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PURE BENDING OF STRIP (BEAM) WITH CRACK IN STRIP OF TENSILE STRESS WITH ALLOWANCE FOR PLASTIC STRIPS NEAR CRACK TIPS

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Abstract: In the article, the pure bending problem for strip (beam) with straight, perpendicular to its axis crack located in the zone of tensile stresses is investigated on the assumption of narrow plastic strips near crack tips. Using methods of the theory of functions of a complex variable and complex potentials, the problem is reduced to the several linear conjunction problems. The solutions of latter problems are obtained in the class of functions confined in the edges of plastic strips. Formulas for the calculation of their lengths are derived. Expressions for the determination of crack tip opening values are written. Numerical analysis of the problem is performed.

Key words: Pure Bending, Strip (Beam), Crack, Plastic Strips, Linear Conjugation Problems

1. INTRODUCTION

Beams are one of the main structural elements that are common in engineering, especially in construction practice. Cracks, which are strong stress concentrators, may appear in them during various operations and, as a result, may lead to destruction of such structural elements. Therefore, it is very important to carefully evaluate the reliability of the beam operation in the presence of such defects.

In an article, Monfared et al. (2018) investigated the configuration of arbitrary crack configurations in the orthotropic strip. Fourier transformation was used to construct a system of singular integral equations that was numerically solved using the Chebyshev quadrature formula for the density of dislocation on a crack face. Effects of crack geometry and parameter of nonhomogeneity of material on the stress intensity, energy release and energy density were considered. In the work of Pavazza (2000), approximate analytical formulas for stresses and displacements in thin rectangular orthotropic or isotropic strips subjected to tension are presented.

Shi (2015) devoted his investigation to analytical and numerical analyses of the doubly periodic arrays of cracks and proposes a precise solution procedure for describing the interaction effect in the doubly periodic rectangular-shaped arrays of cracks. Fan et al. (2014) investigated the elastic-plastic fracture behaviour of an interaction between Zener–Stroh crack and coated inclusion in composite materials with regard to crack tip plastic zones. The sizes of plastic zone at the both crack tips were determined by the generalised Irwin model. In the article by Prawoto (2012), an approach of classical fracture mechanics is used for calculating the near crack tip plastic zones in heterogeneous or composite materials. In the research by Unger (2007), the Dugdale model of plasticity is used for a static crack instead of Tresca plasticity theory.

An analysis of stress–strain state under combined bending and tension of an isotropic plate with crack is represented by Sulym et al. (2018) in the assumption of line and constant width area contact between crack faces but with no plastic zones near the crack tips. In the articles by Nykolyshyn et al. (2010, 2015), the tension of homogeneous isotropic plate weakened by two through cracks with plastic zones is considered. By using the method of complex potentials, the solution of the problem is reduced to linear conjugation problems and the explicit expressions for complex potentials of plane problem. The length of the plastic zone and crack opening displacement are obtained analytically and the numerical analysis of them is performed at various parameters. In the particular case, the known results are obtained.

Kuz et al. (2015, 2019) investigated the influence of stress concentrators (square hole, cut or rigid inclusion) on the strength of a plate under uniaxial tension using the numerical solution of boundary-value problems of the theory of small elasto-plastic deformations for a linear hardening material. The growth of microstructurally short and physically small cracks in the fatigue process zone and the initiation of macrocrack in notched tension specimens were investigated in the work of Ostash (2017).

In this article, the problem of pure bending of strip (beam) with crack perpendicular to its axis and located in tensile stresses area is investigated. We assumed that at the crack tips, the narrow plastic strips are formed, as in the works of Panasyuk (1968) and

Savruk et al. (1989). Solving of the problem is based on the methods of the theory of functions of complex variable and complex potentials and is reduced to the problems of linear conjunction. Their solutions are obtained in class of functions confined at the tips of plastic strips. The method used in the article is given in the work of Muskhelishvili (1966). System of transcendental equations for calculating lengths of plastic zones is written. Expressions for calculating of crack tip opening values are also obtained. The numerical analysis of the problem is carried out. Graphic dependencies of the length of plastic strips and crack tip opening values are constructed at various values of problem parameters.

