INTERACTION OF ORTHOGONAL-POLARIZED WAVES IN 1D METAMATERIAL WITH KERR NONLINEARITY

DMITRII AMPILOGOV

Immanuel Kant Baltic Federal University
Nevskogo 14, 236004 Kaliningrad, Russia

(received: 24 February 2017; revised: 24 March 2017; accepted: 27 March 2017; published online: 19 April 2017)

Abstract: The investigation of the wave propagation in a 1D metamaterial is continued in this paper. A nonlinear evolution equation of the wave interaction of two polarizations by means of the projection operator method is obtained and a particular solution in the case of slow-varying envelopes is found.

Keywords: metamaterials, wave interaction, orthogonal polarized waves

DOI: https://doi.org/10.17466/tq2017/21.2/q

1. Introduction

1.1. On nonlinear pulse propagation theory

The nonlinear behavior of electromagnetic (EM) wave propagation depends on the relations between the field and induced polarization. It is obvious that it is necessary to use either a numerical scheme or approximations to obtain an analytical solution of a nonlinear problem. The first successful approach to such reduction was to use a set of slowly varying envelopes. The simplest model scalar equation for directed wave propagation, based on this approach, has the form of a nonlinear Schrödinger equation derived by Zakharov in 1968 [1]. Its integrability [2] has made the model very attractive because of a rich "zoo" of explicit solutions of the equation [3].

A natural step of an integrable generalization of such a model lies in a plane of better approximations of dispersion, dissipation [4, 5] and nonlinearity (modified nonlinear Schrödinger (MNS) equations, see *e.g.* [3]) that allows extending pulse durations down to picoseconds.

There are plenty of alternative ideas on the few-cycle pulse soliton-type description in different media [6, 7]

The next step of this movement to a description of ultrashort pulses that maintains integrability is made in the works of Shäfer-Wayne [8, 9]. The Short Pulse Equation (SPE) again relates to unidirectional propagation for which a special kind of the dispersion law and nonlinearity action has been accounted for in a rescaled evolution. A generalization that allows including a description and interaction of opposite directed waves is connected with the idea of a joint account of the correspondent spaces of "hybrid" electric-magnetic amplitudes [10–12]. The projecting operator (PO) method [10] works at arbitrary dispersion and nonlinearity. A similar universality is demonstrated by a method of [13]. The PO technique gives a systematic transition to hybrid fields with a simultaneous superposition of nonlinear terms that effectively approximate weak nonlinearity, arriving at the mentioned celebrated model equations at subspaces of directed waves [14]. The field hybridization may account for the ab initio dispersion and dissipation [4, 5] and nonlinearity by the iteration procedure [15].

Next, a natural step for the electromagnetic field accounts for the polarization and leads to double component vector equations [16], a similar vector equation is studied in [17]. Both the direction and polarization are studied theoretically and, which is of importance, experimentally in [18].

1.2. On Drude model

One of the important applications of such a model relates to metamaterials that are characterized by negative values of the constitutive parameters ε and μ that must be dispersive, *i.e.*, their permittivity and permeability must be frequency dependent, otherwise, they would not be causal [19]. The two-time derivative Lorentz material model encompasses the most commonly discussed metamaterial models; it has the susceptibility of a frequency domain [20]:

$$\chi = \frac{\omega_p^2\chi_a + i\omega_p\chi_\beta\omega - \chi_\gamma\omega^2}{\omega_0^2 + i\omega\Gamma - \omega^2} \tag{1}$$

its particular case used in [9]. There would be independent models for the permittivity and permeability. $\varepsilon(\omega) = \omega_0(1+\chi_e)$ and $\mu(\omega) = \mu_0(1+\chi_m)$. This 2TDLM model produces a resonant response at $\omega = \omega_0$ when $\Gamma = 0$. It recovers the Drude model when this resonant frequency goes to zero, and the constants $\chi_\alpha = 1,\ \chi_\beta = \chi_\gamma = 0$. For this model Kanattsikov and Pietrzyk have shown that the propagation of ultra-short pulses could be described by the short pulse Schäfer-Wayne equation [21]

1.3. Aim and scope

A systematic application of the projecting approach originating from [10] for a 1D metamaterial with an account of both polarizations of the EM wave is continued. The technique and results of our previous work [22] on nonlinear evolution equations of opposite directed waves with one polarization in the Drude 1D-metamaterial is developed.

In this paper the application of the projecting operator method for this case is demonstrated. The obtained nonlinear equation for a metamaterial is compared with our previous results and the SPE vector. On the basis of the resulting equations, the wave packets for linear and nonlinear cases are studied.

