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DESCRIPTOR FRACTIONAL CONTINUOUS - TIME LINEAR SYSTEMS

Abstract

A new method for computation of the solution to the state equation of descriptor fractional continuous-time linear systems with regular pencils is proposed. The derivations of the solution is based on the application of the Laplace transform and the convolution theorem. A procedure for computation of the transition matrices is proposed and demonstrated on simple numerical example.

INTRODUCTION

Descriptor (singular) linear systems with regular pencils have been considered in many papers and books [1-5, 10-12, 15, 17, 18, 20]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [1-4, 10, 11] and the realization problem for singular positive continuous-time systems with delays in [15]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [20]. A delay dependent stability criterion for a class of descriptor systems with delays varying in intervals has been proposed in [2].

Fractional differential equations have being analyzed in [19]. Fractional positive continuous-time linear systems have been addressed in [9] and positive linear systems with different fractional orders in [8, 13]. A new concept of the practical stability of the positive fractional 2D systems has been proposed in [14]. An analysis of fractional linear electrical circuits has been presented in [6] and some selected problems in theory of fractional linear systems in the monograph [16].

A new class of descriptor fractional linear systems and electrical circuits has been introduced, their solution of state equations has been derived and a method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts has been proposed in [6]. Positive fractional continuous-time linear systems with singular pencils has been considered in [7]. Fractional-order iterative learning control for fractional-order systems has been addressed in [21].

In this paper a method for finding of the solution of the state equations of descriptor fractional continuous-time linear system with regular pencils will be proposed.

The paper is organized as follows. In section 2 the solution to the state equation of the descriptor system is derived using the method based on Laplace transform and the convolution theorem. A method for computation of the transition matrix is proposed and illustrated on a simple numerical example in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ - the set of $n\times n$ nonnegative integers, I_n - the $n\times n$ identity matrix.

1. CONTINUOUS-TIME FRACTIONAL LINEAR SYSTEMS

Consider the descriptor fractional continuous-time linear system described by the state equation

$$E\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1 \tag{1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $E, A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

It is assumed that det E = 0 and the pencil (E, A) is regular, i.e.

$$\det[E\lambda - A] \neq 0 \text{ for some } \lambda\in \mathrm{C} \text{ (the field of complex numbers).} \tag{2}$$

The following Caputo definition of the fractional derivative will be used [16]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0 < \alpha < 1 \tag{3}$$

where $\alpha\in\Re$ is the order of fractional derivative and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Applying to (1) and (3) the Laplace transform ($\mathcal{L}$) and taking into account that [16]

$$\mathcal{L}\left[\frac{d^\alpha x(t)}{dt^\alpha}\right] = s^\alpha X(s) - s^{\alpha-1}x(0), \quad 0 < \alpha < 1 \tag{4}$$

we obtain

$$X(s) = [Es^\alpha - A]^{-1}[BU(s) + s^{\alpha-1}x(0)] \tag{5}$$

where $X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$ and $U(s) = \mathcal{L}[u(t)]$.

Let

$$[Es^\alpha - A]^{-1} = \sum_{k=-\mu}^{\infty}\Phi_k s^{-(k+1)\alpha} \tag{6}$$

where μ is a nonnegative integer defined by the pair of matrices (E, A) [11]. Comparison of the coefficients at the same powers of s^α of the equality

$$[Es^\alpha - A]\left(\sum_{k=-\mu}^{\infty}\Phi_k s^{-(k+1)\alpha}\right) = \left(\sum_{k=-\mu}^{\infty}\Phi_k s^{-(k+1)\alpha}\right)[Es^\alpha - A] = I_n \tag{7}$$

yields

$$E\Phi_{-\mu} = \Phi_{-\mu}E = 0 \tag{8a}$$

and

$$E\Phi_k - A\Phi_{k-1} = \Phi_k E - \Phi_{k-1}A = \begin{cases} I_n & \text{for } k = 0 \\ 0 & \text{for } k = 1-\mu, 2-\mu, \ldots, -1, 1, 2, \ldots \end{cases} \tag{8b}$$

