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Mean residual lifetime assessment approach for a multi-state standby system

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Highlights

- The mean residual lifetime functions of a three-state standby system are obtained.
- In MSS, a performance measures are obtained in case a system is at state *"j or above"* at *t*.
- The main contribution of the study; MRL is obtained when the system is at state *"j"*.
- The effect of different degradation rates of each state on the MRL is investigated.
- Optimization problem finds the average replacement costs, the optimal replacement times.

Abstract

In this paper, a new MRL assessment approach for a multi-state standby system is considered. The three-state system is backed up with a binary cold standby unit. Given that the system is at a specific state at time *t*, obtaining the MRL is worth considering in conducting the maintenance and repair plans of the system. For different degradation rates and time points, MRL results are examined. An HCTMP is considered for the degradation. Therefore, when the system is observed to be at its perfect state, the MRL decrease with an increase in all the failure rates of the system. However, when the system is observed to be at its partial state, the MRL is not affected by the increase in the failure rate pertained to the perfect state. The MRL when the system has known to be failed before time *t* and backed up with the standby unit increases with the time increase whereas the MRL when the system is at its perfect(or partial) state is constant when time increases. Moreover, cost evaluation of the system is analyzed. The results are supported with numerical examples and graphical representations.

Keywords
reliability analysis, standby systems, multi-state systems, mean time to failure, mean residual lifetime

1. Introduction

In the theory of reliability analysis and the reliability applications of many industrial systems, to enhance the redundancy of the system or bring down the costs in the maintenance activities, a standby redundancy is practically used. Many structures used in engineering can fail due to many external factors besides getting age or as a result of their natural degradation process. For instance, a computer can be affected by a virus, an energy supplier can be influenced by a change in a voltage or a metal component can be affected by temperature changes [4]. Then, the degradation as a result of those external factors for the components or the system is inevitable in many cases. Therefore, to attain the reliability of many systems such as computers, telecommunication systems or electric power supply systems, standby redundancy is used. It also has critical importance in some fields to use a standby component, for instance, in airplane control systems or space works. Standby systems are mostly studied in the literature when the components or the systems have binary states. Pen, Zichun and Bin [27] dealt with the reliability analysis of a standby system in which one of its components is a cold standby. The results

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were illustrated by an exponentially distributed lifetimes of the components. The matrix analysis method was used to deal with the reliability analysis of such systems and some performance characteristics of those systems were obtained. Chen, et al. [5], worked on a binary-state standby system in one of the system's components was cold standby and unrepairable. A binary decision diagram was suggested for the reliability analysis of the system discussed. Also, it has been noted that the lifetime of the system decreases when a specific type of failure mechanism is considered based on a simulation study. Li, et al. [23] have worked on a reliability analysis of an air cooling system of a nuclear power plant in which one of the system components is unrepairable cold standby. They took into account two cases as the activation time of the standby component is negligible or non-negligible. Zhai, et al. [37] considered the reliability analysis of a 1-out-of-n cold standby system and proposed a multiple-valued decision diagram for the reliability evaluation of the system. The findings of the study were also verified by Monte Carlo simulations. Fathizadeh and Khorshidian [12] used an alternative approach based on a matrix repair fuction in obtaining the reliability of a standby system which is binary. In this case, they considered the system with two identical cold standby components have a Semi-Markov degradation process. Some reliability measures such as availability and mean time to failure were also obtained within their study. Chen, et al. [4] considered a two unit cold standby system and evaluated its reliability. The failure of the system resulted from the natural degradation process or external factors. Therefore, they took into account the Markov process, Laplace transformation techniques and Tauberian theory. Some indices were derived for the system availability, system reliability, failure rate and mean time to the first failure. Yuan and Meng [35] evaluated the reliability analysis of a repairable warm standby system with two nonidentical components. Some important robust state indices were derived by using a Markov process theory and Laplace transformations. Numerical illustrations were also given to support the findings in the same work. Leung, Zhang and Lai [20] worked on a repairable cold standby system that has binary components and obtained its reliability. The optimal T minimizing the mean loss/cost rate per time in the long term was determined. The expression for the mean loss rate of the system was obtained explicitly in their study. Su [29] examined the reliability of the repairable and unrepairable cold standby systems which have binary states. New reliability indices and some computing methods for those indices in case of repairable and unrepairable failures of the system were proposed. Yaghoubi and Akhavan [33] examined the reliability of a system with two units. A dependency has been assumed between the switch and its associated active component. Marshall-Olkin bivariate exponential distribution was used to model the dependency and the continuous Markov chain method was considered in the reliability analysis. Wang, et al. [31] studied the availability of a multi-component series system with k-out-of-n:G standby subsystems which are warm standby. Some recent works have also discussed some reliability performance characteristics of standby systems. Some of those characteristics taken into account in the study of Tuncel [30] were the mean residual or mean past lifetimes. In the same work, the mean residual lifetime of a system with one unit and a binary cold standby component was obtained.

Multi-state systems were practically used instead of binary-state systems due to their flexibility in the field of engineering. Many systems can have more than two possible states except just failure and working states. The system can work with a partial performance which is between the failure and the perfect state. Therefore, a multi-state system(MSS) includes m states. State "m" indicates the full performance of the system, besides state "0" denotes the failure of the system. Owing to the system's stochastic behaviour, at the beginning the system starts with its perfect functioning state and when time increases the other states of the system start to be observed as a result of the deterioration. In the literature, many articles deal with the reliability modeling multi-state systems and their analysis. For instance; El-Neweihi, Proschan and Sethuraman [7] worked on the development of the basic theory of multi-state structures. Hudson and Kapur [13] suggested reliability models for multi-state systems and discussed several applications of those models. Two definitions were given for the multi-state systems by Ebrahimi [6] and some features of them were examined within the same work. Boedigheimer and Kapur [2] worked on customer-focused multi-state systems. Brunelle and Kapur [3] suggested a new classification plan for some reliability measures and generalized these measures to multi-state systems. Eryilmaz [8] obtained the results regarding one of the important performance characteristics, the mean residual lifetime, of a MSS. The stochastic behavior of multi-state systems has also worthy of attention in the literature. There are many researches on the stochastic reliability modeling of multi-state systems and its evaluation problem. Iscioglu [14] studied some performance characteristics of two types of multi-state k-out-of-n:G systems under the assumption that the components are independent and identically distributied via using order statistics. Besides, in [15], a similar problem is discussed for the two types of three-state systems under non-identically ditributed components of the system using permanent-based representations for the survival probabilities. The first studies in which the relability measures for a MSS are proposed to be considered at some intermediate state j or above at time t are the studies of Liu and Kapur [25-26]. A detailed literature review of the dynamic reliability of multi-state systems can be found in [24]. Shu, et al. [28] also proposed several new measures for the tool performance. They used non-homogeneous continuous time Markov process(NHCTMP) for the tool degradation. It is stated that in the evaluation of the state probabilities of the multi-state tools, it is important to consider how long the tool has been at its current state. Xue and Yang [32] generalized some parameters in the binary state structures to the multi-state case. Measures for obtaining the performance degradation are developed for the MSS structures. Continuous time Markov and Semi-Markov processes were used in modeling the lifetime distributions of multi-state systems and for those models, performance measures such as $R(t), F(t), \lambda(t)$ were derived in [34]. In many of those works as we have mentioned the degradation of multi-state systems follows a continuous time homogeneous or a nonhomogeneous Markov process assumption. Besides the extensive literature based on multi-state systems, there are not many articles considering multi-state standby systems. The reliability evaluation problem of standby systems with multi-state elements with constant transition rates and absorbing failure states was considered in [17]. A new iterative algorithm based on the element state probabilities were developed. The results were discussed for cold, warm and hot standby systems in the same work. Standby systems with multi-state elements which have constant transition rates were also considered in Levitin et al. [22]. Another iterative algorithm for the reliability of the mentioned systems were proposed in [22]. A modification of a generalized reliability block diagram method was suggested for the reliability analysis and performance measures of multi-state systems with imperfect multi-fault coverage in [21]. Kim et al. [19] considered the reliability analysis of a multi-state parallel system structure with a multi-functional standby component which is redundant and preferred to be used in the systems such as aircrafts. Jia et al. [18] dealt with the reliability evaluation of power systems with multi-state warm standby and multi-state performance sharing mechanism. Multi-state decision diagram was proposed to obtain the reliability of the system. In the study of Eryilmaz [9], a multi-state cold standby system having one active and one standby unit was considered. The activation time for a standby component was determined by state "j" of an active component in the system. When the active component is below the state "j", the standby component starts to work. As the switching mechanism among the working components and the standby units are important in balancing the component's degradation and extending the lifetime of many systems, Zhao et al. [38] proposed a joint optimization model that both captures component switching and mission abort policies and minimizes the long-run expected total economic loss of multistate warm standby systems. In these studies considering multi-state standby systems, many performance characteristics are obtained given the information that the system is at state "j or above" at time t. Different from them, the scientific contribution of this research is the evaluation of the MRL function of a standby system given the information that the system is at state "j" not "j or above" at time t. The theoretical achievements are worth considering and are practically applicable in the field of engineering. In addition, an optimization problem is taken into account to strengthen the applicability of the proposed model in the maintenance and repair plans of the system.