In 1983 Kaplan showed saving the arrangements of polarization for an ordinary nonlinear Kerr material [23]. However, the situation is different for a metamaterial, as shown in our work [22] for unique polarization. We discover a change of the arrangements of wave modes. Now, the questions to be answered is: what happens with the account of polarization and what the interactions of all four modes in a metamaterial look like. The contents of the paper are as follows:

- Section 2: Statement of the boundary regime problem.
- Section 3: 4×4 matrix projection operators with an arbitrary dispersion account are built for the case of two polarizations.
- Section 4: Derivation of a general linear system of equations for two left and two right waves with orthogonal polarizations.
- Section 5: A general nonlinear system of equations for the left and right waves and two polarizations is obtained.
- Section 6: A novel system of short pulse equations is obtained for a particular case of the Kerr nonlinearity within the approximate Drude dispersion that is reduced to the Shäfer-Wayne equation for unique polarization.
- In Section 7 attention is focused on wave trains, starting from linear ones, and a plane wave is obtained with a wavelength depending on the amplitude taking into account the nonlinearity.

2. Maxwell's equations. Boundary regime problem

The starting point are the Maxwell equations for linear isotropic dispersive dielectric media in the SI unit system:

$$\operatorname{div}\vec{D}(\vec{r},t)=0 \tag{2}$$

$$\operatorname{div}\vec{B}(\vec{r},t)=0 \tag{3}$$

$$\operatorname{rot}\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \tag{4}$$

$$\operatorname{rot}\vec{H}(\vec{r},t)=\frac{\partial\vec{D}(\vec{r},t)}{\partial t} \tag{5}$$

Restricting ourselves to a one-dimensional model, similarly to Shäfer, Wayne [8], where the x-axis is chosen as the direction of wave propagation assuming zero longitudinal field components allows us to write the Maxwell equations with the arbitrary polarization account:

$$\begin{aligned} \frac{\partial D_y}{\partial t} &= -\frac{\partial H_z}{\partial x} \\ \frac{\partial D_z}{\partial t} &= \frac{\partial H_y}{\partial x} \\ \frac{\partial B_y}{\partial t} &= \frac{\partial E_z}{\partial x} \\ \frac{\partial B_z}{\partial t} &= -\frac{\partial E_y}{\partial x} \end{aligned} \tag{6}$$

y-projections will be hereinafter denoted by index "1" and other projections – by index "2". To close the system (6) we need to add the material relations:

$$D_i(x,t) = \hat{\varepsilon} E_i(x,t), \quad i = 1,2 \tag{7}$$

$$B_i(x,t) = \hat{\mu} H_i(x,t), \quad i = 1,2 \tag{8}$$

where $\hat{\mu}$ and $\hat{\varepsilon}$ are integral convolution-type operators [22]:

$$\hat{\varepsilon}\psi(x,t) = \int_{-\infty}^{\infty} \tilde{\varepsilon}(t-s)\psi(x,s)ds \tag{9}$$

$$\hat{\mu}\psi(x,t) = \int_{-\infty}^{\infty} \tilde{\mu}(t-s)\psi(x,s)ds \tag{10}$$

with kernels

$$\tilde{\varepsilon}(t-s) = \frac{\varepsilon_0}{2\pi} \int_{-\infty}^{\infty} \varepsilon(\omega) \exp\big(i\omega(t-s)\big)\, d\omega \tag{11}$$

$$\tilde{\mu}(t-s) = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} \mu(\omega) \exp\big(i\omega(t-s)\big)\, d\omega \tag{12}$$

Hence, the operator form of the equation (6) is:

$$\partial_t(\hat{\varepsilon} E_{1,2}) = \mp \partial_x(\hat{\mu}^{-1} B_{2,1}) \tag{13}$$

$$\partial_t B_{1,2} = \pm \partial_x E_{2,1} \tag{14}$$

Here it is marked:

$$\frac{\partial}{\partial x} \equiv \partial_x \tag{15}$$

$$\frac{\partial}{\partial t} \equiv \partial_t \tag{16}$$

Adding the boundary conditions to state the problem:

$$E_{1,2}(0,t) = j_{1,2}(t), \qquad B_{1,2}(0,t) = \ell_{1,2}(t) \tag{17}$$

j_i and ℓ_i are arbitrary functions, continued to the half space $t < 0$ antisymmetrically:

$$j_i(-t) = -j_i(t), \qquad \ell_i(-t) = -\ell_i(t), \quad i = 1,2 \tag{18}$$

3. Dynamic projecting operators

Doing the Fourier transformations like in [22] and plugging them into the system of equations (13) we have a closed system:

$$\partial_t \left(\int_{-\infty}^{\infty} \varepsilon(\omega)\mathcal{E}_{1,2}(x,\omega)\exp(i\omega t)d\omega \right) = \mp \frac{1}{\mu_0\varepsilon_0}\partial_x \left(\int_{-\infty}^{\infty} \frac{\mathcal{B}_{2,1}(x,\omega)}{\mu(\omega)}\exp(i\omega t)d\omega \right) \qquad (19)$$