From the equality (8) we have the matrix equation

$$G\begin{bmatrix}\Phi_{0\mu} \\ \Phi_{1N}\end{bmatrix} = \begin{bmatrix} V \\ 0\end{bmatrix} \tag{9a}$$

where

$$G=\begin{bmatrix} G_1 & 0 \\ G_{21} & G_2 \end{bmatrix} \in \Re^{(N+\mu+1)n\times(N+\mu+1)n}, \quad G_{21}=\begin{bmatrix} 0 & \dots & 0 & A \\ 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \in \Re^{Nn\times(\mu+1)n},$$

$$G_1=\begin{bmatrix} E & 0 & 0 & \dots & 0 & 0 & 0 \\ -A & E & 0 & \dots & 0 & 0 & 0 \\ 0 & -A & E & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -A & E & 0 \\ 0 & 0 & 0 & \dots & 0 & -A & E \end{bmatrix} \in \Re^{(\mu+1)n\times(\mu+1)n},$$

$$G_2=\begin{bmatrix} E & 0 & 0 & \dots & 0 & 0 & 0 \\ -A & E & 0 & \dots & 0 & 0 & 0 \\ 0 & -A & E & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -A & E & 0 \\ 0 & 0 & 0 & \dots & 0 & -A & E \end{bmatrix} \in \Re^{Nn\times Nn},$$

$$\Phi_{0\mu}=\begin{bmatrix} \Phi_{-\mu} \\ \Phi_{1-\mu} \\ \vdots \\ \Phi_0 \end{bmatrix} \in \Re^{(\mu+1)n\times n}, \quad \Phi_{1N}=\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{bmatrix} \in \Re^{Nn\times n}, \quad V=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_n \end{bmatrix} \in \Re^{(\mu+1)n\times n} \tag{9b}$$

The equation (9a) has the solution $\begin{bmatrix} \Phi_{0\mu} \\ \Phi_{1N} \end{bmatrix}$ for given G and V if and only if

$$\operatorname{rank}\left\{G, \begin{bmatrix} V \\ 0 \end{bmatrix}\right\} = \operatorname{rank} G\,. \tag{10}$$

It is easy to show that the condition (10) is satisfied if the condition (2) is met. Substituting (6) into (5) we obtain

$$X(s)=\left(\sum_{k=-\mu}^{\infty}\Phi_k s^{-(k+1)\alpha}\right)[BU(s)+s^{\alpha-1}x(0)]. \tag{11}$$

Applying the inverse Laplace transform $\mathcal{L}^{-1}$ and the convolution theorem to (11) we obtain

$$x(t)=\mathcal{L}^{-1}[X(s)]=\overline{\Phi}_0(t)x_0+\int_0^t \overline{\Phi}(t-\tau)Bu(\tau)d\tau \tag{12a}$$

where

$$\overline{\Phi}_0(t)=\sum_{k=-\mu}^{\infty}\Phi_k\mathcal{L}^{-1}[s^{-(k\alpha+1)}]=\sum_{k=-\mu}^{\infty}\frac{\Phi_k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{12b}$$

$$\overline{\Phi}(t)=\sum_{k=-\mu}^{\infty}\Phi_k\mathcal{L}^{-1}[s^{-(k+1)\alpha}]=\sum_{k=-\mu}^{\infty}\frac{\Phi_k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{12c}$$

since $\mathcal{L}^{-1}[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ [16].

2. COMPUTATION OF THE TRANSITION MATRICES

To compute the matrices Φ_k for $k = -\mu, 1-\mu, ..., 1, 2, ...$ the following procedure is recommended.

Procedure 1.

Step 1. Find a solution $\Phi_{0\mu}$ of the equation

$$G_1 \Phi_{0\mu} = V \tag{13}$$

where G_1, $\Phi_{0\mu}$ and V are defined by (9b). Note that if the matrix E has the form

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \in \Re^{n \times n}, \quad E_1 \in \Re^{r \times r} \text{ and rank } E_1 = \text{rank } E = r < n. \tag{14}$$

then the first r rows of the matrix $\Phi_{0\mu}$ are zero and the last its $n - r$ rows are arbitrary.

Step 2. Choose $n - r$ arbitrary rows of the matrix Φ_0 so that

$$\text{rank}\left\{\begin{bmatrix} E & 0 \\ A & E \end{bmatrix}, \begin{bmatrix} I_n + A\Phi_{-1} \\ 0 \end{bmatrix}\right\} = \text{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} \tag{15}$$

and the equation

$$\begin{bmatrix} E & 0 \\ -A & E \end{bmatrix}\begin{bmatrix} \Phi_0 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} I_n + A\Phi_{-1} \\ 0 \end{bmatrix} \tag{16}$$

has a solution with arbitrary last $n - r$ rows of the matrix Φ_1.

