DOI 10.2478/ama-2024-0030

# THERMAL STRESSES IN A MULTI-LAYERED SPHERICAL TANK WITH A SLOWLY GRADED STRUCTURE

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*received 05 June 2023, revised 26 Septemeber 2023, accepted 28 September 2023*

**Abstract:** The central-symmetrical problem of thermoelasticity for a multi-layered spherical tank is considered. The thermal stresses were caused by a temperature difference between the inner and outer surfaces of the tank. Two approaches to solving this problem have been proposed. In the first approach, the boundary problem defined in the components of a considered inhomogeneous spherical tank was solved. In the second approach, the homogenization method with microlocal parameters was used. Good agreement between the solutions was obtained.

**Key words:** temperature, displacement, thermal stress, composite material, functionally graded material, multi-layered structure, spherical tank

## 1. INTRODUCTION

The progress of technology has created a wide range of design possibilities for modern materials combining the positive properties of metals (high tensile strength and good resistance to damage and cracking) and ceramic materials (high hardness, high compressive strength, resistance to surface wear and good thermal insulation). In particular, these materials are used to create protective coatings: thermal barrier coatings and barrier coatings for wear and corrosion protection (Lee et al., 1996; Wang et al., 2000; Schulz et al., 2003; Chen and Tong, 2004). With the possibility of producing complex multilayer structures, the importance of effective mathematical models for predicting the mechanical and thermal properties of these structures is growing.

The multi-layer structures described above are two- or multi-component solids. Classical formulations of a problem based on the theory of thermoelasticity lead to boundary value problems for partial differential equations with discontinuous, strongly oscillating coefficients or considering a large number of elastic components between which certain conditions of mechanical and thermal contact are fulfilled. The solution for these problems can be extremely labour-consuming or sometimes simply impossible to accomplish. For the sake of simplicity, these problems are usually replaced by ones for approximated models, in which the characteristic parameters are calculated on the basis of mechanical and geometrical properties of the components. These models were derived by using some averaging procedures. In the case of the composite that has the properties of an isotropic solid at the macro level, classical averaging methods are still often used: the Voigt estimation (Voigt, 1889) or the Reuss estimation (Reuss, 1929). These estimations are based only on the volume content of the components in a representative cell of the composite. They do not take into account the actual geometry and the distribution of components. This means that these estimations are insufficient to properly describe the composites that have the properties of an anisotropic solid at the macro level (Ganczarski and Skrzypek, 2013), in particular the multi-layer structures.

The configuration, geometry and concentration of the component phases of the composite are taken into account in newer homogenization models. We will list only some of them: asymptotic homogenization (Bensoussan et al., 1978; Sanchez-Palencia, 1989; Jihov et al., 1994; Manevitch et al., 2002), based on the concept applied in the theory of thick plates (Achenbach, 1975), based on the mixture theory (Bedford and Stern, 1971), based on matrix methods (Bufler and Krennerknecht, 1999), the generalized method of cells (Paley and Aboudi, 1992) and the strain compatible method of cells (Gan et al., 2000). Particular attention should be paid to the method of homogenization with microlocal parameters (Woźniak, 1987; Matysiak and Woźniak, 1987), which is used to solve problems for multi-layer media with a periodic structure. This method makes it possible to evaluate not only the mean but also local values of strains and stresses in every layer of the periodicity cell.

It has been shown that the solutions of the theories of heat conduction, elasticity or thermoelasticity for a non-homogenous multi-layer medium with a periodical structure (Kulchytsky-Zhyhailo et al., 2005, 2007) are well compatible with the corresponding solutions of problems in which the medium is replaced by a homogeneous medium whose mechanical and thermal properties are described by the method of homogenization with microlocal parameters. Small deviations were obtained in the calculation of both continuous stresses at the layer-separation boundary and stresses undergoing jumps at the interfaces. This is a strong argument for the use of this homogenization method to describe the mechanical and thermal properties of multi-layer media with a periodic structure.

