Opuscula Math. 39, no. 2 (2019), 159–174
https://doi.org/10.7494/OpMath.2019.39.2.159 Opuscula Mathematica

NONLINEAR ELLIPTIC EQUATIONS INVOLVING THE p-LAPLACIAN WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS

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Communicated by Dušan Repovš

Abstract. In this paper, a nonlinear differential problem involving the p-Laplacian operator with mixed boundary conditions is investigated. In particular, the existence of three non-zero solutions is established by requiring suitable behavior on the nonlinearity. Concrete examples illustrate the abstract results.

Keywords: mixed problem, critical points.

Mathematics Subject Classification: 35J25, 35J20.

1. INTRODUCTION

Elliptic differential problems with mixed boundary conditions of Dirichlet–Neumann type can be used to describe many engineering or physical phenomena, acting as models in applied sciences. Stress and strain states on an elastic surface in mechanics as well as solidification and melting of a material in industrial processes are just some examples in which mixed conditions are involved. In particular, an intuitive example is given by an iceberg partially submerged in water, for which mixed conditions must be imposed on its boundary. Precisely, in the portion under the water, one imposes Dirichlet boundary condition, while in the remaining part of the boundary that is in contact with the air, Neumann conditions are used. For further details and more information on physical applications of this argument, we refer to [7, 12, 13] and their references. Starting from such motivations, several authors established regularity, existence and multiplicity results of the solutions (see [1, 5, 8, 9, 11]). We also cite the interesting papers [10,14,15], where further nonlinear differential problems, useful in describing different physical phenomena, have been recently investigated. In particular, with regard to the multiplicity of solutions for mixed problems, we cite [4] and [8] where, under appropriate hypotheses, the existence of two non-zero solutions have been guaranteed. Here, our main aim is to ensure, under a suitable set of assumptions, the existence of three non-zero solutions. Precisely, in this paper, we consider a mixed nonlinear differential problem involving the p-Laplacian even with a nonhomogeneous term in the Neumann condition and we obtain the existence of three weak solutions to the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu g(u) & \text{on } \Gamma_2, \end{cases} \tag{$M_{\lambda,\mu}$}$$

where Ω is a nonempty bounded open subset of the Euclidean space $(\mathbb{R}^n, |\cdot|)$, $n \geq 3$, with boundary of class C^1, Γ_1, Γ_2 two smooth $(n-1)-$dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \Sigma$, with Σ a smooth $(n-2)-$dimensional submanifold of $\partial\Omega$, $q \in L^\infty(\Omega)$ with $q_0 := \operatorname{ess\,inf}_\Omega q > 0$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, with $p > n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, λ and μ are real parameters, with $\lambda > 0$ and $\mu \geq 0$, and ν is the outer unit normal to $\partial\Omega$.

In this paper, we present two main theorems. In the first one (Theorem 3.1), we require on the primitive of the function f a growth, which is more than quadratic in an appropriate interval and less than quadratic at infinity, and an asymptotic condition on g, obtaining so three non-zero solutions (see also Remark 3.2). In the second one (Theorem 3.3), assuming on f a sign hypothesis and a suitable behavior in an appropriate interval, the existence of three solutions, which are in addition uniformly bounded with respect to the parameter, is again ensured, without requiring asymptotic conditions at infinity either on f or on g. Finally, special cases of main results are pointed out (see Theorems 3.4 and 3.5) and some concrete examples are included (see Examples 3.6 and 3.7). By way of example of our results, we present here the following particular case.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and assume that*

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi^{p-1}} = \lim_{\xi \to +\infty} \frac{f(\xi)}{\xi^{p-1}} = 0. \tag{1.1}$$

Then there exists $\lambda^* > 0$ such that for each $\lambda \in]\lambda^*, +\infty[$ the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \textit{on } \Gamma_2, \end{cases} \tag{AM_λ}$$

admits at least three nonnegative weak solutions.

The paper is organized as follows. In Section 2, we present preliminaries and main tools, while Section 3 is devoted to main results.

2. PRELIMINARIES AND BASIC NOTATIONS

In this paper, the multiplicity of solutions to the problem $(M_{\lambda,\mu})$ is achieved by variational methods and in particular by exploiting critical point theorems for differentiable functionals of type $\Phi-\lambda\Psi$ defined on a real Banach Space X. To this end, we introduce some basic tools.

Let X be a subset of the Sobolev space $W^{1,p}(\Omega)$, we mean

$$X = W^{1,p}_{0,\Gamma_1}(\Omega) = \{u \in W^{1,p}(\Omega) \ s.t. \ u_{|\Gamma_1} = 0\}$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx \right)^{1/p}.$$

A weak solution of problem $(M_{\lambda,\mu})$ is any $u \in X$ such that

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)dx \\ &= \lambda \int_\Omega f(x,u(x))v(x)dx + \mu \int_{\Gamma_2} g(\gamma(u(x)))\gamma(v(x))d\sigma, \end{aligned}$$

for all $v \in X$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator.