The inverse Fourier transformation yields the four equations of (13), written in the short form:

$$\partial_x \mathcal{B}_{2,1} = \mp i\omega\mu_0\varepsilon_0\mu(\omega)\varepsilon(\omega)\mathcal{E}_{1,2} \qquad (20)$$

$$\partial_x \mathcal{E}_{2,1} = \pm i\omega\mathcal{B}_{1,2} \qquad (21)$$

Let us define the column of the component transform field

$$\tilde{\Psi} = \begin{pmatrix} \mathcal{B}_2 \\ \mathcal{B}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_1 \end{pmatrix} \qquad (22)$$

and the matrix operator with the obvious elements from (20)–(21)

$$\mathcal{L} = \begin{pmatrix} \hat{0} & \mathcal{L}_1 \\ \mathcal{L}_2 & \hat{0} \end{pmatrix} \qquad (23)$$

arriving at

$$\partial_x \tilde{\Psi} = \mathcal{L}\tilde{\Psi} \qquad (24)$$

By the explicit form of the matrix L we write the eigenvector problem:

$$\begin{pmatrix} 0 & 0 & 0 & -i\omega a^2 \\ 0 & 0 & i\omega a^2 & 0 \\ 0 & i\omega & 0 & 0 \\ -i\omega & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \qquad (25)$$

with the standard condition on λ:

$$\det(\mathcal{L} - \lambda I) = 0, \qquad (26)$$

where I – identity matrix. It is the biquadratic equation for λ:

$$\lambda^2 + 2\omega^2 a^2 \lambda^2 + \omega^4 a^4 = 0 \qquad (27)$$

$$(\lambda^2 + \omega^2 a^2)^2 = 0 \qquad (28)$$

The solutions of this equation are:

$$\lambda_{1,2} = \pm i a\omega \qquad (29)$$

that yields two different eigenvalues. The degeneration means the existence two of linear independent eigenvectors for each value of λ:

- for $\lambda_1 = ai\omega$:

$$\Psi_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \frac{1}{a} \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{a} \\ 0 \end{pmatrix} \tag{30}$$

- $\lambda_2 = -ai\omega$:

$$\Psi_3 = \begin{pmatrix} 0 \\ -a \\ 1 \\ 0 \end{pmatrix}, \qquad \Psi_4 = \begin{pmatrix} a \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{31}$$

From these matrices we construct an auxiliary matrix Φ:

$$\Phi = \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & 1 & -a & 0 \\ 0 & \frac{1}{a} & 1 & 0 \\ \frac{1}{a} & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

An inverse matrix:

$$\Phi^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & 1 & a & 0 \\ 0 & -\frac{1}{a} & 1 & 0 \\ \frac{1}{a} & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

The structure of the projectors is described by the relation:

$$\mathbf{P}^{(i)}_{jk} = \Phi_{ji} \Phi^{-1}_{ik} \tag{34}$$

that yields four matrix projecting operators in t-representation by the standard general formula (see again [22]):

$$\mathbf{P}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -\hat{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\hat{a}^{-1} & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

$$\mathbf{P}^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \hat{a} & 0 \\ 0 & \hat{a}^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{36}$$

$$\mathbf{P}^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\hat{a} & 0 \\ 0 & -\hat{a}^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

$$\mathbf{P}^{(4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \hat{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{a}^{-1} & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

Projectors $\mathbf{P}^{(1,2)}$ correspond to λ_1 and two other ones – to λ_2.
The operators $\hat{a}$, $\hat{a}^{-1}$ are defined as [22]:

$$\hat{a}\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\eta(x,\tau)\int_{-\infty}^{\infty} a(\omega)\exp\big(i\omega(t-\tau)\big)\,d\omega\right]d\tau \tag{39}$$

$$\hat{a}^{-1}\xi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\xi(x,\tau)\int_{-\infty}^{\infty} \frac{1}{a(\omega)}\exp\big(i\omega(t-\tau)\big)\,d\omega\right]d\tau \tag{40}$$

where $a(\omega)$ is a positive solution of the quadratic equation (28):

$$\mu_0\varepsilon_0\varepsilon(\omega)\mu(\omega) \equiv c^{-2}\varepsilon(\omega)\mu(\omega) \equiv a^2(\omega) \tag{41}$$

where $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ is the velocity of light in vacuum.