It is generally accepted that the condition for using the homogenization method is the existence of a representative unit cell that is repeated periodically. The layer that is formed as a result of homogenization of a multi-layer solid with a periodic structure is a homogeneous isotropic or anisotropic layer. If this layer is used as a protective coating, we will obtain a large difference in thermomechanical constants between the layer and the substrate. The difference in constants of the materials often causes cracking of the protective coating and subsequent delamination between the coating and the substrate (Kirchhoff et al., 2004). The solution to this problem is the use of a gradient multilayer structure that causes a gradual change in thermomechanical properties. The protective thermal coating is constructed in such a way that on the surface of the coating we obtain, for example, the properties of a thermal insulator, and at the interface the properties of the substrate. The described structure may have a representative cell, but the configuration, geometry and concentration of the constituent phases of the cell will vary along the thickness of the considered structure. Such a structure is called a slowly graded structure (Szymczyk and Woźniak, 2006). If the coating under consideration contains a significant number of representative cells, the possibility of replacing the multi-layer structure with a gradient coating with a continuous change of thermomechanical properties should be considered. The most reasonable way to do this is to use a homogenization method of choice, for example homogenisation with microlocal parameters.

The ring-multi-layered slowly graded structure has been investigated by Kulchytsky-Zhyhailo et al. (2021). The thermal stresses in the considered long pipe were caused by a temperature difference between its inner and outer surfaces. It has been shown that the homogenization approach with microlocal parameters allows for the correct calculation of radial and circumferential stresses in the ring layer with the slowly graded structure. The effect of material inhomogeneity on the temperature and stress distributions in a multi-layer functionally graded hollow sphere has been analyzed by Kushnir et al. (2022). The numerical solving algorithm was based on the finite difference method.

In this paper, the stress field in the multi-layered spherical tank with the slowly graded structure is investigated. The considered structure contains a certain number of representative cells consisting of two homogeneous isotropic layers. As in the article by Kulchytsky-Zhyhailo et al. (2021), thermal stresses are caused by a temperature difference between its inner and outer surfaces. In the first part of the article, the stress distribution in the tank with the periodical structure is analyzed. The problem based on the homogenization method has a simple analytical solution. Obtaining a good agreement between this solution and the solution to the problem for a heterogeneous multi-layered tank, we can convince ourselves of the reliability of the algorithms used. In the second part of the article, the tested reservoir has the slowly graded structure described above. If the use of constitutive relationships based on the homogenization method with microlocal parameters may raise certain doubts, the created algorithm for solving the problem for a multi-layered spherical tank does not introduce any restrictions on the order and the method of arrangement and the materials and geometrics characteristics of the considered spherical layers. The compatibility of both the solutions will be a clear indication that the chosen homogenization method can also be used effectively for slowly graded structures.

## 2. FORMULATION OF THE PROBLEM

The state of stress in a spherical nonhomogeneous tank with the internal radius R0 and external radius R1 is investigated. The stress field is caused by a temperature difference T0 between its inner and outer surfaces. The inner and external surfaces of the tank are unloaded. The considerations will be led using the dimensionless spherical coordinates (r, $\varphi$, $\theta$) related to the radius R1. The section of the tank with a plane containing its center is shown in Fig. 1.