We recall that, since $p > n$, $W^{1,p}(\Omega)$ is embedded in $C_0(\overline{\Omega})$ so X is embedded in $C_0(\overline{\Omega})$. Therefore, by setting

$$k = \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{\frac{1}{p}}},$$

we have

$$\|u\|_\infty \leq k\|u\|, \tag{2.1}$$

where $\|\cdot\|_\infty$ is the usual norm in $L^\infty(\Omega)$.

We recall that if Ω is convex, an explicit upper bound for the constant k is

$$k_1 = 2^{\frac{p-1}{p}} \max\left\{ \left(\frac{1}{\int_\Omega q(x)dx} \right)^{\frac{1}{p}}, \frac{diam(\Omega)}{n^{\frac{1}{p}}} \left(\frac{p-1}{p-n} meas(\Omega) \right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\int_\Omega q(x)dx} \right\},$$

where $diam(\Omega)$ is the diameter of Ω, $meas(\Omega)$ is the Lebesgue measure of Ω and, obviously, $k \leq k_1$ (see [2, Remark 1]).

Throughout the sequel, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and λ is a positive real parameter and μ a nonnegative real parameter.

We recall that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if:

1. $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;
2. $\xi \mapsto f(x, \xi)$ is continuous for almost every $x \in \Omega$;
3. for every $s > 0$ there is a function $l_s \in L^1(\Omega)$ such that

$$\sup_{|\xi| \le s} |f(x, \xi)| \le l_s(x),$$

for a.e. $x \in \Omega$.

Put

$$F(x,t) = \int_0^t f(x,\xi) d\xi \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all} \quad t \in \mathbb{R}.$$

Now, we define the following functionals. For any $u \in X$, set

$$\Phi(u) := \frac{1}{p}\|u\|^p,$$

and

$$\Psi(u) := \int_\Omega F(x, u(x))dx + \frac{\mu}{\lambda}\int_{\Gamma_2} G(\gamma(u(x)))d\sigma,$$

for all $u \in X$.

Now, denoting the Euler function by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \quad \text{for all } t > 0,$$

we define

$$\sigma(p,n) := \inf_{\mu \in]0,1[} \frac{1-\mu^n}{\mu^n(1-\mu)^p},$$

and consider $\overline{\mu} \in]0,1[$ such that $\sigma(p,n) = \frac{1-\overline{\mu}^n}{\overline{\mu}^n(1-\overline{\mu})^p}$. Moreover, let

$$R := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega).$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$, and, for $\overline{\mu} \in]0,1[$, one has $B(x_0, \overline{\mu}R) \subset B(x_0, R)$. Further, put

$$g_{\overline{\mu}}(p,n) := \overline{\mu}^n + \frac{1}{(1-\overline{\mu})^p} n B_{(\overline{\mu},1)}(n, p+1),$$

where $B_{(\overline{\mu},1)}(n,p+1)$ denotes the generalized incomplete beta function defined as follows

$$B_{(\overline{\mu},1)}(n,p+1) := \int\limits_{\overline{\mu}}^{1} t^{n-1}(1-t)^{(p+1)-1}dt.$$

We also denote by

$$\omega_R := \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}R^n,$$

the measure of the n-dimensional ball of radius R, and

$$a(\Gamma_2) = \int\limits_{\Gamma_2} d\sigma.$$

The following lemma guaranties the nonnegativity of the weak solution under appropriate hypothesis on the nonlinear term.

Lemma 2.1 ([4, Lemma 2.3])**.** *If we assume $f(x,0) \geq 0$ for a.e. $x \in \Omega$, then the weak solutions of problem $(M_{\lambda,\mu})$ are nonnegative.*

As said before, our main tools are three critical point theorems that we recall here. The first one has been obtained in [6], and it is a more precise version of Theorem 3.2 of [3]. The second one has been established in [3].

Theorem 2.2 ([6, Theorem 2.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0. \tag{2.2}$$

Assume that there exist $r > 0$ and $\bar{u} \in X$, with $r < \Phi(\bar{u})$, such that:

(a_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(a_2) *for each $\lambda \in \Lambda_r = \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.3 ([3, Corollary 3.1])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0. \tag{2.3}$$

Assume that there exist two positive constant r_1, r_2 and $\bar{u} \in X$, with $2r_1 < \Phi(\bar{u}) < \frac{r_2}{2}$, such that:

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(b_3) *for each*

$$\lambda \in \Lambda_{r_1,r_2} = \left]\frac{3}{2}\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \min\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{r_2}{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right[$$

and for every u_1, $u_2 \in X$, which are local minima for the functional $I_\lambda = \Phi - \lambda\Psi$, and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$ one has $\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0$.