In the present paper, therefore, it is aimed to achieve first the survival functions and then, the MRL functions of a multi-state standby system under the knowledge regarding the state of the system at $\forall t$. This information is noteworthy and can be used by an expert in designing maintenance schedules and in planning the investments of a multi-state standby system. Thus, with regard to the results obtained in this work, we also want to investigate the effect of different degradation rates of the lifetimes spent at each state on the MRL values of the multi-state standby system. Besides, we deal with the optimization problem regarding the average replacement costs of the system in the long run in which simultaneously we search for the optimal replacement times of the system. When the frame of the work is given, firstly in section II, the research methodology used to find out the MRL functions following the conditional survival probabilities for a three-state standby system are provided by introducing the system and its assumptions considered in this work. Also, in the same section the MTTF functions of the system are considered. In Section III, some numerical examples are presented to illustrate the theoretical findings based on several features such as various degradation rates of the three-state component, the standby unit and the changes in time. In Section IV, the problem of determining the optimal replacement time which minimizes the total long-run average cost per unit time for the system is discussed by considering different degradation parameters and replacement costs for the different situations of the system. To make the model proposed in this study be comprehended well, a case study of a smart house that supplies its own energy by its own wind turbine located on its roof is comprised in Section V. Finally in section VI, the study is concluded with some foremost findings obtained and some future research problems are pointed out.

2. Research Methodology

2.1. The System and Its Assumptions

A multi-state standby system is considered. The system has just one component and that component has three states such as; "2" indicates the perfect functioning state, "1" refers to the partially working state and "0" represents the failure state. There is a cold standby unit in the system, when the system fails the standby unit starts to work. This standby unit is considered as binary. Thus, it has just two states such as; "1" shows the working state and "0" indicates the failure state of a standby component. Hence, we call this system a multi-state standby system. Throughout the paper, the following assumptions are considered regarding the system.

* T^1 and T^2 denote the lifetime of the component spent at state "1", and state "2", respectively.

* $T^1 + T^2$ is the lifetime of the component (represented as $T^{\geq 1}$, as well).

* T^2 is the lifetime spent at state "2"(represented as $T^{\geq 2}$, as well).

* The system degrades by time t from its perfect state, which is state "2" , to the lower states.

* The system is at its perfect state when time equals "0".

* A non-repairable system is considered.

* The degradation process of the system follows an HCTMP (homogeneous continuous time Markov process).

* X denotes the lifetime of the cold-standby unit in the system.

In the reliability evaluation of this three-state standby system, we consider especially the mean residual lifetimes of the system except just finding out the survival probabilities of the system for an arbitrary time point t. Especially, during inspections of the system, when someone observes the state of the system, under this knowledge making some predictions about the survival probability or the mean residual lifetime of the system is remarkable and becomes really important for the system experts to make their maintenance plans of the system. Therefore, some residual lifetime representations are given under some conditions for the system as follows;

$$\{T^1 + T^2 + X - t | T^{\geq 2} > t\} \quad (1)$$

(1) represents the residual lifetime of the three-state standby system under the knowledge that the system is at its perfect functioning state at $\forall t$.

$$\{T^1 + T^2 + X - t | T^{\geq 2} \leq t, T^{\geq 1} > t\} \quad (2)$$

(2) indicates the residual lifetime of the three-state standby system under the knowledge that the system is at its partially working at $\forall t$. In the representations (1) and (2), the conditions can also be written as $\{T^{\geq 2} > t, X > t\}$ and $\{T^{\geq 2} \leq t, T^{\geq 1} > t, X > t\}$ since in the case of cold standby redundancy, $P\{T^{\geq 2} > t\} > 0$ and $P\{T^{\geq 2} \leq t, T^{\geq 1} > t\} > 0$ implies respectively $P\{X > t\} = 1$ for $t > 0$.

$$\{T^1 + T^2 + X - t | T^{\geq 1} < t, X > t\} \quad (3)$$

(3) shows the residual lifetime of the three-state standby system under the knowledge that, the three-state system has failed before time t, however, the standby component has been activated and the system is working with its standby unit at time t. Thus, under these three constructions, one can achieve the survival probabilities and the mean residual lifetimes(MRLs) of the three-state standby system.

2.2. Evaluation of the MRL and the Conditional Survival Functions

Let T^1 and T^2 follow a joint cumulative distribution function; $H(t_1, t_2) = P\{T^1 \leq t_1, T^2 \leq t_2\}$ and the marginal distribution functions; $F_1(t_1) = P\{T^1 \leq t_1\}$ $F_2(t_2) = P\{T^2 \leq t_2\}$, respectively, for $t_1, t_2 > 0$. Let X follows a marginal distribution function as; $F_X(x) = P(X \leq x)$. Let $\phi(t)$ denotes the structure function of the system which itself models the functioning of a system for given states of its components. In this study, the system is a one-component system, therefore, $\phi(t)$ denotes the state of the component, as well and takes the values "2,1 and 0". For instance; when the system is at its perfect functioning state at time t, $\phi(t)$ takes the value "2"or when the system is at its partially working state at time t, $\phi(t)$ takes the value "1".

By the following theorems, the conditional survival functions of a three-state standby system are provided. Then, following the theorems, the MRL functions are also achieved.

Theorem 1: Given the information that the three-state standby system is at its perfect functioning state at $\forall t$, the system's conditional survival function is obtained as;

$$P\{T^1 + T^2 + X > s|\phi(t) = 2\} = P\{T^1 + T^2 + X > s|T^{\geq 2} > t\} \quad (4)$$

$$= \frac{1}{\bar{F}_2(t)}\{\bar{F}_2(s) + \int_t^s P\{T^1 + X > s - y|T^2 = y\}dF_2(y)\}, for\ \ s > t (5)$$

Proof: We can express (4) as;

$$P\{T^1 + T^2 + X > s|T^2 > t\} \quad (6)$$

To solve (6) the following equation is established;

$$P\{T^1 + T^2 + X > s|T^2 > t\} = \frac{P\{T^1+T^2+X>s, T^2>t\}}{P\{T^2>t\}} \quad for\ \ s > t \quad (7)$$

By conditioning on T^2 in both numerator and denominator of equation (7), one can obtain;

$$P\{T^1 + T^2 + X > s, T^2 > t\} = \int_{y>t} P\{T^1 + X > s - y|T^2 = y\}dF_2(y)$$

$$= \int_{\substack{y>t, \\ s-y<0}} dF_2(y) + \int_{\substack{y>t, \\ s-y>0}} P\{T^1 + X > s - y|T^2 = y\}dF_2(y)$$

$$= \bar{F}_2(s) + \int_t^s P\{T^1 + X > s - y|T^2 = y\}dF_2(y),$$

and $\bar{F}_2(t)$, respectively. Therefore, by taking into account these two functions, equation (5) is obtained.