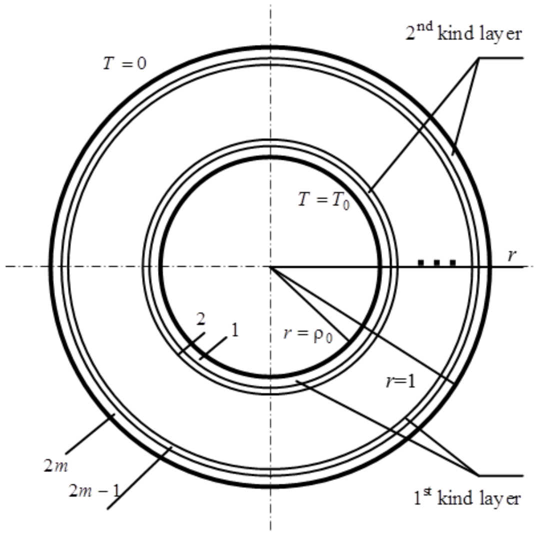


**Fig. 1.** The scheme of the problem

The cross-section of the investigated tank is composed of n = 2m spherical layers, where m is the number of representative cells. The representative cell contains two homogeneous spherical layers with the thermal conductivity coefficients K1, K2; Young modules E1, E2; Poisson ratios $\nu$1, $\nu$2; the coefficients of linear thermal expansion $\alpha$1, $\alpha$2 and dimensionless thickness $\delta$1 = $\eta\delta$, $\delta$2 = (1 – $\eta$)$\delta$, where $\delta$ = (1 – $\rho$0)/m ($\rho$0 = R0/R1) is the dimensionless cell thickness. The parameter $\eta \in (0, 1)$ describes the content of the first type of material in a representative cell and can vary along the thickness of the tank. The thermal and mechanical contact between the tank components is ideal.

The considered problem is a central-symmetrical problem. Unknown state functions (the radial displacement u, the radial stress $\sigma$r, the circumferential stress $\sigma\theta$ and the temperature T) can be calculated by solving the following boundary value problem (Timoshenko and Goodier, 1951; Nowacki, 1986) equations:

$$\frac{d}{dr}\left(r^2\frac{dT^{(i)}}{dr}\right) = 0\ ,\ r \in (r_{i-1}, r_i)\ ,\ i = 1,2,\dots,n \quad (1a)$$

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2u^{(i)}\right) - \frac{1+\nu^{(i)}}{1-\nu^{(i)}}\alpha^{(i)}T^{(i)}\right) = 0\ ,\ r \in (r_{i-1}, r_i)\ ,\ i = 1,2,\dots,n \quad (1b)$$

boundary conditions on the internal and external surfaces of the tank:

$$\sigma_r^{(1)}(r_0) = 0,\ \sigma_r^{(n)}(r_n) = 0 \quad (2a)$$

$$T^{(1)}(r_0) = T_0,\ T^{(n)}(r_n) = 0 \quad (2b)$$

DOI 10.2478/ama-2024-0030

interface conditions of the ideal mechanical and thermal contact between the tank components:

$$u^{(i)}(r_i-0)=u^{(i+1)}(r_i+0)\,,\quad \sigma_r^{(i)}(r_i-0)=\sigma_r^{(i+1)}(r_i+0),\ i=1,2,\dots,n-1 \tag{3a}$$

$$T^{(i)}(r_i-0)=T^{(i+1)}(r_i+0)\,,\quad K^{(i)}\frac{dT^{(i)}}{dr}(r_i-0)=K^{(i+1)}\frac{dT^{(i+1)}}{dr}(r_i+0),\ i=1,2,\dots,n \tag{3b}$$

where the index i corresponds to the number of the spherical layer (see Fig. 1), $\nu^{(2j-1)}=\nu_1$, $\nu^{(2j)}=\nu_2$, $\alpha^{(2j-1)}=\alpha_1$, $\alpha^{(2j)}=\alpha_2$, $K^{(2j-1)}=K_1$, $K^{(2j)}=K_2$, $r_0=\rho_0$, $r_{2j}=\rho_0+j\delta$, $r_{2j-1}=r_{2j}-\delta_2$, $j=1,2,\dots,m$.

## 3. CASE 1. MULTI-LAYERED TANK WITH PERIODIC STRUCTURE

The solution of the problem for a multilayered tank with a periodic structure (parameter $\eta$ = const.) will be compared with the solution of the problem of the tank made of a material with a transversely isotropic homogeneous structure. The thermomechanical properties will be determined by using the method of homogenization with microlocal parameters (Woźniak, 1987; Matysiak and Woźniak, 1987). The received boundary value problem has the form:

– equations:

$$\frac{d}{dr}\left(r^2 q_{\mathrm{hom}}\right)=0\,,\ r\in(\rho_0,1) \tag{4a}$$

$$\frac{d\sigma_r^{\mathrm{hom}}}{dr}+\frac{2}{r}\left(\sigma_r^{\mathrm{hom}}-\sigma_\theta^{\mathrm{hom}}\right)=0,\ ,\ r\in(\rho_0,1) \tag{4b}$$