Then, for each $\lambda \in \Lambda_{r_1,r_2}$, the functional $I_\lambda = \Phi - \lambda\Psi$ admits three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

3. MAIN RESULTS

In this section, we present our main results. To this end, consider the problem $(M_{\lambda,\mu})$ as given in the Introduction and put

$$\delta := \frac{R}{k\left[\omega_R\overline{\mu}^n\sigma(p,n) + \frac{1}{(1-\overline{\mu})^p}\int\limits_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} q(x)\,|R-|x-x_0|^p|\,dx + R^p\int\limits_{B(x_0,\overline{\mu}R)} q(x)dx\right]^{\frac{1}{p}}},$$

where R, k, ω_R, $\bar{\mu}$, $\sigma(p,n)$ and $q(x)$ have been defined in Section 2.

Here and in the sequel we assume that $f(x,0) \geq 0$ for a.e $x \in \Omega$. The first main result of this section is the following theorem.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Assume that there exist two nonnegative constants c and d, with $c < d$ such that*

$$\frac{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)dx}{c^p} < \delta^p\frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p}, \tag{3.1}$$

$$F(x,t) \geq 0, \quad \text{for all} \quad (x,t) \in \Omega \times [0,d], \tag{3.2}$$

$$\limsup_{|\xi|\to+\infty}\left[\sup_{x\in\Omega}\frac{F(x,\xi)}{\xi^p}\right] = 0, \tag{3.3}$$

and

$$\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{\xi^p} < +\infty. \tag{3.4}$$

Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{1}{pk^p\delta^p} \frac{d^p}{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}, \frac{1}{pk^p} \frac{c^p}{\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx} \right[$$

there exists $\overline{\eta} > 0$ *with*

$$\overline{\eta} = \min\left\{ \frac{c^p - \lambda pk^p \int_\Omega \max_{|\xi|\le c} F(x,\xi)dx}{pk^p a(\Gamma_2)G(c)}, \frac{1}{\max\left\{0, pk^p a(\Gamma_2) \limsup_{|\xi|\to+\infty} \frac{G(\xi)}{\xi^p}\right\}} \right\},$$

such that for each $\mu \in [0, \overline{\eta}[$ *the problem* $(M_{\lambda,\mu})$ *admits at least three weak solutions.*

Proof. Our aim is to apply Theorem 2.2. To this end, fix λ, μ and g satisfying our assumptions. So, our end is to verify conditions (a_1) and (a_2) of Theorem 2.2. We observe that Φ and Ψ, as given in Section 2 satisfy all regularity assumptions requested in Theorem 2.2 and that the critical points in X of the functional $I_\lambda = \Phi - \lambda\Psi$ are precisely the weak solutions of problem $(M_{\lambda,\mu})$. Now, define

$$\overline{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \frac{d}{R(1-\overline{\mu})}(R - |x-x_0|) & \text{if } x \in B(x_0,R) \setminus B(x_0,\overline{\mu}R), \\ d & \text{if } x \in B(x_0,\overline{\mu}R), \end{cases}$$

and put $r = \frac{1}{p}\left(\frac{c}{k}\right)^p$.

Clearly, $\overline{u} \in X$, and one has

$$\begin{aligned} \Psi(\overline{u}) &= \int\limits_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} F\left(x, \frac{d}{R(1-\overline{\mu})}(R - |x-x_0|)\right) dx + \int\limits_{B(x_0,\overline{\mu}R)} F(x,d)\,dx \\ &\ge \int\limits_{B(x_0,\overline{\mu}R)} F(x,d)\,dx, \end{aligned}$$

and

$$\begin{aligned} \Phi(\overline{u}) = \frac{1}{p}\left(\frac{d}{R}\right)^p \Bigg[&\omega_R \overline{\mu}^n \sigma(p,n) + \frac{1}{(1+\overline{\mu})^p} \int\limits_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} q(x)\,|R - |x-x_0|^p|\,dx \\ &+ R^p \int\limits_{B(x_0,\overline{\mu}R)} q(x)dx \Bigg] = \frac{1}{p}\left(\frac{d}{k\delta}\right)^p. \end{aligned}$$