Corollary 1: If T^1, T^2 and X are independent, then

$$P\{T^1 + T^2 + X > s|T^2 > t\} = \frac{\bar{F}_2(s)}{\bar{F}_2(t)} + \frac{\int_t^s P\{T^1+X>s-y\}dF_2(y)}{\bar{F}_2(t)} \quad (8)$$

for $s > t$.

Given the information that the three-state standby system is at its perfect functioning state, the MRL of the system is the expected value of the conditional random variable $\{T^1 + T^2 + X - t|T^2 > t\}$ and obtained as;

$$m_1(t) = E(T^1 + T^2 + X - t|T^2 > t) \quad (9)$$
$$= \int_0^\infty P\{T^1 + T^2 + X > t + v|T^2 > t\}dv.$$

Using equation (5),

$$m_1(t) = \int_0^\infty \frac{\bar{F}_2(t+v)}{\bar{F}_2(t)}dv + \frac{1}{\bar{F}_2(t)}\int_0^\infty \int_t^{t+u} P\{Z_1 > t + v - y|T^2 = y\}dF_2(y)dv \quad (10)$$

where $Z_1 = T^1 + X$. Thus, equation (10) actually is used to calculate the MRL of the standby system when someone observes the system at its perfect functioning state at time t.

Corollary 2: If T^1, T^2 and X are independent, then the MRL when the system is observed to be at its perfect functioning state at time t is;

$$m_1(t) = \int_0^\infty \bar{F}_{Z_1}(v)dv + \int_t^\infty \frac{\bar{F}_2(v)}{\bar{F}_2(t)}dv \quad (11)$$

Proof: If T^1, T^2 and X are independent, then the MRL is written as;

$$m_1(t) = E(T^1 + X) + E(T^2 - t|T^2 > t) \quad (12)$$

Let Z_1 denotes $T^1 + X$, then it is obvious that

$$E(Z_1) = \int_0^\infty P(Z_1 > v)dv = \int_0^\infty \bar{F}_{Z_1}(v)dv$$

and

$$E(T^2 - t|T^2 > t) = \int_0^\infty P(T^2 > t + v|T^2 > t)dv$$

$$= \int_0^\infty \frac{P(T^2 > t + v)}{P(T^2 > t)}dv = \int_0^\infty \frac{\bar{F}_2(t + v)}{\bar{F}_2(t)}dv$$

Thus the proof is completed.

Theorem 2: Given the information that the three-state standby system is at its partially working state at $\forall t$, the system's conditional survival function is obtained as;

$$P\{T^1 + T^2 + X > s|\phi(t) = 1\} = P\{T^1 + T^2 + X > s|T^2 \leq t, T^1 + T^2 > t\} \quad (13)$$

$$= \frac{\iint_{\substack{y<t, \\ x>0}} P\{T^1 > s - y - x, T^1 > t - y|T^2 = y, X = x\}dF_{X,T^2}(x,y)\,dx\,dy}{\int_0^t P\{T^1 > t - y|T^2 = y\}d\,F_2(y)} \quad (14)$$

for $s > t$.

Proof: We can express (13) as;

$$P\{T^1 + T^2 + X > s|T^2 \leq t, T^1 + T^2 > t\} =$$

$$\frac{P\{T^1+T^2+X>s, T^2\leq t, T^1+T^2>t\}}{P\{T^2\leq t, T^1+T^2>t\}} \quad (15)$$

and to solve equation (15), it is required to condition on T^2 and X for the numerator, and condition on T^2 for the denominator.

Corollary 3: If T^1, T^2 and X are independent, then

$$P\{T^1+T^2+X>s|\phi(t)=1\} \tag{16}$$

$$=\frac{\int_0^t\int_0^{s-t}\bar{F}_1(s-y-x)f_X(x)\,f_2(y)dx\,dy+\int_0^t\int_{s-t}^{\infty}\bar{F}_1(t-y)f_X(x)f_2(y)dx\,dy}{\int_0^t\bar{F}_1(t-y)f_2(y)\,dy}$$

Proof: The denominator of equation (15) can be written by conditioning on T^2 as follows;

Also, the numerator of equation (15) can be written by conditioning on T^2 and X as follows;

$$\int_{y<t}P\{T^2\le t,T^1+T^2>t|T^2=y\}\,dF_2(y)=\int_0^t\bar{F}_1(t-y)f_2(y)\,dy$$

$$=\int_{y<t}P\{T^1+T^2+X>s,T^2\le t,T^1+T^2>t|T^2=y\}\,dF_2(y)$$

$$=\int_{y<t}P\{T^1+X>s-y,T^1>t-y\}\,dF_2(y)$$

$$=\int_0^t\int_0^{\infty}P\{T^1>s-y-x,T^1>t-y|X=x\}\,dF_X(x)\,dF_2(y)$$

$$=\int_0^t\int_0^{s-t}P\{T^1>s-y-x\}\,dF_X(x)\,dF_2(y)+\int_0^t\int_{s-t}^{\infty}P\{T^1>t-y\}\,dF_X(x)\,dF_2(y)$$

$$=\int_0^t\int_0^{s-t}\bar{F}_1(s-y-x)f_X(x)\,f_2(y)dx\,dy+\int_0^t\int_{s-t}^{\infty}\bar{F}_1(t-y)f_X(x)f_2(y)\,dx\,dy$$

Given the information that the three-state standby system is at its partially working state, the MRL of the system is the expected value of the conditional random variable $\{T^1+T^2+X-t|T^2\le t,T^1+T^2>t\}$ and obtained as;

$$m_2(t)=E(T^1+T^2+X-t|T^2\le t,T^1+T^2>t) \tag{17}$$

$$=\int_0^{\infty}P(T^1+T^2+X>t+v|T^2\le t,T^1+T^2>t)\,dv$$

$$=\int_0^{\infty}\frac{P(T^1+T^2+X>t+v,T^2\le t,T^1+T^2>t)}{P(T^2\le t,T^1+T^2>t)}dv$$

By taking into account equation (17), under the information that a partially working state is observed for the system at $\forall t$, one can obtain the MRL of the three-state standby system as;

$$m_2(t)=\frac{\int_0^{\infty}\iint_{y<t,x>t}P\{T^1>t+v-y-x,T^1>t-y|T^2=y,X=x\}dF_{X,T^2}(x,y)\,dv}{\int_0^tP\{T^1>t-y|T^2=y\}dF_2(y)} \tag{18}$$

Thus, equation (18) is basically used to calculate the MRL of the three-state standby system when someone observes the system at its partially working state at $\forall t$.

Corollary 4: If T^1, T^2 and X are independent, then the MRL when the system is observed to be at its partial woking state at time t is;

$$m_2(t)=\frac{\int_0^{\infty}\left(\int_0^t\int_0^{s-t}\bar{F}_1(t+v-y-x)f_X(x)f_2(y)\,dxdy+\int_0^t\int_{s-t}^{\infty}\bar{F}_1(t-y)f_X\,f_2(y)\,dxdy\right)dv}{\int_0^t\bar{F}_1(t-y)f_2(y)\,dy} \tag{19}$$

Theorem 3: Given the information that the three-state system has failed and been backed up with a standby component at $\forall t$, the system's conditional survival function is obtained as;

$$P\{T^1+T^2+X>s|T^{\ge1}\le t,X>t\} \tag{20}$$

$$=\frac{\int_0^tP\{X>s-z|Z=z\}\,dF_z(z)}{\int_0^tP\{X>t|Z=z\}\,dF_z\,(z)},\qquad s\ge 2t$$

$$=\frac{\int_0^{s-t}P\{X>s-z|Z=z\}\,dF_z(z)+\int_{s-t}^tP\{X>t|Z=z\}\,dF_z\,(z)}{\int_0^tP\{X>t|Z=z\}\,dF_z\,(z)},t<s\le 2t$$

where $T^1 + T^2 = Z$.