4. Separated equations and definition for left and right waves

Let us return to the time-domain. Let us write the matrix equation (24) in this representation:

$$\partial_x\Psi = \widehat{L}\Psi \tag{42}$$

where

$$\Psi = \begin{pmatrix} B_2 \\ B_1 \\ E_2 \\ E_1 \end{pmatrix} \tag{43}$$

$$\widehat{L} = \begin{pmatrix} 0 & 0 & 0 & -\partial_t\hat{a}^2 \\ 0 & 0 & \partial_t\hat{a}^2 & 0 \\ 0 & \partial_t & 0 & 0 \\ -\partial_t & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

The action of projectors $P^{(1)}$ and $P^{(3)}$ on (42) yields to hybrid waves Λ_1 and Λ_2:

$$\Lambda_1 = \frac{1}{2}(B_2 - \hat{a}E_1) \tag{45}$$

$$\Lambda_2 = \frac{1}{2}(B_1 - \hat{a}E_2) \tag{46}$$

The action of projectors $P^{(4)}$ and $P^{(2)}$ on (42) yields to hybrid waves Π_1 and Π_2:

$$\Pi_1 = \frac{1}{2}(B_2 + \hat{a}E_1) \tag{47}$$

$$\Pi_2 = \frac{1}{2}(B_1 + \hat{a}E_2) \tag{48}$$

Waves Π_1 and Λ_1 are introduced for the case of unique polarization [22] and describe a propagation of y-polarization. The other two waves do the same for z-polarization.

$$\partial_x \Pi_1 = -\partial_t \hat{a} \Pi_1 \tag{49}$$

$$\partial_x \Pi_2 = \partial_t \hat{a} \Pi_2 \tag{50}$$

$$\partial_x \Lambda_1 = \partial_t \hat{a} \Lambda_1 \tag{51}$$

$$\partial_x \Lambda_2 = -\partial_t \hat{a} \Lambda_2 \tag{52}$$

Using the definitions of (45)–(48) and (17) we derive the boundary regime conditions for left and right waves:

$$\Lambda_1(0,t) = \frac{1}{2}\big(B_2(0,t) - \hat{a}E_1(0,t)\big) = \frac{1}{2}\big(k_z(t) - \hat{a}j_1(t)\big) \tag{53}$$

$$\Lambda_2(0,t) = \frac{1}{2}\big(B_1(0,t) - \hat{a}E_2(0,t)\big) = \frac{1}{2}\big(k_1(t) - \hat{a}j_z(t)\big) \tag{54}$$

$$\Pi_1(0,t) = \frac{1}{2}\big(B_2(0,t) + \hat{a}E_1(0,t)\big) = \frac{1}{2}\big(\ell_z(t) + \hat{a}j_1(t)\big) \tag{55}$$

$$\Pi_2(0,t) = \frac{1}{2}\big(B_1(0,t) + \hat{a}E_2(0,t)\big) = \frac{1}{2}\big(\ell_1(t) + \hat{a}j_z(t)\big) \tag{56}$$

5. General nonlinearity account

Let us consider a nonlinear problem. The starting point are the Maxwell's equations (6) again with generalized nonlinear material relations:

$$\begin{aligned} D_i &= \hat{\varepsilon} E_i + P_i^{(NL)} \\ B_i &= \hat{\mu} H_i + M_i^{(NL)}, \quad i = 1,2 \end{aligned} \tag{57}$$

P_{NL} – the nonlinear part of polarization (M_{NL} – one for magnetization). The linear parts of polarization and magnetization have already been taken into account. In the time-domain a closed nonlinear version of (13) is:

$$\partial_t(\hat{\varepsilon} E_{1,2}) + \partial_t P_{1,2}^{(NL)} = \mp \partial_x \hat{\mu}^{-1} B_{2,1} \mp \partial_x \hat{\mu}^{-1} M_{2,1}^{(NL)} \tag{58}$$

$$\partial_t B_{2,1} = -\partial_x E_{1,2} \tag{59}$$

The action of the operator $\hat{\mu}$ on the first pair of equations of system (58) and the use of the same notations Ψ and $\widehat{L}$ from (43), (44) once more produce a nonlinear analogue of the matrix equation (42):

$$\partial_x \Psi - \widehat{L}\Psi = \partial_x \begin{pmatrix} M_2^{(NL)} \\ M_1^{(NL)} \\ 0 \\ 0 \end{pmatrix} - \hat{\mu}\partial_t \begin{pmatrix} -P_2^{(NL)} \\ P_1^{(NL)} \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