2. FORMULATION OF THE PROBLEM

Consider an elastic isotropic strip (beam) perpendicular to its axis straight through crack of length $2l$, centre of which is at the distance x_0 from the axis. Let $2h$ be the height of the beam transverse section and $2\tilde{a}$ be the width of the beam. Assume that the beam is under pure bending with the bending moment M. Let us introduce a Cartesian coordinate system Oy-axis, which is directed along the beam centre line, and Ox-axis, which is directed along the crack. The crack is located in the tensile stress zone and its faces are free of external loads. Assume that near crack tips with coordinates $(a, 0)$ and $(b, 0)$ narrow plastic zones (plastic srips) have been formed on the extending of the crack. In these zones, normal stresses are equal to the yield strength of beam material (Panasyuk, 1968; Savyn, 1968). Let the coordinates of edges of the plastic strips are (d_2, a) and (b, d_1), L denotes the projection of the crack onto the Ox-axis and L_1 and L_2 denote plastic strips of length $\Delta_2 = a - d_2$ and $\Delta_1 = d_1 - b$, respectively (Fig. 1). In addition, we introduce the notations $\tilde{L} = L + L_1 + L_2$ and $\tilde{L}_1 = L_1 + L_2$.

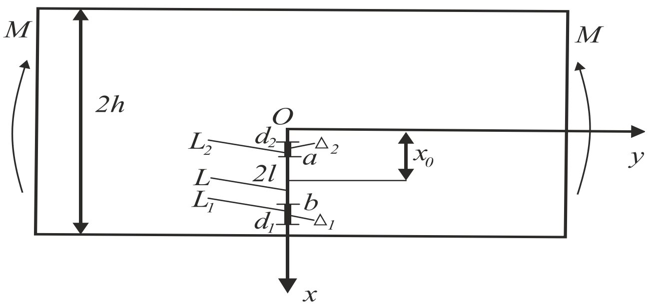

Fig. 1. Beam loading scheme and location of crack with plastic strips

The following boundary conditions correspond to the formulated problem:

$$\sigma_{xy}^{\pm} = 0, x \in \tilde{L};$$
$$\sigma_{yy}^{\pm} = 0, x \in L; \sigma_{yy}^{\pm} = \sigma_Y, x \in \tilde{L}_1, \tag{1}$$

where σ_{yy} and σ_{xy} are the components of stress tensor and marks '+' and '–' mean limits as $y \to \pm 0$.

3. CONSTRUCTION OF THE SOLUTION OF THE PROBLEM

Let us introduce the complex potentials $\Phi(z)$ and $\Omega(z)$ (Muskhelishvili, 1966) and use the expressions

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{2}$$

$$2\mu(u + iv)' = \kappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}, \tag{3}$$

where μ is the shearing modulus; κ is the Muskhelishvili's constant; $u' = \partial u/\partial x$, $v' = \partial v/\partial x$; u and v are the components of displacement vector of a beam point on axes Ox and Oy; $z = x + iy$, $i^2 = -1$.

According to Panasyuk and Lozovyy (1961) and Savyn (1968), functions $\Phi(z)$ and $\Omega(z)$ at large $|z|$ can be presented as

$$\Phi(z) = Cz/4 + O(1/z^2)$$
$$\Omega(z) = 3Cz/4 + O(1/z^2) \tag{4}$$
$$C = M/I$$

where $I = 4\tilde{a}h^3/3$ is an inertia moment of the beam about neutral line of its transverse section.

From the boundary condition,

$$(\sigma_{yy} - i\sigma_{xy})^+ - (\sigma_{yy} - i\sigma_{xy})^- = 0, x \in \tilde{L}$$

taking into account (2), we obtain

$$[\Phi(x) - \Omega(x)]^+ - [\Phi(x) - \Omega(x)]^- = 0, x \in \tilde{L}. \tag{5}$$

The solution of the linear conjunction problem (5), (4) is

$$\Phi(z) - \Omega(z) = -\frac{C}{2}z. \tag{6}$$

On the basis of (1), we can write the following boundary condition:

$$(\sigma_{yy} - i\sigma_{xy})^+ + (\sigma_{yy} - i\sigma_{xy})^- = \begin{cases} 2\sigma_Y, x \in \tilde{L}_1; \\ 0, x \in L. \end{cases} \tag{7}$$

Taking into account (2), from (7), we obtain

$$[\Phi(x) + \Omega(x)]^+ + [\Phi(x) + \Omega(x)]^- = \begin{cases} 2\sigma_Y, x \in \tilde{L}_1, \\ 0, x \in L. \end{cases} \tag{8}$$

The solution of this linear conjunction problem is

$$\Phi(z) + \Omega(z) = \frac{\sigma_Y X(z)}{\pi i}\int_{\tilde{L}_1}\frac{dt}{X^+(t)(t-z)} + c_1 X(z)L, \tag{9}$$

where

$$X(z) = \sqrt{(z - d_1)(z - d_2)}, X^+(t) = i\sqrt{(d_1 - t)(t - d_2)},$$

c_1 is an unknown constant.