Proof: We can express (20) as;

$$P\{T^1 + T^2 + X > s | T^1 + T^2 \leq t, X > t\} \quad (21)$$

To solve equation (21) the following equation is established;

$$P\{T^1 + T^2 + X > s | T^1 + T^2 \leq t,\ X > t\} = \frac{P\{T^1+T^2+X>s, T^1+T^2<t, X>t\}}{P\{T^1+T^2<t, X>t\}}, \quad (22)$$

and then to solve equation (22), respectively, the following equations needs to be dealt with;

$$P\{T^1 + T^2 \leq t, X > t\} = P\{Z \leq t, X > t\} \quad (23)$$

$$P\{T^1 + T^2 + X > s, T^1 + T^2 \leq t, X > t\} = P\{Z + X > s, Z \leq t, X > t\} \quad (24)$$

where $T^1 + T^2 = Z$. Thus, equations (23) and (24) are solved respectively;

$$P\{Z \leq t, X > t\} = \int_{z \leq t} P\{X > t | Z = z\}\, dF_z(z) \quad (25)$$

$$P\{Z + X > s, Z \leq t, X > t\} = \int_{z \leq t} P\{X > s - z, X > t | Z = z\}\, dF_z(z)$$

$$= \int_{\substack{z \leq t \\ s-z>t}} P\{X > s - z | Z = z\}\, dF_z(z) + \int_{\substack{z \leq t \\ s-z<t}} P\{X > t | Z = z\}\, dF_z(z)$$

$$= \int_0^{\min(t, s-t)} P\{X > s - z | Z = z\}\, dF_z(z) + \int_{s-t}^{t} P\{X > t | Z = z\}\, dF_z(z) \quad (26)$$

Therefore, by considering $s - t < t$ or $s - t \geq t$ in equation (26) and the other result obtained in equation (25), the required result is provided.

Corollary 5: If T^1, T^2 and X are independent, then the survival probability equation can be written as;

$$P\{T^1 + T^2 + X > s | T^1 + T^2 < t, X > t\} \quad (27)$$

$$= \frac{\int_0^t \bar{F}_X(s - z) f_Z(z)\, dz}{\int_0^t \bar{F}_X(t) f_Z(z)\, dz}, \qquad s \geq 2t$$

$$= \frac{\int_0^{s-t} \bar{F}_X(s - z) f_Z(z)\, dz + \int_{s-t}^t \bar{F}_X(t) f_Z(z)\, dz}{\int_0^t \bar{F}_X(t) f_Z(z)\, dz}, \qquad t \leq s \leq 2t$$

The MRL of the three-state standby system when the system has known to be failed and the standby component has been activated at $\forall t$ is provided by taking into account the expectation of the random variable $\{T^1 + T^2 + X - t | T^1 + T^2 \leq t, X > t\}$ as follows;

$$m(t) = E(T^1 + T^2 + X - t | T^1 + T^2 \leq t, X > t) \quad (28)$$

$$= \int_0^\infty P(T^1 + T^2 + X > t + v | T^1 + T^2 \leq t, X > t)\, dv$$

Using equation (28),

$$m(t) = \frac{\int_0^\infty \int_0^t P\{X > t + v - z | Z = z\}\, dF_Z(z)\, dv}{\int_0^t P\{X > t | Z = z\}\, dF_Z(z)}, s \geq 2t \quad (29)$$

$$m(t) = \frac{\int_0^\infty \left[\int_0^{s-t} P\{X > t + v - z | Z = z\}\, dF_Z(z) + \int_{s-t}^t P\{X > t | Z = z\}\, dF_Z(z)\right] dv}{\int_0^t P\{X > t | Z = z\}\, dF_Z(z)},$$

$$t \leq s < 2t \quad (30)$$

Thus equations (29) and (30) are substantially used to achieve the system's MRL values when someone observes the three-state system has failed and the standby component has been activated at $\forall t$.

Corollary 6: If T^1, T^2 and X are independent, then the mean residual lifetime when the system has known to be failed before time t and the standby component has been activated is;

$$m(t) = \frac{\int_0^\infty \int_0^t \bar{F}_X(t + v - z) f_Z(z)\, dz\, dv}{\int_0^t \bar{F}_X(t) f_Z(z)\, dz}, \quad s \geq 2t \quad (31)$$

$$m(t) = \frac{\int_0^\infty \left[\int_0^t \bar{F}_X(t + v - z) f_Z(z)\, dz + \int_{s-t}^t \bar{F}_X(t) f_Z(z)\, dz\right] dv}{\int_0^t \bar{F}_X(t) f_Z(z)\, dz}, t \leq s < 2t \quad (32)$$

2.3. Assessment of the MTTF of a Three-state Standby System

Mean time to failure is one of the important characteristics of the system in the dynamic reliability analysis. MTTF denotes actually the expected time to failure for a binary system. In a binary context, for a system with lifetime T, the MTTF is $E(T) = P\{T > t\}$. However, in a MSS, because the lifetime of the system is defined in a specific state j or above, the MTTF is obtained as $E(T^{\geq j}) = P\{T^{\geq j} > t\}$. Given the information that the three-state standby system is at its perfect or partial working states in the beginning of the degradation process, where $t = 0$, the MRLs of the system equal to the MTTF of the system, as well. As a result of Theorem 1, the results obtained via equation (9) when $t = 0$, actually is the MTTF value of the system. Therefore, the MTTF of the three-state standby system where it is known to be at perfect state at the beginning(where $t = 0$), can be obtained by solving $\int_0^\infty P\{T^1 + T^2 + X > x\} \partial x$ as follows;

$$= \int_0^\infty \int_y P\{T^1 + X > x - y | T^2 = y\} dF_2(y)\, dx$$

$$= \int_0^\infty \int_{y>0,x-y<0} dF_2(y)dx + \int_0^\infty \int_{y>0,x-y>0} P\{T^1 + X > x - y|T^2 = y\}dF_2(y)dx \quad (33)$$

Equation (33) can be given as;

$$= \int_0^\infty \bar{F}_2(x)dx + \int_0^\infty \int_0^x \bar{F}_Z(x-y)f_2(y)dydx \quad (34)$$

where $Z = T^1 + X$.

Further, as a result of Theorem 2, the results obtained via equation (17) when $t = 0$, directly gives the MTTF value of the system. Thus, the MTTF of the three-state standby system where it is known to be at partial working state at the beginning(where $t = 0$), can be obtained by solving $\int_0^\infty P\{T^1 + X > s\}\partial s$ directly as follows;

$$= \int_0^\infty \int_x P\{T^1 + X > s|X = x\}\, dF_X(x)ds$$

$$= \int_0^\infty \int_{s-x<0} dF_X(x)ds + \int_0^\infty \int_{s-x>0} P\{T^1 > s - x\}f_X(x)dxds$$

$$= \int_0^\infty \int_s^\infty f_X(x)\, dxds + \int_0^\infty \int_0^s \bar{F}_{T^1}(s-x)f_X(x)dxds \quad (35)$$

Also by the use of Theorem 3, one can again calculate the MTTF of the system when $t = 0$ in equation (28). Then, the MTTF of the system is obtained by;

$$\int_0^\infty P\{X > x\}dx = \int_0^\infty \bar{F}_X(x)dx \quad (36)$$

In this part of the study, some numerical examples are provided to point out how the theorems work under different distribution assumptions. The results are discussed under independency assumption of the random lifetimes. First, let the lifetime distribution of T^1 and T^2 follows exponential distribution with parameters λ_1 and λ_2, respectively (T^1~Exponential(λ_1),T^2 ~Exponential(λ_2)). Also, the lifetime distribution of the standby component follows exponential distribution with the parameter λ. In this case, because of independency, the distribution of $Z = T^1 + X$ follows Gamma Distribution with the parameters λ_1 and λ. In case of independency, by using the equations given in Corollary 2 and 4, the MRL results of the three-state standby system under the given information of the system regarding states "2" and at state "1" are obtained and they are presented in the following tables I and II, respectively.