Moreover, from $c < d$ and (3.1) one has $\delta c < d$. Indeed, arguing by a contradiction, if $\delta c \ge d$,

$$\begin{aligned} \frac{\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx}{c^p} &\ge \frac{\int_\Omega F(x,c)dx}{c^p} \ge \frac{\int_\Omega F(x,d)dx}{c^p} \\ &\ge \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{c^p} \ge \delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p}, \end{aligned}$$

and this is an absurd for which our claim is proved. Hence, it follows that

$$\Phi(\overline{u}) > r. \tag{3.5}$$

Moreover, for all $u \in X$ such that $u \in \Phi^{-1}\left(]-\infty, r]\right)$ and taking (2.1) into account, one has

$$|u(x)| < k\|u\| < k\,(pr)^{\frac{1}{p}} = c.$$

So, since g in nonnegative and owing to (3.2) one has

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda}\int_{\Gamma_2} G(\gamma(u(x)))d\sigma \le \int_\Omega \max_{|t|\le c} F(x,t)dx + \frac{\mu}{\lambda}\int_{\Gamma_2} \max_{|t|\le c} G(t)d\sigma \\ &= \int_\Omega F(x,c)dx + \frac{\mu}{\lambda}a(\Gamma_2)G(c),\end{aligned}$$

for all $u \in X$ such that $u \in \Phi^{-1}\left(]-\infty, r]\right)$. Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) \le \int_\Omega F(x,c)dx + \frac{\mu}{\lambda}a(\Gamma_2)G(c). \tag{3.6}$$

Therefore, since $\mu \in [0, \overline{\eta}[$, and owing to (3.5) and (3.6), one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} &\le pk^p \frac{\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx}{c^p} + pk^p\frac{\mu}{\lambda}a(\Gamma_2)\frac{G(c)}{c^p} < \frac{1}{\lambda} \\ &< pk^p\delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F\,(x,d)\,dx}{d^p} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}.\end{aligned}$$

Therefore, hypothesis (a_1) of Theorem 2.2 is verified.

Now, since $\mu < \overline{\eta}$, we can fix $l > 0$ such that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{\xi^p} < l,$$

and $\mu l < \frac{1}{pk^p a(\Gamma_2)}$. Therefore, there exists $T \in \mathbb{R}$ such that

$$G(\xi) \le l\xi^p + T,$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$.

Finally, fix $0 < \epsilon < \frac{1}{p\lambda k^p meas(\Omega)} - \frac{\mu l a(\Gamma_2)}{\lambda meas(\Omega)}$. From (3.3) there is a function $h_\epsilon \in L^1(\Omega)$ such that

$$F(x,\xi) \le \epsilon\xi^p + h_\epsilon(x),$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$. It follows that, for each $u \in X$,

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \ge \left[\frac{1}{p} - \lambda\epsilon k^p meas(\Omega) - \mu l k^p a(\Gamma_2)\right]\|u\|^p - \lambda\|h_\epsilon\|_1 - \mu T a(\Gamma_2).$$

This leads to the coercivity of I_λ, and condition of Theorem 2.2 is verified. Since, from (3.7), Theorem 2.2 assures the existence of three critical points for the functional I_λ, and the proof is complete. □

Remark 3.2. We explicitly observe that if we consider

$$\widetilde{\delta} := \frac{R}{k\left[\omega_R\left(\overline{\mu}^n\sigma(p,n) + \|q\|_\infty R^p g_{\overline{\mu}}(p,n)\right)\right]^{\frac{1}{p}}},$$

clearly we have $\widetilde{\delta} \leq \delta$. So if we assume

$$\frac{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)dx}{c^p} < \widetilde{\delta}^p \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p}, \tag{3.7}$$

the assumption (3.1) is satisfied and, by using (3.7), the coefficient $\widetilde{\delta}$ is easier to calculate. From (3.7), simple cases can be obtained, see Theorem 3.4.

We also emphasize that the assumption (3.1) allow us to assume in addition $f(x,0) \neq 0$, $x \in \Omega$, for which all the three obtained solutions are non-zero.

Now, we state a second result on the existence of three solutions. Here no asymptotic condition on g and f are requested.

Theorem 3.3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that there exist three nonnegative constants c_1, c_2 and d, with $2^{\frac{1}{p}}\delta c_1 < d < 2^{-\frac{1}{p}}\delta c_2$ such that*

$$f(x,\xi) \geq 0, \quad \textit{for all} \quad (x,\xi) \in \Omega \times [0, c_2], \tag{3.8}$$

$$\frac{\int_\Omega F(x,c_1)dx}{c_1^p} < \frac{2}{3}\delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p}, \tag{3.9}$$

and

$$\frac{\int_\Omega F(x,c_2)dx}{c_2^p} < \frac{1}{3}\delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p}. \tag{3.10}$$

Then, for each

$$\lambda \in \Lambda_2 := \left]\frac{3}{2pk^p\delta^p}\frac{d^p}{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}, \frac{1}{pk^p}\min\left\{\frac{c_1^p}{\int_\Omega F(x,c_1)dx}, \frac{c_2^p}{\int_\Omega F(x,c_2)dx}\right\}\right[,$$

and for each nonegative continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\eta > 0$ with

$$\eta = \min\left\{\frac{c_1^p - \lambda pk^p \int_\Omega F(x,c_1)dx}{pk^p a(\Gamma_2)G(c_1)}, \frac{c_2^p - 2\lambda pk^p \int_\Omega F(x,c_2)dx}{2pk^p a(\Gamma_2)G(c_2)}\right\},$$

such that, for each $\mu \in [0,\eta[$, the problem $(M_{\lambda,\mu})$ admits at least three weak solutions u_i, $i = 1,2,3$, such that $0 \leq u_i < c_2$ for all $i = 1,2,3$.