On the basis of (4) at large $|z|$, we can write

$$\Phi(z) + \Omega(z) = Cz + O(1/z^2). \tag{10}$$

As at large $|z|$

$$X(z) = z - \frac{1}{2}(d_1 + d_2) - \frac{1}{8}(d_1 - d_2)^2\frac{1}{z} + \ldots,$$

$$\frac{X(z)}{t - z} = -1 + \frac{1}{z}\left(\frac{1}{2}(d_1 + d_2) - t\right) + \ldots,$$

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expanding right and left parts of formulas (9) into series at large $|z|$ and making equal the coefficients at equal powers of z, we receive

$$\begin{aligned} &c_1 = C \\ &-\frac{\sigma_Y}{\pi i}\int_{\tilde{L}_1}\frac{dt}{X^+(t)} - C\frac{1}{2}(d_1+d_2) = 0 \\ &\frac{\sigma_Y}{\pi i}\int_{\tilde{L}_1}\frac{1}{X^+(t)}\left(\frac{1}{2}(d_1+d_2)-t\right)dt \\ &-\frac{C}{8}(d_1-d_2)^2 = 0 \end{aligned} \tag{11}$$

Using calculated integrals from Bronshteyn and Semendyaev (1967) formula (11) leads us to the system of transcendental equations for finding lengths $\Delta_i (i = 1,2)$ of plastic strips at the crack tips:

$$\begin{aligned} &\frac{2}{\pi}[\arccos((2+w_1-w_2)/\gamma_1) \\ &\arccos((2+w_2-w_1)/\gamma_1)] = \tilde{\sigma}(2\tilde{x}+w_1-w_2) \\ &\tilde{\sigma}\gamma_1^2 = \frac{16}{\pi}\frac{w_1-w_2}{\sqrt{w_2(2+w_1)}+\sqrt{w_1(2+w_2)}} \end{aligned} \tag{12}$$

where

$w_i = \Delta_i/l, \tilde{x} = x_0/l, \tilde{\sigma} = Ml/(I\sigma_Y), \gamma_1 = 2 + w_1 + w_1.$

Now, on the basis of (6) and (9), we have

$$\Phi(z) = \frac{C}{2}\left(X(z) - \frac{1}{2}z\right) + \frac{\sigma_Y X(z)}{2\pi i}\int_{\tilde{L}_1}\frac{dt}{X^+(t)(t-z)}, \tag{13}$$

$$\Omega(z) = \Phi(z) + \frac{C}{2}z. \tag{14}$$

Taking into account (3), we can write the expression for the derivative of the opening of the crack faces

$$\begin{aligned} &2\mu[(u+iv)'^+ - (u+iv)'^-] = \\ &\kappa[\Phi^+(x) - \Phi^-(x)] + \Omega^+(x) - \Omega^-(x), x \in \tilde{L}_1. \end{aligned} \tag{15}$$

From here, using (14),

$$\begin{aligned} &2\mu\delta'_x(x) = 2\mu[v'^+ - v'^-] = \\ &(\kappa+1)\mathrm{Im}[\Phi^+(x) - \Phi^-(x)], x \in \tilde{L}_1. \end{aligned} \tag{16}$$