According to Table I, in case the three-state standby system is observed to be at its perfect functioning state at time t , there is no change in the MRL value of the system with an increase in time. This is an expected result via using exponential lifetime distributions for each state and due to the independency among the states. When the results based on the parameters are examined, when λ_1 increases whereas λ_2 and λ are constant, the MRL value of the three-state standby system decreases. The similar result is obtained for MRL when λ_1 and λ are constant and λ_2 increases. Both the changes in λ_1 and λ_2 has an effect on the related MRL. The influences of the degradation rate of the standby component on the MRL values of the system are also examined. When we consider both cases in which λ_1 and λ_2 are taken as; one is constant and the other one is increasing, respectively, while the values of λ are constant, the MRL values do not change. Let us explain it with a specific example; when λ_2 =0.5 , $\lambda_1 = 0.8$ and λ =0.7, in case the system is observed to be at its perfect state at time "1.0", the MRL of the system is 4.679. Besides, the result is the same when λ_2=0.8 , $\lambda_1 = 0.5$ and $\lambda = 0.7$. However, for a similar situation (λ_2 =0.5(1.2) and, $\lambda_1 = 1.2(0.5)$), if only the value of the parameter λ increases to 0.8, the related MRL of the system decreases to "4.083". Thus, we can easily say that when λ increases, it has an decreasing effect on the MRL of the system. In Table 1, the values obtained when t=0 are also the MTTF values of the system. They can also be achieved by the use of equation (34), as well.

In accordance with the results in Table 2, it can be pointed out the MRL value of the three-state standby system given the information that it is at its partially working state at t, does not change in accordance with the time change. The MRL of the system is observed to decrease when λ_2 and λ are constant and λ_1 increases. However, when λ_1 and λ are constant and λ_2 increases, the MRL do not change. Thus, it is worth prominent that the MRL obtained when the system is observed to be at its partial working state at time t is not affected by the parameter λ_2. When an increasing effect of the failure rate parameter λ of the standby component is sought, it is obvious to say that it has a decreasing effect on the related MRL of the system. In Table 2, the values obtained when $t = 0$ are also the MTTF values of the system. They can be obtained via equation (35), as well.

Also the MRL results of the system when the system has known to be failed and the standby component has been activated are obtained by using equations (31) and (32) in case of independency. In this case, because of independency, the distribution of $Z = T^1 + T^2$ follows Gamma Distribution with the parameters λ_1 and λ_2 and the MRL results are obtained and presented in the following Table 3.

Table 1. The MRL values of a three-state standby system given that the system is at its perfect functioning state

λ_2	λ_1	λ	$t=0$	$t=1.0$	$t=2.0$
			$m_1(t)$	$m_1(t)$	$m_1(t)$
0.5	0.8	0.7(0.7)	4.68(4.679)	4.679(4.679)	4.679(4.679)
0.5	1.2	0.7(0.8)	4.263(4.083)	4.262(4.083)	4.262(4.083)
0.5	2.0	0.7(0.9)	3.93(3.611)	3.929(3.611)	3.929(3.611)
0.5	3.5	0.7(1.2)	3.715(3.119)	3.714(3.119)	3.714(3.119)
0.5	5.5	0.7(1.5)	3.611(2.848)	3.61(2.848)	3.61(2.848)
0.8	0.5	0.7(0.7)	4.68(4.679)	4.679(4.679)	4.679(4.679)
1.2	0.5	0.7(0.8)	4.263(4.083)	4.262(4.083)	4.262(4.083)
2.0	0.5	0.7(0.9)	3.93(3.611)	3.929(3.611)	3.929(3.611)
3.5	0.5	0.7(1.2)	3.715(3.119)	3.714(3.119)	3.714(3.119)
5.5	0.5	0.7(1.5)	3.611(2.848)	3.61(2.848)	3.61(2.848)

Table 2. The MRL values of a three-state standby system given that the system is at its partially working state.

λ_2	λ_1	λ	$t=0$	$t=1.0$	$t=2.0$
			$m_2(t)$	$m_2(t)$	$m_2(t)$
0.5	0.8	0.7(0.7)	2.679(2.679)	2.679(2.679)	2.679(2.679)
0.5	1.2	0.7(0.8)	2.262(2.083)	2.262(2.083)	2.262(2.083)
0.5	2.0	0.7(0.9)	1.929(1.611)	1.929(1.611)	1.929(1.611)
0.5	3.5	0.7(1.2)	1.714(1.119)	1.714(1.119)	1.714(1.119)
0.5	5.5	0.7(1.5)	1.61(1.015)	1.61(1.015)	1.61(1.015)
0.8	0.5	0.7(0.7)	3.429(3.429)	3.429(3.429)	3.429(3.429)
1.2	0.5	0.7(0.8)	3.429(3.25)	3.429(3.25)	3.429(3.25)
2.0	0.5	0.7(0.9)	3.429(3.111)	3.429(3.111)	3.429(3.111)
3.5	0.5	0.7(1.2)	3.429(2.833)	3.429(2.833)	3.429(2.833)
5.5	0.5	0.7(1.5)	3.429(2.667)	3.429(2.667)	3.429(2.667)

Table 3. The MRL values of a three-state standby system given that the system has failed and the standby component has been activated.

λ_2	λ_1	λ	$t=0$	$t=1.0$	$t=2.0$
			$m_2(t)$	$m_2(t)$	$m_2(t)$
0.5	0.8	0.7(0.7)	1.429(1.43)	2.05(2.058)	2.609(2.609)
0.5	1.2	0.7(0.8)	1.429(1.252)	2.047(1.868)	2.567(2.389)
0.5	2.0	0.7(0.9)	1.429(1.113)	2.027(1.71)	2.502(2.184)
0.5	3.5	0.7(1.2)	1.429(0.834)	1.998(1.403)	2.426(1.83)
0.5	5.5	0.7(1.5)	1.429(0.667)	1.971(1.209)	2.375(1.613)
0.8	0.5	0.7(0.7)	1.429(1.43)	2.05(2.058)	2.609(2.609)
1.2	0.5	0.7(0.8)	1.429(1.252)	2.047(1.868)	2.567(2.389)
2.0	0.5	0.7(0.9)	1.429(1.113)	2.027(1.71)	2.502(2.184)
3.5	0.5	0.7(1.2)	1.429(0.834)	1.998(1.403)	2.426(1.83)
5.5	0.5	0.7(1.5)	1.429(0.667)	1.971(1.209)	2.375(1.613)

When the results are considered, the MRL values are observed to increase by the time increase. When the effect of the parameters are examined, some interesting results are observed. For instance, when λ_2 and λ are constant and λ_1 increases, MRL values obtained when $t = 0$ do not change except the other time points. The interpretation is the same when λ_1 and λ are constant and λ_2 increases. For the same values of $\lambda_1 + \lambda_2$ only the decrease in the value of the parameter λ has a decreasing effect on the MRL of the system for different values of t except "0". In Table III, the values obtained when $t = 0$ are also the MTTF values of the system. Equation (36) can also be used to calculate the related time to failure values, as well.

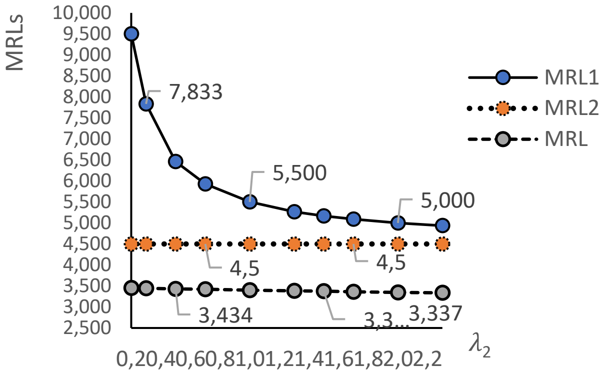

Figure 1. Plots of $m_1(1.5), m_2(1.5)$ and $m(1.5)$ based on the change in the parameter λ_2 where $\lambda_1 = 0.5$, $\lambda = 0.4$.