Proof. Without loss of generality, we can assume $f(x,t) \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. Our aim is to apply Theorem 2.3. To this end, fix λ, μ and g satisfying our assumptions and take X, Φ and Ψ as in the proof of Theorem 3.1.

We observe that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.3 and that the critical points in X of the functional $\Phi - \lambda\Psi$ are precisely the weak solutions of problem $(M_{\lambda,\mu})$. So, our end is to verify conditions (b_1)–(b_3) of Theorem 2.3.

We define $\overline{u}(x)$ as in Theorem 3.1 and put $r_1 = \frac{1}{p}\left(\frac{c_1}{k}\right)^p$, and $r_2 = \frac{1}{p}\left(\frac{c_2}{k}\right)^p$.

Arguing as in the proof of Theorem 3.1, we obtain that

$$\Psi(\overline{u}) \geq \int\limits_{B(x_0,\overline{\mu}R)} F\left(x,d\right)dx,$$

and

$$\Phi(\overline{u}) = \frac{1}{p}\left(\frac{d}{k\delta}\right)^p.$$

Therefore, from $2^{\frac{1}{p}}\delta_1 c_1 < d < 2^{-\frac{1}{p}}\delta_2 c_2$ one has

$$2r_1 < \Phi(\overline{u}) < \frac{r_2}{2}. \tag{3.11}$$

Moreover, for all $u \in X$ such that $u \in \Phi^{-1}\left(]-\infty, r_i[\right)$ with $i = 1, 2$ and taking (2.1) into account, one has

$$|u(x)| < k\|u\| < k\left(pr_1\right)^{\frac{1}{p}} = c_i, \quad \text{with} \quad i = 1, 2.$$

So, since g in nonegative and owing (3.8) one has

$$\begin{aligned}\Psi(u) &= \int\limits_{\Omega} F(x,u(x))dx + \frac{\mu}{\lambda}\int\limits_{\Gamma_2} G(u(x))dx \leq \int\limits_{\Omega}\sup_{|t|<c_i} F(x,t)dx + \frac{\mu}{\lambda}\int\limits_{\Gamma_2}\sup_{|t|<c_i} G(t)dx \\ &= \int\limits_{\Omega} F(x,c_i)dx + \frac{\mu}{\lambda}a(\Gamma_2)G(c_i)dx,\end{aligned}$$

for all $u \in X$ such that $u \in \Phi^{-1}\left(]-\infty, r_i[\right)$ with $i = 1, 2$. Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \Psi(u) \leq \int\limits_{\Omega} F(x,c_i)dx + \frac{\mu}{\lambda}a(\Gamma_2)G(c_i). \tag{3.12}$$

Therefore, since $\mu \in [0, \eta[$, and owing to (3.11) and (3.12), one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} &\leq pk^p\frac{\int_\Omega F(x,c_1)dx}{c_1^p} + pk^p\frac{\mu}{\lambda}a(\Gamma_2)\frac{G(c_1)}{c_1^p} < \frac{1}{\lambda} \\ &< \frac{2}{3}pk^p\delta^p\frac{\int_{B(x_0,\overline{\mu}R)} F\left(x,d\right)dx}{d^p} < \frac{2}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})},\end{aligned}$$

and

$$\begin{aligned}2\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} &\leq 2pk^p\frac{\int_\Omega F(x,c_2)dx}{c_2^p} + 2pk^p\frac{\mu}{\lambda}a(\Gamma_2)\frac{G(c_2)}{c_2^p} < \frac{1}{\lambda} \\ &< \frac{2}{3}pk^p\delta^p\frac{\int_{B(x_0,\overline{\mu}R)} F\left(x,d\right)dx}{d^p} < \frac{2}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}.\end{aligned}$$

Therefore, hypothesis (b_1) and (b_2) of Theorem 2.3 are verified.

Finally, we verify that satisfies assumption (b_3) of Theorem 2.3. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem $(M_{\lambda,\mu})$. For every positive parameter λ and for every $(x,t) \in \Omega \times [0,+\infty[$ one has $\lambda f(x,t) \geq 0$, hence, owing to the Weak Maximum Principle (see for instance Lemma 2.1) we obtain $u_1(x) \geq 0$ and $u_2(x) \geq 0$, for all $x \in \Omega$. Then, it follows that $su_1(x) + (1-s)u_2(x) \geq 0$, for all $s \in [0,1]$, and that $\lambda f(x, su_1(x) + (1-s)u_2(x)) + \mu g(su_1(x) + (1-s)u_2(x)) \geq 0$, and, hence, $\Psi(su_1(x) + (1-s)u_2(x)) \geq 0$.