Taking into account (13) and calculating integrals (Bronshteyn, Semendyaev, 1967, Savruk et al. 1989), we obtain

$$\begin{aligned} &\tilde{\delta}_1 = \frac{\gamma_1}{2}\left[\frac{\gamma_1\tilde{\sigma}}{4}\left(a_1\sqrt{1-a_1^2} - \arccos a_1\right) - \right. \\ &\left. -\frac{1}{\pi}(\gamma_3 + \gamma_2\arccos a_1)\right], \end{aligned} \tag{17}$$

$$\tilde{\delta}_2 = \frac{\gamma_1}{2}\left[\frac{\gamma_1\tilde{\sigma}}{4}\left(a_2\sqrt{1-a_2^2} + \arccos\tilde{a}_2\right) + \frac{1}{\pi}(\gamma_3 + \gamma_2\arccos\tilde{a}_2)\right],$$

where

$a_1 = (2 + w_2 - w_1)/\gamma_1, a_2 = (2 + w_1 - w_2)/\gamma_1, \tilde{a}_2 = -a_2, \gamma_2 = \sqrt{1-a_1^2} - \sqrt{1-a_2^2},$

$$\gamma_3 = \frac{1}{2}(a_1 - a_2)\ln\frac{1-a_1a_1-\sqrt{(1-a_1^2)(1-a_2^2)}}{1-a_1a_1+\sqrt{(1-a_1^2)(1-a_2^2)}},$$

$$\tilde{\delta}_i = \frac{2\mu\delta_i}{\sigma_Y(1+\kappa)l}, \delta_2 = \delta_a, \delta_1 = \delta_b. \tag{18}$$

Account for (2), (7) and (8), we calculate stress tensor components on the faces of crack and on its extending using the formulas

$$\begin{aligned} &\sigma_{yy}^* = \sigma_{yy}/\sigma_Y = \\ &-\tilde{\sigma}X(x_1) + \frac{1}{\pi}(\arccos\tilde{\gamma}_1 + \arccos\tilde{\gamma}_2); \sigma_{xx}^* = \sigma_{xx}/\sigma_Y = \\ &-\tilde{\sigma}x_1 + \sigma_{yy}^*, x_1 < \tilde{x} - 1 - w_2, \\ &\sigma_{yy}^* = 1, \sigma_{xx}^* = \sigma_{xx}^{*\pm} = -\tilde{\sigma}x_1 + 1, \\ &\tilde{x} - 1 - w_2 < x_1 < \tilde{x} - 1, 1 + \tilde{x} < x_1 < 1 + \tilde{x} + w_1; \\ &\sigma_{yy}^* = 0, \sigma_{xx}^* = \sigma_{xx}^{*\pm} = -\tilde{\sigma}x_1, \tilde{x} - 1 < x_1 < \tilde{x} + 1; \\ &\sigma_{yy}^* = \sigma_{yy}/\sigma_Y = \tilde{\sigma}X(x_1) + \frac{1}{\pi}(\arccos\tilde{\gamma}_1 + \arccos\tilde{\gamma}_2), \\ &\sigma_{xx}^* = \sigma_{xx}/\sigma_Y = -\tilde{\sigma}x_1 + \sigma_{yy}^*, x_1 > 1 + \tilde{x} + w_1, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\tilde{\gamma}_1 = 1 - \frac{2w_1(x_1-\tilde{d}_2)}{(x_1-\tilde{d}_1+w_1)(\tilde{d}_1-\tilde{d}_2)}, \\ &\tilde{\gamma}_2 = 1 - \frac{2w_2(\tilde{d}_1-x_1)}{(\tilde{d}_2+w_2-x_1)(\tilde{d}_1-\tilde{d}_2)}, \\ &X(x_1) = \sqrt{(x_1-\tilde{d}_1)(x_1-\tilde{d}_2)}, \end{aligned} \tag{20}$$

$\tilde{d}_1 = d_1/l = 1 + \tilde{x} + w_1, \tilde{d}_2 = d_2/l = \tilde{x} - 1 - w_2,$
$x_1 = x/l.$

Note that based on (12), we can find out the condition under which $w_2 = 0$, that is, the length of the plastic strip at the nearest to the beam axe crack tip is zero. For example, given the external load $\tilde{\sigma}$, we obtain the following expression from the second equation (12):

$$\tilde{\sigma}(2 + w_1)^2 = \frac{8\sqrt{2}}{\pi}\sqrt{w_1}, \tag{21}$$

for calculating the length of the plastic strip at another crack tip. And from the first equation (12) at given $\tilde{\sigma}$ and obtained w_1, we determine the coordinate of crack centre $\tilde{x}_1 = x_0/l$

$$\tilde{x}_1 = \frac{1}{2}\left[\frac{2}{\pi\tilde{\sigma}}\arccos\frac{2-w_1}{2+w_1} - w_1\right]. \tag{22}$$

Note also that equation (21) in w_1 has a solution up to some certain value $\tilde{\sigma}_0$ only. At $\tilde{\sigma} > \tilde{\sigma}_0$, it does not have the solution and satisfying the condition $w_2 = 0$ is impossible.