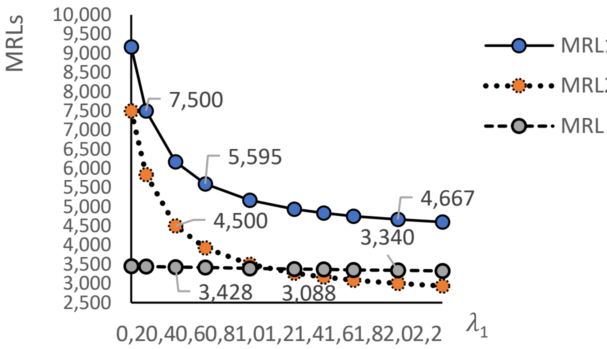

Figure 2. Plots of $m_1(1.5), m_2(1.5)$ and $m(1.5)$ based on the change in the parameter λ_1 where $\lambda_2 = 0.6$, $\lambda = 0.4$.

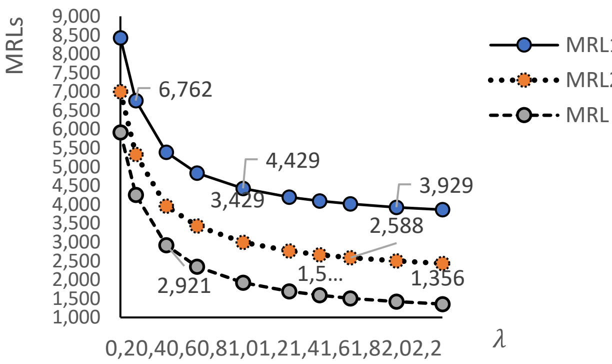

Figure 3. Plots of $m_1(1.5), m_2(1.5)$ and $m(1.5)$ based on the change in the parameter λ where $\lambda_2 = 0.7$, $\lambda_1 = 0.5$.

In Figures I-III, the behaviors of the mean residual lifetime functions can be seen more explicitly. In the figures, MRL1 denotes $m_1(1.5)$, MRL2 and MRL indicates $m_2(1.5)$ and $m(1.5)$, respectively. In Figure I, for the given values of λ_1 and λ, the increase in the parameter λ_2 has no effect on $m_2(1.5)$ whereas it has a decreasing effect on both $m_1(1.5)$ and $m(1.5)$. In Figure II, for the given values of λ_2 and λ, the increase in the parameter λ_1 has a decreasing effect on all the MRLs at time point 1.5. Besides, the decrease in $m(1.5)$ is slightly smaller than the decrease in $m_2(1.5)$. Eventually, the increase in the parameter λ, where the other parameters are taken as constants, has again a decreasing effect on all the MRLs of the system.

3. Optimal Replacement Time

For an expert to decide the optimal replacement time of the system which minimizes the total long-run average cost per unit time is quite important in scheduling the maintenance activities of the system. Therefore, in this section, we aim to determine the optimal replacement time of a multi-state standby system which supplies at the same time the minimum average cost per unit time of the system. According to the classical age replacement policy considered in Ahmad&Kamaruddin [1], the system is replaced upon its failure or upon its reaching age t. Then, the mean cost rate per unit time can be calculated via,

$$C(t) = \frac{c_I P(T>t) + c_{II} P(T \le t)}{E(\min(T,t))} \tag{37}$$

where T represents the lifetime of the system , c_I and c_{II} denotes the costs of replacing a non-failed and a failed system ($c_I < c_{II}$), respectively [10]. The value of the replacement age t^* which minimizes the average cost is determined by the equation (37).

Equation (37) just considers that the system is working or failed at an arbitrary time t. However, for an arbitrary time point t when we consider the system is working, there can be two situations. The first one is a three-state system can be in a working state at time t. The second one is; the three-state system can be failed and the two-state standby unit can be activated at time t, thus the system is still in a working state. Therefore, for both of these cases the three-state standby system is working. Thus, the replacement costs for the cases where the standby unit is activated and is not activated differes. Instead of equation (37), the following mean cost per unit time is proposed for the model we consider,

$$C(t) = \frac{c_1 P(T^{\geq 1}+X>t, T^{\geq 1}>t)+c_2 P(T^{\geq 1}+X>t, T^{\geq 1}\leq t)+c_3 P(T^{\geq 1}+X\leq t)}{E(min(T^{\geq 1}+X,t))} \quad (38)$$

where $T^{\geq 1}$ and X represent the lifetime of the three-state system and the standby component, respectively, c_1 denotes the cost of replacing a non-failed three-state system, c_2 indicates the cost of replacing the failed three-state system whereas the standby component has been activated , and finally, c_3 denotes the cost of replacing the failed three-state standby system. When the replacement costs are considered, the cost of replacement for the failed three-state standby system, c_3, is greater than the costs of replacement for the non-failed three-state standby system, c_1 and c_2 ($c_3 > c_2$ and $c_3 > c_1$). Also, the cost of replacement when the standby component is activated is always greater than the cost when the standby component is not activated ($c_1 < c_2$). Therefore, the following relation among the replacement costs are considered; $c_1 < c_2 < c_3$ within the study.

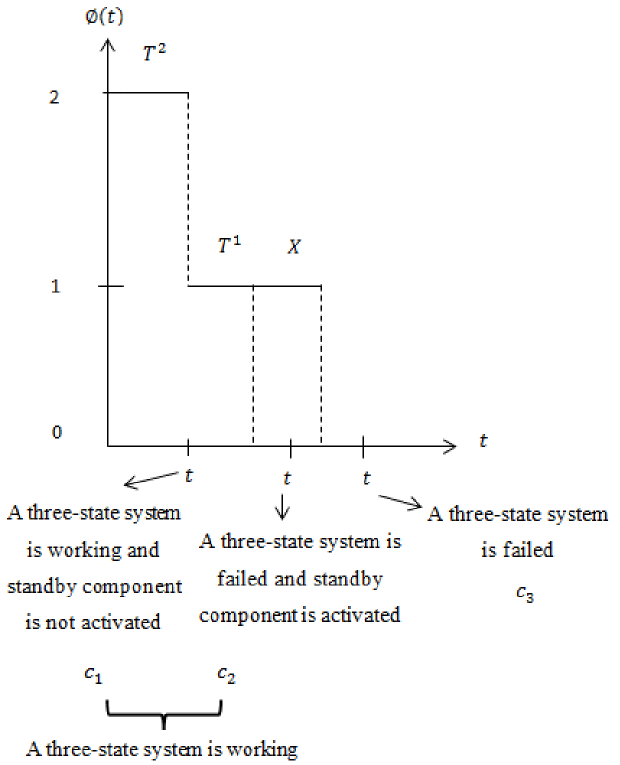

Figure 4. Athree-state standby system's degradation process.

By taking into account the mentioned replacement costs, which are also shown in the Figure 4, the value of the replacement age t^* which minimizes the average cost is determined by solving the equation (38). To solve equation (38), one needs to deal with;

$$P\{T^1+T^2+X>t, T^1+T^2>t\} = \int_0^\infty \bar{F}_Z(t) f_X(x)dx \quad (39)$$

$$P\{T^1+T^2+X>t, T^1+T^2\leq t\} = \int_{z\leq t} \bar{F}_X(t-z) f_Z(z)dz \quad (40)$$

where $Z = T^1 + T^2$ and $Z\sim F_Z(z)$, $X\sim F_X(x)$,

Als$T^2\sim F_{T^2}(t)$, $T^1\sim F_{T^1}(t)$. o, it is considered as

; $T^1\sim Exponential(\lambda_1)$, $T^2\sim Exponential(\lambda_2)$, $X\sim Exponential(\lambda)$

and $Z\sim Gamma(\lambda_1, \lambda_2)$.

$$P\{T^1+T^2+X\leq t\} = 1-(\bar{F}_{T^2}(t)+\int_0^t \bar{F}_Z(t-y) f_{T^2}(y)dy) \quad (41)$$

$$E(\min(T^1+T^2+X,t)) = \int_0^t(\bar{F}_{T^2}(x)+\int_0^x \bar{F}_Z(x-y) f_{T^2}(y)dy)dx \quad (42)$$

where $Z = T^1 + X$ and and $Z\sim F_Z(z)$, $X\sim F_X(x)$, $T^2\sim F_{T^2}(t)$, $T^1\sim F_{T^1}(t)$. T^1 and T^2 are assumed to distribute with exponential distributions with λ_1 and λ_2 parameters, respectively, whereas $Z\sim Gamma(\lambda_1, \lambda)$.