In the r.h.s there is a vector of nonlinearity for the case of the opposite directed 1D-waves:

$$\mathbb{N} = \begin{pmatrix} \partial_x M_1^{(NL)} + \hat{\mu}\partial_t P_2^{(NL)} \\ \partial_x M_2^{(NL)} - \hat{\mu}\partial_t P_1^{(NL)} \\ 0 \\ 0 \end{pmatrix} \tag{61}$$

Next, acting by operators $\widehat{\mathbf{P}}^{(1,2,3,4)}$ (35) on Equation (60) one can find:

$$\partial_x \Pi_1 + \partial_t \hat{a} \Pi_1 = \partial_x M_2^{(NL)} + \hat{\mu}\partial_t P_1^{(NL)} \tag{62}$$

$$\partial_x \Pi_2 - \partial_t \hat{a} \Pi_2 = \partial_x M_1^{(NL)} - \hat{\mu}\partial_t P_2^{(NL)} \tag{63}$$

$$\partial_x \Lambda_1 - \partial_t \hat{a} \Lambda_1 = \partial_x M_2^{(NL)} + \hat{\mu}\partial_t P_1^{(NL)} \tag{64}$$

$$\partial_x \Lambda_2 + \partial_t \hat{a} \Lambda_2 = \partial_x M_1^{(NL)} - \hat{\mu}\partial_t P_2^{(NL)} \tag{65}$$

Generally the r.h.s. of each equation (62) depends on the field vectors $\vec{E}, \vec{B}$ that should be presented in terms of the fields $\vec{\Pi}, \vec{\Lambda}$ to close the system. The vector components are expressed by means of the inverse transformation of (45)–(48).

6. Kerr nonlinearity account for lossless Drude metamaterials

6.1. Equations of interaction of waves via Kerr effect

For nonlinear Kerr materials [24], the third-order nonlinear part of polarization [16, 24] has the form:

$$P_{1,2}^{(NL)} = \varepsilon_0 \chi_e^{(3)} \left(E_{1,2}^3 + E_{1,2} E_{2,1}^2 \right) \tag{66}$$

From (61), deleting magnetic nonlinearity, one can find the vector N:

$$\mathbb{N} = -\varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \begin{pmatrix} -E_2^3 - E_2 E_1^2 \\ E_1^3 + E_1 E_2^2 \\ 0 \\ 0 \end{pmatrix} \tag{67}$$

An account for the definitions of the hybrid fields as (45) gives:

$$E_1 = \hat{a}^{-1}(\Pi_1 - \Lambda_1)(68)$$

$$E_2 = \hat{a}^{-1}(\Pi_2 - \Lambda_2)(69)$$

The system for left and right waves with two polarization equations in a medium with the Kerr nonlinearity:

$$\partial_x \Pi_1 + \partial_t \hat{a} \Pi_1 = \varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[\left(\hat{a}^{-1}(\Pi_1 - \Lambda_1) \right)^3 + \hat{a}^{-1}(\Pi_1 - \Lambda_1) \left(\hat{a}^{-1}(\Pi_2 - \Lambda_2) \right)^2 \right] \tag{70}$$

$$\partial_x \Pi_2 - \partial_t \hat{a} \Pi_2 = -\varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[\left(\hat{a}^{-1}(\Pi_2 - \Lambda_2) \right)^3 + \hat{a}^{-1}(\Pi_2 - \Lambda_2) \left(\hat{a}^{-1}(\Pi_1 - \Lambda_1) \right)^2 \right] \tag{71}$$

$$\partial_x \Lambda_1 - \partial_t \hat{a} \Lambda_1 = \varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[\left(\hat{a}^{-1}(\Pi_1 - \Lambda_1) \right)^3 + \hat{a}^{-1}(\Pi_1 - \Lambda_1) \left(\hat{a}^{-1}(\Pi_2 - \Lambda_2) \right)^2 \right] \tag{72}$$

$$\partial_x \Lambda_2 + \partial_t \hat{a} \Lambda_2 = -\varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[\left(\hat{a}^{-1}(\Pi_2 - \Lambda_2) \right)^3 + \hat{a}^{-1}(\Pi_2 - \Lambda_2) \left(\hat{a}^{-1}(\Pi_1 - \Lambda_1) \right)^2 \right] \tag{73}$$