From Theorem 2.3, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of problem $(M_{\lambda,\mu})$ and the conclusion is achieved. □

Now, we point out some results in the autonomous case. To be precise, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the following mixed boundary value problem involving the p-Laplacian

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases} \tag{AM_λ}$$

Theorem 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and assume that there exist two positive constants c and d, with $c < d$ such that, fixed $\widehat{k} = \frac{\delta^p \omega_{\overline{\mu}R}}{meas(\Omega)}$*

$$\frac{F(c)}{c^p} < \widehat{k}\frac{F(d)}{d^p}, \tag{3.13}$$

and

$$\limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^p} = 0. \tag{3.14}$$

Then, for each

$$\lambda \in \widetilde{\Lambda}_1 := \left] \frac{1}{pk^p \delta^p \omega_{\overline{\mu}R}} \frac{d^p}{F(d)}, \frac{1}{pk^p meas(\Omega)} \frac{c^p}{F(c)} \right[,$$

the problem (AM_λ) admits at least three weak solutions.

Proof. Our aim is to apply Theorem 3.1 with $g = 0$ and f depending only on the second variable. Since f is a nonnegative function, one has

$$\frac{\int_\Omega \max_{|\xi| \leq c} F(\xi) dx}{c^p} = \frac{\int_\Omega F(c) dx}{c^p} = meas(\Omega) \frac{F(c)}{c^p},$$

and

$$\delta^p \frac{\int_{B(x_0, \overline{\mu}R)} F(d) dx}{d^p} = \delta^p \omega_{\overline{\mu}R} \frac{F(d)}{d^p},$$

from condition (3.13) we obtain condition (3.1) of Theorem 3.3. Finally, condition (3.14) follows from (3.3). Then, for each

$$\lambda \in \widetilde{\Lambda}_1 := \left] \frac{1}{pk^p \delta^p \omega_{\overline{\mu}R}} \frac{d^p}{F(d)}, \frac{1}{pk^p meas(\Omega)} \frac{c^p}{F(c)} \right[$$

the problem (AM_λ) admits at least three weak solutions. In particular, by Lemma 2.1 (see also [4, Lemma 2.3]) we obtain that the solutions are nonnegative. □

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and assume that there exist three positive constants c_1, c_2 and d, with $2^{\frac{1}{p}}\delta c_1 < d < 2^{-\frac{1}{p}}\delta c_2$ such that*

$$\frac{F(c_1)}{c_1^p} < \frac{2}{3}\frac{\delta^p \omega_{\overline{\mu}R}}{m(\Omega)}\frac{F(d)}{d^p}, \tag{3.15}$$

$$\frac{F(c_2)}{c_2^p} < \frac{1}{3}\frac{\delta^p \omega_{\overline{\mu}R}}{m(\Omega)}\frac{F(d)}{d^p}. \tag{3.16}$$

Then, for each

$$\lambda \in \widetilde{\Lambda}_2 := \left] \frac{3}{2pk^p\delta^p\omega_{\overline{\mu}R}}\frac{d^p}{F(d)}, \frac{1}{pk^p m(\Omega)} \min\left\{\frac{c_1^p}{F(c_1)}, \frac{c_2^p}{F(c_2)}\right\}\right[,$$

the problem (AM_λ) admits at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 \le u_i < c_2$ for all $i = 1, 2, 3$.

Proof. Our aim is to apply Theorem 3.3 with $g = 0$ and f depending only of the second variable. Since f is a nonnegative function, one has

$$\frac{\int_\Omega \max_{|\xi|\le c_1} F(\xi)dx}{c_1^p} = \frac{\int_\Omega F(c_1)dx}{c_1^p} = meas(\Omega)\frac{F(c_1)}{c_1^p},$$

$$\frac{\int_\Omega \max_{|\xi|\le c_2} F(\xi)dx}{c_2^p} = \frac{\int_\Omega F(c_2)dx}{c_2^p} = meas(\Omega)\frac{F(c_2)}{c_2^p},$$

and

$$\delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p} = \delta^p \frac{\int_{B(x_0,\overline{\mu}R)} F(d)dx}{d^p} = \delta^p \omega_{\overline{\mu}R}\frac{F(d)}{d^p},$$

from which and from conditions (3.15) and (3.16) we have conditions (3.9) and (3.10). Then, for each

$$\lambda \in \widetilde{\Lambda}_2 := \left] \frac{3}{2pk^p\delta^p\omega_{\overline{\mu}R}}\frac{d^p}{F(d)}, \frac{1}{pk^p m(\Omega)} \min\left\{\frac{c_1^p}{F(c_1)}, \frac{c_2^p}{F(c_2)}\right\}\right[$$

the problem (AM_λ) admits at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 \le u_i < c_2$ for all $i = 1, 2, 3$. □

Finally, we point out the proof of Theorem 1.1, in the Introduction.