Table 4. Minimum Average Cost per Unit Time and Optimal Replacement Time of the System.

λ_1	λ_2	λ	c_1	c_2	c_3	$C(t^*)$	t^*
0.8	0.5	0.7	3	4	5	1.069	16
0.8	0.5	0.7	3	4	10	1.994	4
0.8	0.5	0.7	3	10	5	1.069	19
0.8	0.5	0.7	8	4	5	1.069	18
1.2	0.5	0.7	3	4	5	1.173	17
0.8	0.9	0.7	3	4	10	2.464	3
0.8	0.5	1.5	3	4	10	2.412	3

When we consider Table IV, for all the given parameters, the increase in the value of t , the optimal replacement time, is observed with an increase in c_3 while c_1 and c_2 are constant. The minimum average cost per unit time increases with an increase in c_3. The increases in both c_1 and c_2 do not have any effect on the minimum average cost whereas have an increasing effect on the optimal replacement time. When the results are examined considering the changes in the parameters of the lifetime distributions, when λ_2 and λ are constant and λ_1 increases, the average cost per unit time increases. Also, we obtain a slight increase in the optimal replacement time of the system. Furthermore, when λ_1 and λ are constant and λ_2 increases, the system's average replacement cost increases. Similarly the optimal replacement time decreases slightly. The same interpretation can be done when λ_2 and λ_1 are taken as constant values and λ increases.

4. A Case Study

To give a more specific example relates with the model which is proposed within this study, we consider the problem of an energy supply issue of a smart house. For instance, let us consider a smart house providing its own energy by a wind turbine installed in its own roof. Kinetic energy is converted to electrical energy by the wind turbines. Due to the wind speed, the power is generated by the turbine which is working properly. Therefore, an energy is produced by the turbine based on the speed of the wind. Both the wind speed and the power are random variables. The maximum power generated by the turbine at a specific wind speed, is called the nominal wind speed for the turbine. For example, if the turbine generates at the maximum level, 1.5 megawatts energy, at a wind speed of 12.5 meter/second, then the nominal wind speed for the turbine is 12.5 meter/second. The studies considering the reliability analysis of wind turbines generally take into account the turbine as a binary system structure. However, there are some recent studies regarding wind turbines as multi-state structures. One of the studies in which a wind turbine is taken into consideration as a three-state system structure is the study of Eryilmaz [11]. In that study, Eryilmaz defines three states for a single wind turbine such as; State "2" indicates that the wind turbine generates power at nominal(rated) power, state "1" denotes that the wind turbine generates power at a rate which depends on the wind speed and as a last, state "0" is the failure state and it means the wind turbine generates no power.

A wind turbine supported the smart home system and the direct current which the turbine generates charges the storage battery. The direct current is then converted to alternative current by a rectifier, so as to provide the energy requirement of the smart house. At the time when the turbine stops working properly or failes, a generator system which is a standby unit becomes involved in the system.This generator system only activates when the turbine is failed. It has just two states such as; state 1 indicates "working" and state 0 implies "failure" states. When this generator starts working, it directly generates alternative current.

Therefore, a three-state turbine being backed up with a two-state generator can be a good representative of the model proposed within this study. Also, owing to the fact that a hybrid energy models given in Figure 5, which includes solar panels along with turbines are attracting attention recently, a specific hybrid energy modeling can also be suggested for the proposed method as one further research problem arised from this study.

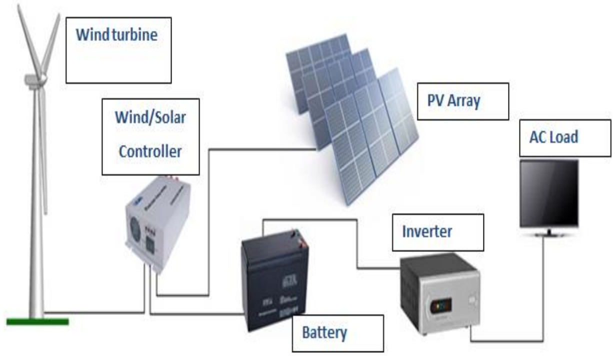

Figure 5. A hybrid energy model working with a turbine and a solar panel.

Let us consider tha data set related to the one of 177 USW 56-100 type wind turbines included in the Sakskaya wind plant that is found in the Crime peninsula given in the study of Zaliskyi et al [36]. The data set includes 36 observed failures (i=1,2,...,36) of the wind turbine #112 during operation. Although the data set relates with the total times between failures, meaining a binary state turbine system has been considered within that work, we just use the mentioned data set to estimate the lifetime distributions for both states of the turbine considered in this work. For this reason, we assume the failure times in Table 5 are the lifetimes of the turbine when the wind turbine generates power at nominal(rated) power. Also, one more assumption is that the lifetimes obtained when the wind turbine generates power at nominal rate and at a rate which depends on the wind speed have identical distributions.

According to the goodness of fit tests, Anderson Darling test statistic shows exponential distribution fit to the data set with the p value 0.483. The estimated degradation rate for both states of the turbine is found to be as; $3.302 \times 10^{-4}(\lambda_1 = \lambda_2$=0.0003302). The MTTF for each state of the system is 3028.3 hours. We assume the generator as the standby unit in the system also has an exponential distribution with a failure rate of $\lambda = 0.015$ hours. Therefore, its MTTF is 66.67 hours. Moreover, the MTTF for this three-state standby system is 6.124×10^3. Then, for the given estimated and the assumed parameters of the lifetimes, the related MRLs are calculated as; $m_1(1500) = 6.124 \times 10^3$, $m_2(1500) = 3.095 \times 10^3$ and $m(1500) = 1.024 \times 10^3$ hours of use. Also for another time point, 5000 hours, the estimated MRLs are; $m_1(5000) = 6.124 \times 10^3$, $m_2(5000) = 3.095 \times 10^3$ and $m(5000) = 2.9 \times 10^3$. When the three-state standby turbine system is observed at its perfect functioning state after 1500 hours of use, the estimated MRL will be 6124 hours. When it is observed at its partially working state after 1500 hours of use, it has an estimated MRL of 3095 hours. After 5000 hours of use those estimated MRLs given the system is at its perfect and partial working states will be the same with the case of 1500 hours of use. However, when the turbine has failed before 1500 hours and been backed up with a generator then the estimated MRL of the system will be 1024 hours. If the turbine has been failed before 5000 hours and the generator has been activated, then the MRL will be estimated as 2900 hours.

Table 5. Lifetime data set for a turbine when it generates power at nominal rate.

Failure i	Total time x_i(hour)	Failure i	Total time x_i(hour)	Failure i	Total time x_i(hour)
1	2445	13	1885	25	4737
2	330	14	3473	26	397
3	73	15	3032	27	1012
4	992	16	10825	28	1257
5	3736	17	3321	29	1788
6	96	18	5007	30	1884
7	1163	19	11331	31	11
8	3750	20	14493	32	1875
9	589	21	643	33	1739
10	44	22	5566	34	352
11	5986	23	1734	35	1885
12	6223	24	1872	36	3473

5. Results and Discussion

In this research, by the information given regarding the system is at a specific state j at time t, the multi-state standby system's survival and the mean residual lifetime functions are evaluated. A three-state single unit system is considered. The system's states "0, 1 and 2" denote failure, partial and perfect working states, respectively. The considered MSS is a cold standby when the system fails it is directly backed with a binary state unit with the failure and functioning states. In the study, the survival functions of the related system are achieved first and then, the mean residual lifetime functions of the system are obtained, under the cases that the system is known to be at its perfect and the partially working states at $\forall t$. Also, when the system fails and backed up with a cold standby component at time t, the survival and mean residual lifetime functions are obtained, as well. The results obtained are examined under exponential lifetime distributions of the system.

