}+3e^{26}$
-4	$-e-e^{2}$	13	$1+e^{12}+e^{21}+e^{27}$
-3	$2e$	14	$1+e^{13}+e^{23}+e^{30}$
-2	-1	15	$1+e^{14}+e^{25}+e^{33}+e^{38}+3e^{40}$
-1	$-e^{-1}$	16	$1+e^{15}+e^{27}+e^{36}+e^{42}$

□

Basing on Theorem 1, it is not difficult to determine the integrals T_{a,b,k_0} and the limit conditions s_{a,b,k_0} corresponding to an (m,n)-*symmetric difference* with the bases $a=\{a(k)\}$, $b=\{b(k)\}$:

$$S_{a,b}\{x(k)\} := \{a(k)x(k+m)-b(k)x(k-n)\}, \tag{31}$$

where

$$\bigwedge_{k\in\mathbb{Z}} a(k)b(k)>0 \quad \text{(cf. [14]).}$$

Namely, by representing the derivative (31) in the form of

$$S_{a,b}\{x(k)\} = \{a(k)\}\left\{x(k+m)-\frac{b(k)}{a(k)}x(k-n)\right\} = \{a(k)\}S_{b/a}\{x(k\},$$

we obtain

$$T_{a,b,k_0}\{x(k)\} = T_{b/a,k_0}\left\{\frac{x(k)}{a(k)}\right\}, \quad s_{a,b,k_0}\{x(k)\} = s_{b/a,k_0}\{x(k)\},$$

where $S_{b/a}$ and $T_{b/a,k_0}$, $s_{b/a,k_0}$ are operations (14),(24),(23), in which the sequence b needs to be replaced with the sequence b/a.

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MODEL RACHUNKU OPERATORÓW Z RÓŻNICĄ (m, n)-SYMETRYCZNĄ

STRESZCZENIE

W pracy określono model dyskretny nieklasycznego rachunku operatorów Bittnera, w którym pochodna rozumiana jest jako różnica (m, n)-symetryczna $D_{m,n}\{x(k)\} := \{x(k+m) - x(k-n)\}$. Dokonano uogólnienia opracowanego modelu rozważając operację $D_{m,n,b}\{x(k)\} := \{x(k+m) - b(k)x(k-n)\}$.

Słowa kluczowe:

rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica (m, n)-symetryczna.