4. NUMERICAL ANALYSIS OF THE PROBLEM

The results of the calculations are given in Figures 2–7.

Graphical dependencies of dimensionless length of plastic strip and dimensionless opening the displacement of crack faces at crack tips on external load are given in Figures 2 and 3, correspondingly. In these figures, curves with label '1' correspond to the farther to the beam axis crack tip and curves with label '2' correspond to the nearer one. Figures 2(a) and 3(a) are constructed at $\tilde{x} = x_0/l = 1.5$, and Figures 2(b) and 3(b) are constructed at $\tilde{x} = 3$. It is seen from Figures 2 and 3 that, under fixed external load, the length of plastic strip and crack tip opening

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values increases when crack centre moves from beam axis. On the basis of δ_κ model, the beam destruction starts from the farther (with respect to beam axis) crack tip.

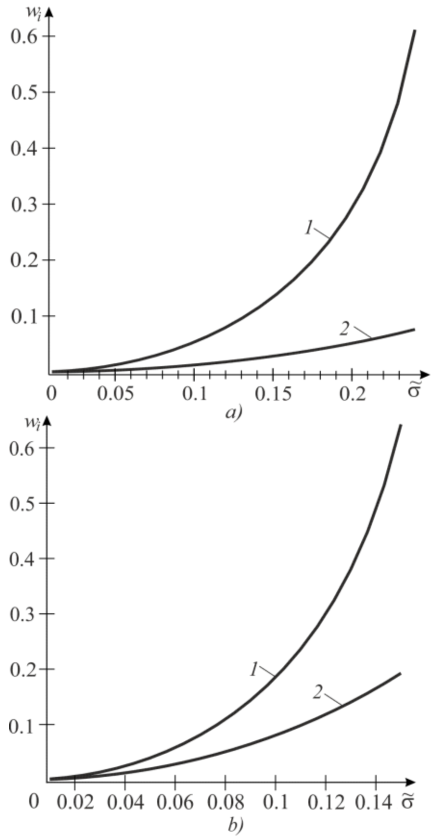

Fig. 2. Graphical dependencies of relative length of plastic strip at crack tips on external load

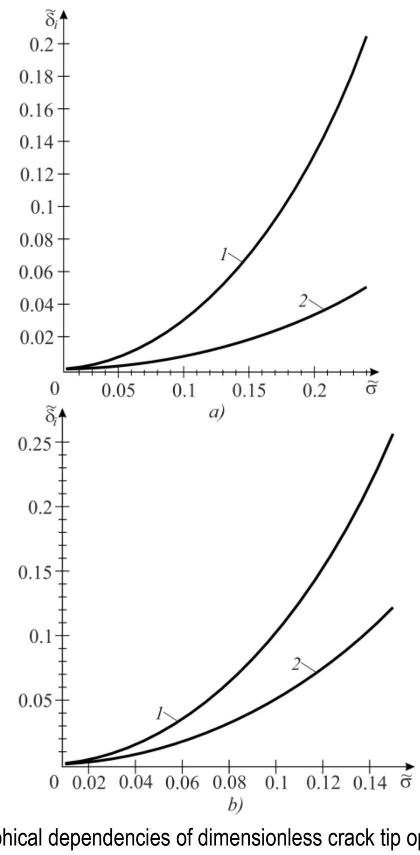

Fig. 3. Graphical dependencies of dimensionless crack tip opening displacement on external load

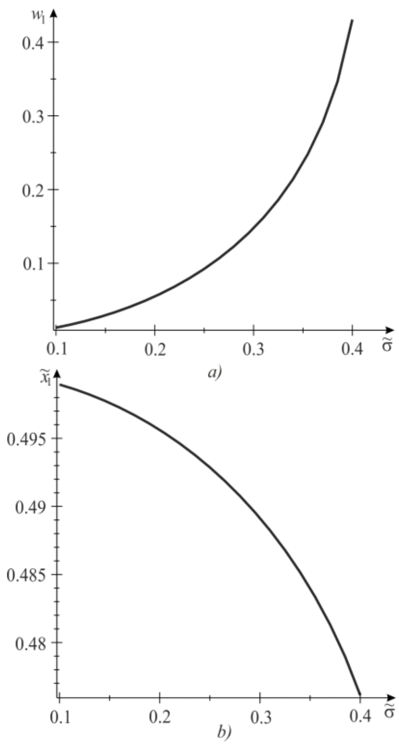

Fig. 4. Graphical dependencies of length of plastic strip at the crack tip (Fig. 4 a) and coordinate of crack centre, at which the length of plastic strip at nearer crack tip is zero, on external load