– boundary conditions:

$$\sigma_r^{\mathrm{hom}}(\rho_0)=\sigma_r^{\mathrm{hom}}(1)=0 \tag{5a}$$

$$T_{\mathrm{hom}}(\rho_0)=T_0,\ T_{\mathrm{hom}}(1)=0 \tag{5b}$$

The stresses in the transversely isotropic homogeneous tank can be calculated using the following equations (Kaczyński, 2004):

$$\sigma_r^{(1)}=\sigma_r^{(2)}=\sigma_r^{\mathrm{hom}}=A_1\frac{du_{\mathrm{hom}}}{dr}+2B\frac{u_{\mathrm{hom}}}{r}-\Lambda_1 T_{\mathrm{hom}} \tag{6a}$$

$$\sigma_\theta^{(j)}=D_j\frac{du_{\mathrm{hom}}}{dr}+\left(E_j+C_j\right)\frac{u_{\mathrm{hom}}}{r}-F_jT_{\mathrm{hom}},\ j=1,2 \tag{6b}$$

In the Eqs. (4) - (6), the following notation is introduced: $\sigma_r^{(j)}$ and $\sigma_\theta^{(j)}$, j = 1,2 denote radial and circumferential stress in the j-th layer of the periodicity cell; uhom, $\sigma_r^{\mathrm{hom}}$, Thom, and qhom are the radial displacement, radial stress, temperature, and heat flux, which are averaged within the periodicity cell;

$$A_1=\frac{(\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} \tag{7a}$$

$$B=\frac{(1-\eta)\lambda_2(\lambda_1+2\mu_1)+\eta\lambda_1(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} \tag{7b}$$

$$\Lambda_1=\frac{(1-\eta)\alpha_2(3\lambda_2+2\mu_2)(\lambda_1+2\mu_1)+\eta\alpha_1(3\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} \tag{7c}$$

$$C_j=\frac{\lambda_j(B+2\mu_j)}{\lambda_j+2\mu_j},\ D_j=\frac{\lambda_jA_1}{\lambda_j+2\mu_j}\ ;\ E_j=\frac{4\mu_j(\lambda_j+\mu_j)+\lambda_jB}{\lambda_j+2\mu_j},j=1,2 \tag{7d}$$

$$F_j=\frac{2\alpha_j(3\lambda_j+2\mu_j)\mu_j+\lambda_j\Lambda_1}{\lambda_j+2\mu_j},j=1,2 \tag{7e}$$

The constants $\lambda$j , $\mu$j in Eqs. (7) are Lame constants of the j-th layer of the periodicity cell:

$$\lambda_j=\frac{E_j\nu_j}{(1+\nu_j)(1-2\nu_j)}\ ,\ \mu_j=\frac{E_j}{2(1+\nu_j)}\ ,\ j=1,2 \tag{8}$$

It should be emphasized that under the proposed homogenization method, we can directly calculate the stresses in each layer of the periodicity cell. The radial stress is continuous at the interfaces: $\sigma_r^{(1)}=\sigma_r^{(2)}=\sigma_r^{\mathrm{hom}}$. The circumferential stress experiences a spike at the interfaces: $\sigma_\theta^{(1)}\neq\sigma_\theta^{(2)}$.

The circumferential stress averaged over the periodicity cell is equal:

$$\sigma_\theta^{\mathrm{hom}}=\eta\sigma_\theta^{(1)}+(1-\eta)\sigma_\theta^{(2)}=B\frac{du_{\mathrm{hom}}}{dr}+(A_2+C)\frac{u_{\mathrm{hom}}}{r}-\Lambda_2T_{\mathrm{hom}} \tag{9}$$

where

$$B=\eta D_1+(1-\eta)D_2,\ A_2=\eta E_1+(1-\eta)E_2 \tag{10a}$$

$$C=\eta C_1+(1-\eta)C_2\,,\ \Lambda_2=\eta F_1+(1-\eta)F_2 \tag{10b}$$

The radial heat flux is continuous at the interfaces:

$$q_r^{(1)}=q_r^{(2)}=q_{\mathrm{hom}}=-K_{\mathrm{hom}}\frac{dT_{\mathrm{hom}}}{dr} \tag{11}$$

where

$$K_{\mathrm{hom}}=\frac{K_1K_2}{\eta K_2+(1-\eta)K_1}. \tag{12}$$

Substituting the constitutive relationships (6), (9), and (11) into Eqs. (4), then Eqs. (4) can be regrouped to form:

$$\frac{d}{dr}\left(r^2\frac{dT_{\mathrm{hom}}}{dr}\right)=0\,,\ r\in(\rho_0,1) \tag{13a}$$

$$A_1\frac{d^2u_{\mathrm{hom}}}{dr^2}+\frac{2A_1}{r}\frac{du_{\mathrm{hom}}}{dr}-2(A_2+C-B)\frac{u_{\mathrm{hom}}}{r^2}=\Lambda_1\frac{dT_{\mathrm{hom}}}{dr}+\frac{2(\Lambda_1-\Lambda_2)}{r}T_{\mathrm{hom}},\ r\in(\rho_0,1). \tag{13b}$$

## 4. CASE 2. THE MODELLING OF GRADIENT TANK

In applications, a gradient material is often used to create a smooth transition from the mechanical and thermal properties of the coating to the properties of the substrate. In order to simplify the analysis of modelling, we consider such a gradient spherical layer independently.

Let the thermomechanical properties on the inner surface of the tank be described by parameters *K1, E1,* $\nu$*1* and $\alpha$*1*. On the external surface of the tank the properties are specified by parameters *K2, E2,* $\nu$*2* and $\alpha$*2.* This means that the parameter $\eta$ assumes limit values: $\eta(r\to\rho0)\to 1$ and $\eta(r\to 1)\to 0$. Inside the tank, the parameter $\eta$ changes along its thickness. We assume that the dependence of the parameter $\eta$ on the coordinate r is described by a linear function:

$$\eta(r)=\frac{1-r}{1-\rho_0}\ ,\ \rho_0<r<1 \tag{14}$$

As follows from the dependencies (7) and (12), the material constants are described by the functions of the r coordinate. Substituting the constitutive relationships (6), (9), and (11) into the Eqs. (4), we obtain the differential equations with variable coefficients:

$$\frac{d}{dr}\left(r^2 K_{\text{hom}}(r)\frac{dT_{\text{hom}}}{dr}\right) = 0 \quad (15a)$$

$$A_1\frac{d^2u_{\text{hom}}}{dr^2} + \left(\frac{dA_1}{dr} + \frac{2A_1}{r}\right)\frac{du_{\text{hom}}}{dr} - 2\left(A_2 + C - \frac{d(rB)}{dr}\right)\frac{u_{\text{hom}}}{r^2} =$$

$$\frac{d(\Lambda_1 T_{\text{hom}})}{dr} + \frac{(\Lambda_1-\Lambda_2)}{r}\, T_{\text{hom}},\ r \in (\rho_0, 1) \quad (15b)$$

The boundary conditions of the resulting boundary problem are described by dependencies Eqs. (5).

## 5. METHOD OF SOLUTION

In the first direct approach, we integrate Eqs. (1). The general solutions can be written in the form:

$$T^{(i)}(r)/T_0 = \theta^{(i)}(r) = t_{2i-1} + t_{2i}r^{-1},\ r_{i-1} \le r \le r_i,\ i = 1,2,\dots,n \quad (16a)$$

$$u^{(i)}(r) = \frac{1-2\nu^{(i)}}{1+\nu^{(i)}}s_{2i-1}r - \frac{1}{2}s_{2i}r^{-2} - \frac{1+\nu^{(i)}}{1-\nu^{(i)}}\alpha^{(i)}T_0\frac{1}{r^2}\int_r^{r_i} x^2\theta^{(i)}(x)dx,$$
$$r_{i-1} \le r \le r_i,\ i = 1,2,\dots,n \quad (16b)$$