In an unidirectional case with $\Pi_2, \Lambda_1 = 0$ one can obtain a system that describes the interaction between hybrid fields with different polarizations:

$$\partial_x \Pi_1 + \partial_t \hat{a} \Pi_1 = \varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[\left(\hat{a}^{-1} \Pi_1 \right)^3 + \hat{a}^{-1} \Pi_1 \left(-\hat{a}^{-1} \Lambda_2 \right)^2 \right] \tag{74}$$

$$\partial_x \Lambda_2 + \partial_t \hat{a} \Lambda_2 = -\varepsilon_0 \hat{\mu} \chi_e^{(3)} \partial_t \left[-\left(\hat{a}^{-1} \Lambda_2 \right)^3 - \hat{a}^{-1} \Lambda_2 \left(-\hat{a}^{-1} \Pi_1 \right)^2 \right] \tag{75}$$

Due to the propagation in one direction, it is useful to mark Π_1 and Λ_2 as:

$$\Pi_1 \equiv R_1, \qquad \Lambda_2 \equiv R_2. \tag{76}$$

Applying the Drude model, we approximately write

$$\hat{a}^{-1} \eta(x,t) \approx \frac{c}{pq} \partial_t^2 \eta(x,t) \tag{77}$$

$$\mu \eta(x,t) \approx -q \partial_t^{-2} \eta(x,t) \tag{78}$$

(see again [22] for details), plugging $\varepsilon_0 \mu_0 = c^{-2}$, finally, we get the SPE system:

$$\frac{c}{pq} \partial_x R_1 + \partial_t^{-1} R_1 = \varepsilon_0 q \chi_e^{(3)} \left(\frac{c}{pq} \right)^3 \partial_t^{-1} \left[\left(\partial_t^2 R_1 \right)^3 + \partial_t^2 R_1 \left(-\partial_t^2 R_2 \right)^2 \right] \tag{79}$$

$$\frac{c}{pq} \partial_x R_2 + \partial_t^{-1} R_2 = \varepsilon_0 q \chi_e^{(3)} \left(\frac{c}{pq} \right)^3 \partial_t^{-1} \left[\left(\partial_t^2 R_2 \right)^3 + \partial_t^2 R_2 \left(\partial_t^2 R_1 \right)^2 \right] \tag{80}$$

Differentiation on t leads to the equation:

$$\frac{c}{pq} \partial_{xt} R_1 + R_1 = \gamma^2 \left[\left(\partial_t^2 R_1 \right)^3 + \partial_t^2 R_1 \left(\partial_t^2 R_2 \right)^2 \right] \tag{81}$$

$$\frac{c}{pq} \partial_{xt} R_2 + R_2 = \gamma^2 \left[\left(\partial_t^2 R_2 \right)^3 + \partial_t^2 R_2 \left(\partial_t^2 R_1 \right)^2 \right] \tag{82}$$

where

$$\gamma^2 = \chi_e^{(3)} \frac{c^2}{p^4 q^2} \tag{83}$$

Introducing the new field functions R_1 and λ and the variable χ as:

$$r_1 \equiv \gamma^{-1} \partial_t^2 R_1 \tag{84}$$

$$r_2 \equiv \gamma^{-1} \partial_t^2 R_2 \tag{85}$$

$$\partial_x = \frac{pq}{c} \partial_\chi \tag{86}$$

we can obtain the system:

$$\partial_{\chi t} r_1 + r_1 = \partial_t^2 \left(r_1^3 + r_1 r_2^2 \right) \tag{87}$$

$$\partial_{\chi t} r_2 + r_2 = \partial_t^2 \left(r_2^3 + r_1^2 r_2 \right) \tag{88}$$

It is a generalization of the Schäfer-Wayne equation for the case of interaction of two right waves which is one of the objectives of this work.