To exhibit the effect of the standby component of the system we compared the findings of this study with the results obtained in one recent study, the study of Iscioglu[16]. In the mentioned study, the three-state system's MRL values are achieved under the information that the system is at a specific state at time t. The MRL values of both a three-state and a three-state standby system in case the systems are known to be at their perfect and partial states at $\forall t$ are shown in Table 6. According to the results, the MRLs obtained for a three-state standby system are higher than the MRLs attained for a three-state system when different state knowledges are given regarding the system. Based on the time increase, both MRLs of the systems do not change due to the effect of the exponential distributed lifetimes spent by the system at each state. When considering the lifetime parameters, such that λ_2 is constant and λ_1 is increasing, the MRL of a three-state system decrease. Similarly, when λ_2 and λ parameters are constant and λ_1 increases, the MRL of a three-state standby system also decrease. Furthermore, when λ_1 is constant and λ_2 increases, the MRL values of the three-state system in case it is at its perfect functioning state decrease whereas, MRL values of the three-state system when it is at its partially working state do not change. Similarly, MRL values of the three-state standby system under the knowledge that it is at its perfect state at time t decrease when λ_1 and λ parameters are constant and λ_2 increases. However, with the given information regarding the system is at its partial state at time t, the MRL values of the three-state standby system do not change when λ_1 and λ parameters are constant and λ_2 increases.

When the optimization problem results are discussed, one can observe that the increase in the degradation rate of the standby unit requires early replacement time of the system and the increase in the cost of replacement of the standby unit results inthe higher average cost of replacement with a lower optimal replacement time of the system.

Table 6. The MRL comparison of a three-state system and a three-state cold standby system.

			$t=0$				$t=1$			
λ_2	λ_1	λ	$m_1^I(t)$	$m_1^{II}(t)$	$m_2^I(t)$	$m_2^{II}(t)$	$m_1^I(t)$	$m_1^{II}(t)$	$m_2^I(t)$	$m_2^{II}(t)$
0.5	0.8	-	3.25	-	1.25	-	3.25	-	1.25	-
0.5	0.8	0.8	-	4.5	-	2.5	-	4.5	-	2.5
0.5	1.2	-	2.83	-	0.833	-	2.833	-	0.833	-
0.5	1.2	0.8	-	4.083	-	2.083	-	4.083	-	2.083
0.5	2.0	-	2.5	-	0.5	-	2.5	-	0.5	-
0.5	2.0	0.8	-	3.75	-	1.75	-	3.75	-	1.75
0.8	0.5	-	3.25	-	2.0	-	3.25	-	2.0	-
0.8	0.5	0.8	-	4.5	-	4.0	-	5.25	-	4
1.2	0.5	-	2.83	-	2.0	-	2.833	-	2.0	-
1.2	0.5	0.8	-	4.083	-	3.25	-	4.083	-	3.25
2.0	0.5	-	2.5	-	2.0	-	2.5	-	2.0	-
2.0	0.5	0.8	-	3.75	-	3.25	-	3.75	-	3.25
$m_1^I(t)$; MRL of three-state system in case the system is at its perfect state at time point t.										
$m_2^I(t)$; MRL of three-state system in case the system is at its partial working state at time point t.										
$m_1^{II}(t)$; MRL of three-state standby system in case the system is at its perfect state at time point t.										
$m_2^{II}(t)$; MRL of three-state standby system in case the system is at its partial working state at time point t.										

6. Conclusions

For a three-state standby system, the MRL of the system and the optimal replacement times that minimize the mean replacement costs of the system under the knowledges regarding the states of the system for arbitrary time points are highly important in the maintenance and repair plans. In this sense, this study contributes some important findings to the literature and as well to the industrial applications considering standby redundancy. The results obtained are supposed to be used in the reliability improvement studies of multi-state standby systems.

Some additional contributions can also arise. We used a binary standby unit in this study for the ease of theoretical achievements. However, the standby unit can also be multi-state. For instance, let's consider the standby unit has also three-states and the related lifetimes at each state are represented as; X^2 is the lifetime of the standby unit spent at state "2" and X^1 is the lifetime of the standby unit spent at state "1", then in order to find out the survival probability functions, one needs to deal with the following probabilities;

$$P(T^{\geq 1}+X^{\geq 1}>s|T^{\geq 2}>t) \tag{43}$$

$$P(T^{\geq 1}+X^{\geq 1}>s|T^{\geq 2}\leq t, T^{\geq 1}>t) \tag{44}$$

$$P(T^{\geq 1}+X^{\geq 1}>s|T^{\geq 1}\leq t, X^{\geq 2}>t) \tag{45}$$

$$P(T^{\geq 1}+X^{\geq 1}>s|T^{\geq 1}\leq t, X^{\geq 2}\leq t, X^{\geq 1}>t) \tag{46}$$

where $P\{T^{\geq 2}>t\}>0$ and $P\{T^{\geq 2}\leq t, T^{\geq 1}>t\}>0$ implies respectively $P\{X^{\geq 1}>t\}=1$ for $t>0$ due to cold standby redundancy assumption. When the component fails the standby component starts to function with its perfect functioning state and by the increase in time it will degradate to a lower state.

Moreover this problem, further studies can also be conducted on the standby redundancy of a MSS based on hot or warm standby components. Also, except considering exponentially distributed lifetimes of a system, Weibull distribution can be one other distribution assumption for the system lifetimes if a NHCTMP assumption is taken into account for the degradation process of the system. However, in that case, in the theoretical achievements of the theorems, although there is an independency assumption among the lifetimes spent at each state and also the lifetime of the standby unit, one problem that can arise is to find the distribution of the sum of the two random lifetimes. Maybe, one other distribution which is most widely used, the normal distribution, can also be used under NHCTMP assumption in the application of the proposed theorems. For instance; if $T^1\sim N(\mu_{T^1},\sigma_{T^1})$, $T^2\sim N(\mu_{T^2},\sigma_{T^2})$ and $X\sim N(\mu_X,\sigma_X)$ then, because the random lifetimes are independent, one can achieve $T^1 + X \sim N(\mu_{T^1} + \mu_X, \sqrt{\sigma_{T^1}^2 + \sigma_X^2})$. Thus, this convolution of the two normally distributed random variables can easily be used in the theoretical evaluations obtained in the study and the interpretations based on normal distributions are worth considering. Besides, the lifetimes at each state are assumed to be independent random variables in this study. However, those lifetimes can be handled as dependent. Therefore, the dependency parameter effect on the survival probabilities and the MRL of the system can also be examined under the case of dependency.

Acronyms and Abbreviations

MSS	multi-state system
MRL	mean residual lifetime
MTTF	mean time to failure
HCTMP	homogeneous continuous time Markov process
NHCTMP	non-homogeneous continuous time Markov process

Notations

$P(.)$	Probability of a random variable
$F(.)$	Cumulative distribution function of a random variable
$\bar{F}(.)$	Survival function of a random variable
$f(.)$	Probability density function of a random variable
$H(.,.)$	Joint cumulative distribution function of random variables
$E(.)$	Expected value of a random variable
$\phi(t)$	Structure function of a system at time t
T^1	Lifetime of the component spent at state "1"
$T^2 (or\ T^{\geq 2})$	Lifetime of the component spent at state "2"
$T^{\geq 1} (or\ T^1 + T^2)$	Lifetime of the component
X	Lifetime of the standby unit
$m_1(t)$	MRL function of the system when the system is observed to be at its perfect functioning state at time t
$m_2(t)$	MRL function of the system when the system is observed to be at its partially working state at time t
$m(t)$	MRL function of the system when the system has known to be failed before time t and the standby component has been activated
$C(.)$	Mean cost per unit time
$c_I(c_{II})$	Cost of replacing a non-failed(failed) binary system
c_1	Cost of replacing a non-failed three-state system
c_2	Cost of replacing a failed three-state system whereas the standby component is activated
c_3	Cost of replacing a failed three-state standby system

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