The radial displacement described by formulas generates the stresses:

$$\frac{\sigma_r^{(i)}(r)}{2\mu^{(i)}} = s_{2i-1} + \frac{s_{2i}}{r^3} + \frac{1+\nu^{(i)}}{1-\nu^{(i)}}\alpha^{(i)}T_0\frac{2}{r^3}\int_r^{r_i} x^2\theta^{(i)}(x)dx,$$
$$r_{i-1} \le r \le r_i,\ i = 1,2,\dots,n \quad (17a)$$

$$\frac{\sigma_\theta^{(i)}(r)}{2\mu^{(i)}} = s_{2i-1} - \frac{s_{2i}}{2r^3} - \frac{1+\nu^{(i)}}{1-\nu^{(i)}}\alpha^{(i)}T_0\frac{1}{r^3}\int_r^{r_i} x^2\theta^{(i)}(x)dx$$
$$-\frac{1+\nu^{(i)}}{1-\nu^{(i)}}\alpha^{(i)}T_0\theta^{(i)}(r), \quad (17b)$$
$$r_{i-1} \le r \le r_i, \qquad i = 1,2,\dots,n,$$

where $\mu(2j-1) = \mu1$, $\mu(2j) = \mu2$, $j = 1, 2, ..., m$.

Equation (16a) contains the unknown parameters *ti*, *i = 1, 2, ..., 2n*. Substituting Eqs. (16a) into the boundary conditions (2b) and (3b), we obtain the following system of the linear equations:

$$t_1 + t_2 r_0^{-1} = 1 \quad (18a)$$

$$t_{2i-1} + t_{2i}r_i^{-1} - t_{2i+1} - t_{2i+2}r_i^{-1} = 0,\ i = 1,2,\dots,n-1 \quad (18b)$$

$$K^{(i)}t_{2i} - K^{(i+1)}t_{2i+2} = 0,\ i = 1,2,\dots,n-1 \quad (18c)$$

$$t_{2n-1} + t_{2n} = 0 \quad (18d)$$

The unknown parameters si, i = 1, 2, ..., 2n in the Eqs. (16b) and (17) are calculated using the boundary conditions (2a) and (3a). The system of the linear equations is obtained:

$$s_1 + \frac{s_2}{r_0^3} = -2\tilde{t}_0, \quad (19a)$$

$$\frac{1-2\nu^{(i)}}{1+\nu^{(i)}}s_{2i-1}r_i - \frac{s_{2i}}{2r_i^2} - \frac{1-2\nu^{(i+1)}}{1+\nu^{(i+1)}}s_{2i+1}r_i + \frac{s_{2i+2}}{2r_i^2} = -\tilde{t}_i r_i,\ i = 1,2,\dots,n-1, \quad (19b)$$

$$\frac{\mu^{(i)}}{\mu^{(i+1)}}s_{2i-1} + \frac{\mu^{(i)}}{\mu^{(i+1)}}\frac{s_{2i}}{r_i^3} - s_{2i+1} - \frac{s_{2i+2}}{r_i^3} = 2\tilde{t}_i,\ i = 1,2,\dots,n-1 \quad (19c)$$

$$s_{2n-1} + s_{2n} = 0 \quad (19d)$$

where

$$\tilde{t}_i = \frac{1+\nu^{(i+1)}}{1-\nu^{(i+1)}}\alpha^{(i+1)}T_0\left(t_{2i+1}\frac{r_{i+1}^3 - r_i^3}{3r_i^3} + t_{2i+2}\frac{r_{i+1}^2 - r_i^2}{2r_i^3}\right),\ i = 0,1,\dots,n-1. \quad (20)$$

By first solving the system of Eqs. (18) and next, the system (19), and after substituting the constants ti, si, i = 1, 2, ..., 2n into Eqs. (16) and (17), the distributions of the temperature, the radial displacement and the thermal stresses in the investigated multi-layered tank will be found.

In the second alternative approach based on the homogenization method, we integrate Eqs. (13) (transversely isotropic homogeneous material) or Eqs. (15) (transversely isotropic gradient material). If the parameter $\eta$ = const., the obtained Eqs. (13) are the differential equations with constant coefficients. The analytical solution of Eqs. (13), satisfying the boundary conditions Eqs. (5), can be written in form:

$$T_{\text{hom}}(r) = t_{\text{hom}}^{(1)} + t_{\text{hom}}^{(2)}r^{-1},\ \rho_0 \le r \le 1 \quad (21a)$$

$$u_{\text{hom}}(r) = u_{\text{hom}}^{(e)}(r) + u_{\text{hom}}^{(th)}(r),\ \rho_0 \le r \le 1 \quad (21b)$$