7. Wave trains

7.1. Linear wave packets for right waves

We start from the linear version of equations (74) that are identical, hence, we take one of them:

$$c\partial_{xt}R_1 + pqR_1 = 0 \tag{89}$$

plugging the wavetrain solution that we prepare for a comparison with the nonlinear case:

$$R_1 = A(x,t)\exp\left[i(kx-\omega t)\right] + c.c. \tag{90}$$

Differentiating

$$\begin{aligned}\partial_{xt}R_1 = A_{xt}\exp\left[i(kx-\omega t)\right] + ikA_t\exp\left[i(kx-\omega t)\right] - i\omega A_x\exp\left[i(kx-\omega t)\right] \\ - i\omega(ik)A\exp(kx-\omega t) + c.c.\end{aligned} \tag{91}$$

putting the result in the equation (89), assuming a slow varying amplitude:

$$A_x \ll kA, \qquad A_t \ll \omega A, \tag{92}$$

to kill the zeroth order term gives the dispersion relation:

$$k(\omega) = -\frac{pq}{c\omega} \tag{93}$$

then, in the first order the equation arrives at

$$A_t - \frac{\omega}{k}A_x = 0 \tag{94}$$

Next, denoting

$$v_g = \frac{\omega}{k} \tag{95}$$

after a conventional change of the variables:

$$\eta = t - \frac{x}{v_g}, \qquad \xi = t + \frac{x}{v_g} \tag{96}$$

$$\partial_t = \partial_\xi + \partial_\eta, \qquad \partial_x = \frac{1}{v_g}\partial_\xi - \frac{1}{v_g}\partial_\eta, \qquad A(x,t) \to \mathbb{A}(\eta,\xi) \tag{97}$$

the equation (94) trivializes as

$$2\mathbb{A}_\eta = 0 \tag{98}$$

It is shown that the amplitude function A is independent from η

$$\mathbb{A} = f(\xi), \qquad A(x,t) = f\left(t + \frac{x}{v_g}\right) \tag{99}$$

Substituting this relation into (94) leads to the definition of v_g:

$$v_g = -\frac{\omega^2 c}{pq} \tag{100}$$

The “minus” sign means the right direction of the wave propagation. To fix the unique solution, it is necessary to add a boundary condition:

$$A(0,t) = A_0\exp\left[-\left(\frac{t}{\tau}\right)^2\right] \tag{101}$$

τ characterizes the wave packet width and ω characterizes the period of oscillation. Accounting for the boundary regime (101), for the R_1-wave, propagated to the right the explicit formula is obtained:

$$R_1(x,t) = A_0 \exp\left[-\left(\frac{t+\frac{x}{v_g}}{\tau}\right)^2 + i(kx-\omega t)\right] + c.c. \tag{102}$$

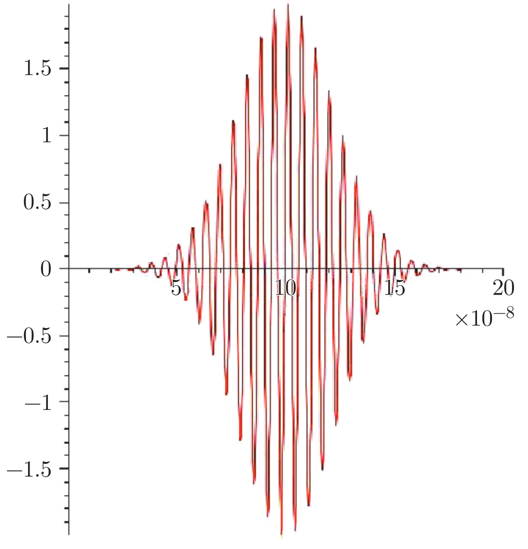

Figure 1. Wave packet of R_1 for case $\omega = 10^9$ Hz, $x = 0$ m, $v_g = 10^8$ m/s, $\tau = 50T$ s

The wavetrain with the other polarization differs only by the component numbers of electric and magnetic fields as is prescribed by (46). The waves directed in the opposite direction are defined by (45), (48), their formulas differ from (102) only by signs of $\frac{x}{v_g}$.

7.2. Dispersionless nonlinear equations for envelopes

For R_1 and R_2 in the wavetrain form with the frequency chosen by the boundary condition as in the linear case:

$$R_1 = A(x,t)\exp\left[i(kx-\omega t)\right] + c.c. \tag{103}$$

$$R_2 = B(x,t)\exp\left[i(kx-\omega t)\right] + c.c. \tag{104}$$

and plugging these relations into the equations (74) together, with an account of (92), the linear independence of complex conjugated parts and strong inequality (92), in the first approximation (keeping the nonlinear resonant terms in the r.h.s) one can obtain

$$c\left(ikA_t - i\omega A_x\right) = \frac{2\chi_e^{(3)} c^2 \omega^6}{p^3 q} A\left(|A|^2 + |B|^2\right) \tag{105}$$

$$c\left(ikB_t - i\omega B_x\right) = \frac{2\chi_e^{(3)}c^2\omega^6}{p^3 q}B\left(|A|^2+|B|^2\right) \tag{106}$$