Proof. Our aim is to apply Theorem 3.5. Fix $\lambda \in]\lambda^*, +\infty[$, then there is $d > 0$ such that

$$\lambda > \frac{3}{2pk^p\delta^p\omega_{\overline{\mu}R}}\frac{d^p}{\int_0^d f(\xi)d\xi}.$$

Then, from (1.1), there is $c_1 < 2^{-\frac{1}{p}}\delta^{-1}d$ such that

$$pk^p m(\Omega)\frac{F(c_1)}{c_1^p} < \frac{1}{\lambda} < \frac{2}{3}pk^p\delta^p\omega_{\overline{\mu}R}\frac{F(d)}{d^p}.$$

Moreover, there is $c_2 > 2^{\frac{1}{p}}\delta^{-1}d$ such that

$$2pk^p m(\Omega)\frac{F(c_2)}{c_2^p} < \frac{1}{\lambda} < \frac{2}{3}pk^p\delta^p\omega_{\overline{\mu}R}\frac{F(d)}{d^p}.$$

Hence, from Theorem 3.5 for each $\lambda \in]\lambda^*, +\infty[$ the problem (AM_λ) admits at least three nonnegative weak solutions. □

Example 3.6. Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$, put $p = 4$, and

$$f(u) = \begin{cases} \frac{e^2}{2^4}t^4 & \text{if } t \le 2, \\ e^t & \text{if } 2 < t \le 23, \\ \frac{e^{23}}{23}t^2 & \text{if } t \ge 23. \end{cases}$$

Consider the following problem

$$\begin{cases} -\Delta_4 u + |u|^2 u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.17}$$

We can choose $\overline{\mu} = \frac{1}{2}$ and, as a simple computation shows, we have

$$B_{(\overline{\mu},1)}(n, p+1) = B_{\left(\frac{1}{2},1\right)}(3,5) = \frac{1}{2^7}\frac{29}{105},$$

$$\sigma(p,n) = \sigma(4,3) = 112,$$

$$\omega_R = \omega_1 = \frac{4}{3}\pi.$$

Since $q(x) = 1$, then

$$\delta = \frac{R}{k\left[\omega_R\left(\overline{\mu}^n\sigma(p,n) + \frac{1}{(1-\overline{\mu})^p}nR^pB_{(\overline{\mu},1)}(n,p+1) + R^p\overline{\mu}^n\right)\right]^{\frac{1}{p}}},$$

with

$$k_1 = \left(\frac{2^5\cdot 3^3}{\pi}\right)^{\frac{1}{4}}.$$

Then $\delta^4 = \frac{5\cdot 7}{2^8\cdot 3^2\cdot 83}$.

Put $c = 1$, $d = 3e^3$ and we prove that all conditions of Theorem 3.1 hold. Indeed,

$$\frac{\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx}{c^p} = \frac{e^2}{2^4\cdot 5}\int_\Omega \max_{|\xi|\le 1}\xi^5 dx = \frac{e^2}{2^2\cdot 5}\frac{\pi}{3}.$$

$$\delta\frac{\int_{B(x_0,\overline{\mu}R)} F(x,d)dx}{d^p} = \frac{5\cdot 7}{2^8\cdot 3^2\cdot 83}\frac{\int_{B\left(0,\frac{1}{2}\right)}\left(e^{3e^2} - \frac{3}{5}e^2\right)dx}{(3e^2)^4}$$
$$= \frac{5\cdot 7}{2^9\cdot 3^8\cdot 83\cdot e^8}\left(e^{3e^2} - \frac{3}{5}e^2\right)\pi,$$

finally

$$\limsup_{|\xi|\to+\infty}\left[\sup_{x\in\Omega}\frac{F(x,\xi)}{\xi^p}\right]=\limsup_{|\xi|\to+\infty}\left[\frac{\frac{2}{5}e^2+e^{23}-e^2+\frac{e^{23}}{3\cdot 23^2}\xi^3-\frac{23}{3}e^{23}}{\xi^4}\right]=0.$$

Then, owing to Theorem 3.1, for each

$$\lambda\in\left]\frac{2^2\cdot 3^5\cdot 83}{5\cdot 7}\frac{e^8}{e^{3e^2}-\frac{3}{5}e^2},\frac{5}{2^5\cdot 3^2e^2}\right[$$

the problem (3.17) admits at least three weak solutions.