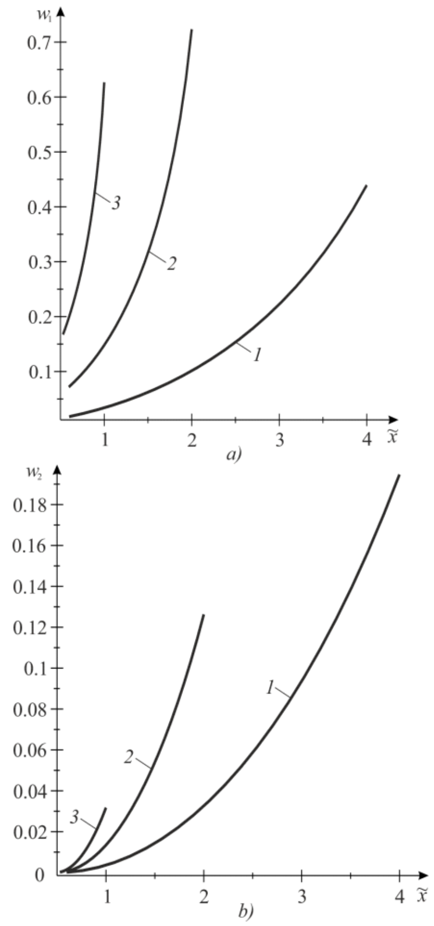

Fig. 5. Graphical dependencies of length of plastic strip at the crack tips on relative coordinate of crack centre $\tilde{x} = x_0/l$ at various values of external load

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Figure 4 deals with case when the length of plastic strip at nearer to the beam axis crack tip is zero. The graphical dependencies of length of plastic strip at farther crack tip are given in Fig. 4(f). Figure 4(b) shows the graphical dependencies of coordinate of crack center for this case on external load attached to the beam. It is can be seen that increasing the value of external load leads to increasing the length of plastic strip at farther crack tip. At the same time, crack centre gets closer to the beam axis and the crack tip with zero plastic strip is strip of compressive stresses for the beam without crack.

Figure 5 demonstrates graphical dependencies of length of plastic strip in farther (Fig. 5a) and nearer (Fig. 5b) to the beam axis crack tips on relative coordinate of crack centre $\tilde{x}$ at several values of external load. Curves labelled as '1', '2' and '3', respectively, correspond to $\tilde{\sigma} = 0.1075$, $\tilde{\sigma} = 0.205$ and $\tilde{\sigma} = 0.3025$. Note that curves are constructed at $\tilde{x} > \tilde{x}_1$ and labels '1', '2' and '3' stand for $\tilde{x}_1 = 0.499$, $\tilde{x}_1 = 0.495$, and $\tilde{x}_1 = 0.489$, correspondingly. It is seen that decreasing the distance between crack centre and beam axis increases the length of the plastic strips at crack tip.

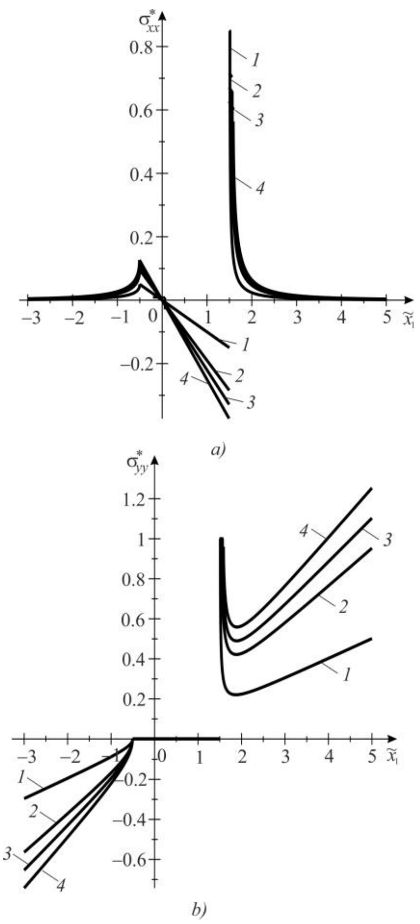

Fig. 6. Stress distribution on crack line at $\tilde{x} = \tilde{x}_1$ and various values of external load $\tilde{\sigma}$