The solution parameter k is chosen to simplify the equations as

$$-ci\omega(ik)A + pqA = \left(k\omega + \frac{pq}{c}\right)A = 0 \tag{107}$$

that is equivalent to the expression

$$k = -\frac{pq}{c\omega} \tag{108}$$

that fixes the phase velocity of the carrier wave as in the linear case. Equations (105) and (106) with an account of the approximation (92) are:

$$\frac{k}{\omega}A_t - A_x = -i\frac{2\chi_e^{(3)}c\omega^5}{p^3 q}A\left(|A|^2+|B|^2\right) \tag{109}$$

$$\frac{k}{\omega}B_t - B_x = -i\frac{2\chi_e^{(3)}c\omega^5}{p^3 q}B\left(|A|^2+|B|^2\right) \tag{110}$$

7.3. On nonlinear dispersion relations

Equation (105) in the suggestion of constant A, B transforms into:

$$-ic\omega(i\kappa) + pq = \chi_e^{(3)}\frac{c^2}{p^3 q}\left(|A|^2+|B|^2\right) \tag{111}$$

One can easily find the same expression from equation (106). The nonlinear dispersion relation $\kappa(\omega)$ will be:

$$\kappa = \omega^5\chi_e^{(3)}\frac{c}{p^3 q}\left(|A|^2+|B|^2\right) - \frac{pq}{c\omega} \tag{112}$$

The particular solutions of (105) and (106) in this case are:

$$R_1 = A\exp\left[i\left(\left(\omega^5\chi_e^{(3)}\frac{c}{p^3 q}\left(|A|^2+|B|^2\right) - \frac{pq}{c\omega}\right)x - \omega t\right)\right] + c.c. \tag{113}$$

$$R_2 = B\exp\left[i\left(\left(\omega^5\chi_e^{(3)}\frac{c}{p^3 q}\left(|A|^2+|B|^2\right) - \frac{pq}{c\omega}\right)x - \omega t\right)\right] + c.c. \tag{114}$$

It is useful to introduce normalized frequencies

$$q = \omega_q p \tag{115}$$

$$\omega = \omega_k p \tag{116}$$

$$w = \frac{\omega_k}{\omega_q} \tag{117}$$

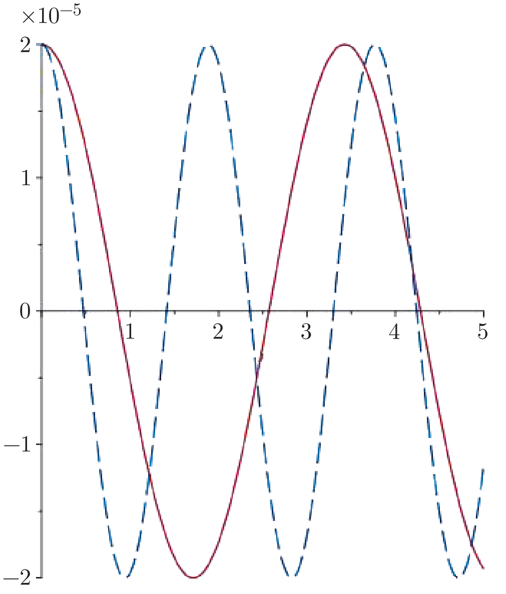

Figure 2. Nonlinear function R_1 (118) (red) and its linear analogue $A\exp(ikx-i\omega t)$ (blue dash) for case: $p=10^9$ Hz, $q=\omega=10^3 p$, $A=10^{-5}$ T, $B=2\cdot 10^{-5}$ T, $t=5(\frac{2\pi}{\omega})$ s

The waves R_1 and R_2 in these terms are:

$$R_1 = A\exp\left[i\left(cp\chi_e^{(3)}\omega_k^4 w\left(|A|^2+|B|^2\right)-\frac{1}{c}\frac{p}{w}\right)x-i\omega_k pt\right]+c.c. \tag{118}$$

$$R_2 = B\exp\left[i\left(cp\chi_e^{(3)}\omega_k^4 w\left(|A|^2+|B|^2\right)-\frac{1}{c}\frac{p}{w}\right)x-i\omega_k pt\right]+c.c. \tag{119}$$

The function R_1 is represented in Figure 2.

8. Conclusion

The wave propagation of two polarizations in a 1D-metamaterial was studied in this work. The general equation of directed wave propagation in a 1D-metamaterial with two polarizations was obtained. It is shown that it has the "SPE vector" form for the Drude metamaterial with the Kerr nonlinearity.

Acknowledgements

We thank S. B. Leble for the important advice and fruitful discussion.

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