Example 3.7. Let $\Omega=\left\{x\in\mathbb{R}^3:|x|<1\right\}$, put $p=4$, and consider the following problem

$$\begin{cases}-\Delta_4 u+|u|^2u=\lambda|x|\left(\frac{|u|^3}{u^2+1}-|u|^3e^{-|u|}\right) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_1,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \Gamma_2.\end{cases}\tag{3.18}$$

We can choose $\overline{\mu}=\frac{1}{2}$ and, as a simple computation shows, we have

$$\lim_{\xi\to 0^+}\frac{f(\xi)}{\xi^{p-1}}=\lim_{\xi\to 0^+}\left(\frac{1}{\xi^2+1}-e^{-\xi}\right)=0.$$

and

$$\lim_{\xi\to+\infty}\frac{f(\xi)}{\xi^{p-1}}=\lim_{\xi\to+\infty}\left(\frac{1}{\xi^2+1}-e^{-\xi}\right)=0.$$

Finally, we observe that

$$\begin{aligned}\lambda^*&=\frac{3}{2pk^p\delta^p\omega_{\overline{\mu}R}}\inf_{d>0}\frac{d^4}{\int_0^d\left(\frac{\xi^3}{\xi^2+1}-\xi^3e^{-\xi}\right)d\xi}\\ &=\frac{3\cdot 83}{2^2\cdot 5\cdot 7}\inf_{d>0}\frac{d^4}{\frac{1}{2}\left(d^2-\ln(d^2+1)\right)+e^{-d}(d^3+3d^2+6d+6)-6}.\end{aligned}$$

Then, owing to Theorem 1.1, for each

$$\lambda>\frac{3\cdot 83}{2^2\cdot 5\cdot 7}\inf_{d>0}\frac{d^4}{\frac{1}{2}\left(d^2-\ln(d^2+1)\right)+e^{-d}(d^3+3d^2+6d+6)-6}$$

the problem (3.18) admits at least three nonnegative weak solutions. In particular, the problem

$$\begin{cases}-\Delta_4 u+|u|^2u=50|x|\left(\frac{|u|^3}{u^2+1}-|u|^3e^{-|u|}\right) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_1,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \Gamma_2,\end{cases}$$

admits at least three nonnegative weak solutions.

Acknowledgements

The authors have been partially supported by the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the "Istituto Nazionale di Alta Matematica" (INdAM). In particular, the last author is holder of a postdoctoral fellowship from the INdAM/INGV research project "Strategic Initiatives for the Environment and Security – SIES".

REFERENCES

[1] G. Barletta, R. Livrea, N.S. Papageorgiou, *Bifurcation phenomena for the positive solutions on semilinear elliptic problems with mixed boundary conditions*, J. Nonlinear Convex Anal. **17** (2016), 1497–1516.

[2] G. Bonanno, P. Candito, *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian*, Arch. Math. **80** (2003), 424–429.

[3] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), 3031–3059.

[4] G. Bonanno, G. D'Aguì, *Mixed elliptic problems involving the p-Laplacian with nonhomogeneous boundary conditions*, Discrete Contin. Dyn. Syst. **37** (2017) 11, 5797–5817.

[5] G. Bonanno, G. D'Aguì, N.S. Papageorgiou *Infinitely many solutions for mixed elliptic problems involving the p-Laplacian*, Adv. Nonlinear Stud. **15** (2015), 939–950.

[6] G. Bonanno, S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. **89** (2010), 1–10.

[7] V. Bonfim, A.F. Neves, *A one-dimensional heat equation with mixed boundary conditions*, J. Differential Equations **139** (1997), 319–338.

[8] E. Colorado, I. Peral, *Semilinear elliptic problems with mixed Dirichlet–Neumann boundary conditions*, J. Funct. Anal. **199** (2003), 468–507.

[9] J. Dávila, *A strong maximum principle for the Laplace equation with mixed boundary condition*, J. Funct. Anal. **183** (2001), 231–244.

[10] G. D'Aguì, S.A. Marano, N.S. Papageorgiou, *Multiple solutions to a Robin problem with indefinite weight and asymmetric reaction*, J. Math. Anal. Appl. **433** (2016) 2, 1821–1845.

[11] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral, *Concentration of solutions for some singularly perturbed mixed problems: Asymptotics of minimal energy solutions*, Ann. I. H. Poincaré – AN **27** (2010), 37–56.

[12] R. Haller-Dintelmann, H.C. Kaiser, J. Rehberg, *Elliptic model problems including mixed boundary conditions and material heterogeneities*, J. Math. Pures. Appl. **89** (2008), 25–48.

[13] I. Mitrea, M. Mitrea, *The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains*, Trans. Amer. Math. Soc. **359** (2007), 4143–4182.

[14] N.S. Papageorgiou, V.D. Radulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlinear Stud. **16** (2016), 737–764.

[15] N.S. Papageorgiou, V.D. Radulescu, *Multiplicity of solutions for nonlinear nonhomogeneous Robin problems*, Proc. Amer. Math. Soc. **146** (2018), 601–611.

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Received: October 23, 2018.
Accepted: October 30, 2018.