In Figures 6 and 7, curves with label '1' are constructed at external load $\tilde{\sigma} = 0.01$, with label '2' at $\tilde{\sigma} = 0.1$, '3' at $\tilde{\sigma} = 0.15$ and '4' at $\tilde{\sigma} = 0.2$. Figures 6(a) and 7(a) show graphical dependencies of stresses $\sigma_{xx}^* = \sigma_{xx}/\sigma_Y$, and Figure 6(b) and 7(b) show graphical dependencies of stresses $\sigma_{yy}^* = \sigma_{yy}/\sigma_Y$. From Figures 6(a) and 7(a), we see that stresses σ_{xx}^* are always positive beyond the crack, but on the crack, they can be both positive and negative, everything depends on the position of crack center. σ_{xx}^* decreases away from the crack tips. From Figures 6(b) and 7(b), we see that stresses σ_{yy}^* beyond the crack tip b are constant in plastic strip at first, then decrease to some value and after that increase and become greater than 1 at a certain distance, that is $\sigma_{yy} > \sigma_Y$ (beam material goes into a plastic state). But beyond the crack tip a, they behave differently. As we can see from Figure 6(b), stresses σ_{yy} are negative when the length of plastic strip at this tip is zero. But for nonzero length, at first, they are equal to σ_Y in plastic strip and then decrease from positive values to negative ones.

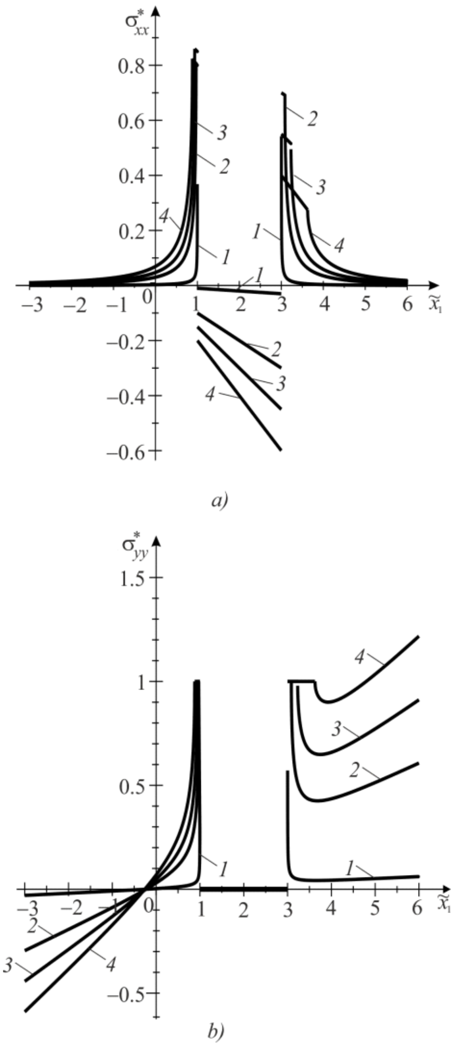

Fig. 7. Stress distribution on crack line at $\tilde{x} = 2$ and various values of external load $\tilde{\sigma}$

5. CONCLUSIONS

Numerical analysis of the problem confirms that the destruction of the strip (beam) with a crack perpendicular to its axis on the grounds of δ_K – fracture model for bodies with crack will start from the crack tip that is farther to the beam axis. A quantitative estimation of length of plastic zone and value of the opening of crack faces are also obtained. Note that the plastic strip is not always formed at the nearer (to the beam axis) crack tip, which is in the strip of compressive stresses for the beam without crack. But, in this case, there is no contact between crack faces, and the length of the plastic strip at this tip decreases when the tip removes from the beam axis. At some distance, this length becomes zero. As the distance increases, quantitative changes turn into qualitative ones, and perhaps, crack faces begin to contact. This requires a further investigation.

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