Step 3. Knowing $\Phi_{0\mu}$ choose the last $n - r$ rows of the matrix Φ_1 so that

$$\text{rank}\left\{\begin{bmatrix} E & 0 \\ -A & E \end{bmatrix}, \begin{bmatrix} A\Phi_0 \\ 0 \end{bmatrix}\right\} = \text{rank}\begin{bmatrix} E & 0 \\ -A & E \end{bmatrix} \tag{17}$$

and the equation

$$\begin{bmatrix} E & 0 \\ -A & E \end{bmatrix}\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}\Phi_0 \tag{18}$$

has a solution with arbitrary last $n - r$ rows of the matrix Φ_2. Repeating the last step for $\begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}, \begin{bmatrix} \Phi_3 \\ \Phi_4 \end{bmatrix}, \ldots$ we may compute the desired matrices Φ_k for $k = -\mu, 1-\mu, \ldots$.

The details of the procedure will be shown on the following example.

Example 1. Find the solution to the equation (1) for $\alpha = 0.5$ with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{19}$$

and the initial condition x_0 and $u(t)$. In this case the pencil (2) of (13) is regular since

$$\det[E\lambda - A] = \begin{vmatrix} \lambda - 1 & 0 \\ -1 & 2 \end{vmatrix} = 2(\lambda - 1) \tag{20}$$

and $\mu = 1$.

Using Procedure 1 we obtain the following.

Step 1. In this case the equation (13) has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{-1} \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

and its solution with the arbitrary second row $[\Phi_{21}^0 \quad \Phi_{22}^0]$ of Φ_0 is given by

$$\begin{bmatrix} \Phi_{-1} \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 1 & 0 \\ \Phi_{21}^0 & \Phi_{22}^0 \end{bmatrix}. \tag{22}$$

Step 2. We choose the row $[\Phi_{21}^0 \quad \Phi_{22}^0]$ of Φ_0 so that (15) holds, i.e.

$$\operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \tag{23}$$

and the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{24}$$

has the solution

$$\begin{bmatrix} \Phi_0 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ \Phi_{21}^1 & \Phi_{22}^1 \end{bmatrix} \tag{25}$$

with the second arbitrary row $[\Phi_{21}^1 \quad \Phi_{22}^1]$ of Φ_1.

Step 3. We choose $[\Phi_{21}^1 \quad \Phi_{22}^1]$ so that the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Phi_{21}^1 & \Phi_{22}^1 \\ 1 & 0 \\ \Phi_{21}^2 & \Phi_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

has the solution

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ \Phi_{21}^2 & \Phi_{22}^2 \end{bmatrix} \tag{27}$$

with arbitrary $[\Phi_{21}^2 \quad \Phi_{22}^2]$ of Φ_2.

Continuing the procedure we obtain

$$\Phi_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \Phi_k = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix} \text{ for } k = 0,1,\ldots \tag{28}$$

Using (12), (19) and (28) we obtain the desired solution of the equation (1) with (19) in the form

$$x(t) = \overline{\Phi}_0(t)x_0 + \int_0^t \overline{\Phi}(t-\tau)\begin{bmatrix}1\\2\end{bmatrix}u(\tau)d\tau \tag{29a}$$

where

$$\overline{\Phi}_0(t) = \sum_{k=-\mu}^{\infty}\frac{\Phi_k t^{k\alpha}}{\Gamma(k\alpha+1)} = \frac{1}{\Gamma(0.5)}\begin{bmatrix}0 & 0\\0 & 0.5\end{bmatrix}t^{-\alpha} + \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}t^{k\alpha} \tag{29b}$$

$$\overline{\Phi}(t) = \sum_{k=-\mu}^{\infty}\frac{\Phi_k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty}\frac{1}{\Gamma[(k+1)\alpha]}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}t^{(k+1)\alpha-1} \tag{29c}$$

since $\Gamma(0) = \infty$.

CONCLUDING REMARKS

A new method for computation of the solutions of the state equation of descriptor fractional continuous-time linear systems with regular pencils has been proposed. Derivation of the solution has been based on the application of the Laplace transform and the convolution theorem. A procedure for computation of the transition matrices has been proposed and its application has been demonstrated on simple numerical example. An open problem is an extension of the method for 2D descriptor fractional discrete and continuous-discrete linear systems.

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SINGULARNE UKŁADY LINIOWE CIĄGŁE RZĘDU NIECAŁKOWITEO

Abstrakt

W pracy podano nową metodę wyznaczania rozwiązania równań stanu singularnego o pęku regularnym ciągłego układu rzędu niecałkowitego. Rozwiązanie to wyprowadzono stosując przekształcenie Laplace'a oraz twierdzenie o transformacie splotu. Zaproponowano procedurę wyznaczania macierzy tranzycji. Proponowaną metodę zlustrowano przykładem numerycznym.

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