where

$$u_{\text{hom}}^{(e)}(r) = s_{\text{hom}}^{(1e)}r^{\gamma_1} + s_{\text{hom}}^{(2e)}r^{\gamma_2},\ \rho_0 \le r \le 1 \quad (22a)$$

$$u_{\text{hom}}^{(th)}(r) = s_{\text{hom}}^{(1th)}r + s_{\text{hom}}^{(2th)},\ \rho_0 \le r \le 1 \quad (22b)$$

$$t_{\text{hom}}^{(1)} = -\frac{T_0\rho_0}{1-\rho_0},\ t_{\text{hom}}^{(2)} = \frac{T_0\rho_0}{1-\rho_0} \quad (22c)$$

$$2\gamma_{1,2} = -1 \pm \sqrt{1+8\gamma_0},\ \gamma_0 = A_1^{-1}(A_2 + C - B) \quad (22d)$$

$$s_{\text{hom}}^{(1th)} = \frac{\Lambda_1-\Lambda_2}{A_1+B-A_2-C}t_{\text{hom}}^{(1)}\ s_{\text{hom}}^{(2th)} = -\frac{\Lambda_1-2\Lambda_2}{2(A_2+C-B)}t_{\text{hom}}^{(2)} \quad (22e)$$

$$s_{\text{hom}}^{(1e)} = \frac{\Lambda_1\rho_0 + (A_1+2B)s_{\text{hom}}^{(1th)}(\rho_0^{\gamma_2}-\rho_0) + 2Bs_{\text{hom}}^{(2th)}(\rho_0^{\gamma_2}-1)}{(A_1\gamma_1+2B)(\rho_0^{\gamma_1}-\rho_0^{\gamma_2})} \quad (22f)$$

$$s_{\text{hom}}^{(2e)} = \frac{\Lambda_1\rho_0 + (A_1+2B)s_{\text{hom}}^{(1th)}(\rho_0^{\gamma_1}-\rho_0) + 2Bs_{\text{hom}}^{(2th)}(\rho_0^{\gamma_1}-1)}{(A_1\gamma_2+2B)(\rho_0^{\gamma_2}-\rho_0^{\gamma_1})} \quad (22g)$$

The radial displacement described by formulas (21b) generates the stresses:

$$\sigma_r^{(1)} = \sigma_r^{(2)} = \sigma_r^{\text{hom}} = (A_1\gamma_1 + 2B)s_{\text{hom}}^{(1e)}r^{\gamma_1-1}$$
$$+(A_1\gamma_2 + B)s_{\text{hom}}^{(2e)}r^{\gamma_2-1} +$$
$$(A_1 + 2B)s_{\text{hom}}^{(1th)} + 2Bs_{\text{hom}}^{(2th)}r^{-1} - \Lambda_1 T_{\text{hom}} \quad (23a)$$

$$\sigma_\theta^{(j)} = \left(D_j\gamma_1 + E_j + C_j\right)s_{\text{hom}}^{(1e)}r^{\gamma_1-1} \quad (23b)$$
$$+\left(D_j\gamma_2 + E_j + C_j\right)s_{\text{hom}}^{(2e)}r^{\gamma_2-1} +$$
$$+\left(D_j + E_j + C_j\right)s_{\text{hom}}^{(1th)} + \left(E_j + C_j\right)s_{\text{hom}}^{(2th)} - F_j T_{\text{hom}},$$
$$j = 1,2.$$

If the parameter $\eta$ changes along the tank thickness, the differential Eqs. (15) are the differential equations with variable coefficients. We can perform analytical integration only in the case of Eq. (15a). The solution of Eq. (15a), which satisfies the boundary conditions (5b), is given in the form:

$$\frac{T_{\text{hom}}(r)}{T_0} = \left(\int_{\rho_0}^1 \frac{dx}{x^2K_{\text{hom}}(x)}\right)^{-1}\left(\int_r^1 \frac{dx}{x^2K_{\text{hom}}(x)}\right),\ \rho_0 \le r \le 1$$

DOI 10.2478/ama-2024-0030

Taking into account the relations Eqs. (14), the function $T_{hom}(r)$, after integration in the Eq. (24), can be written in the form:

$$\frac{T_{\mathrm{hom}}(r)}{T_0} = \frac{(1-r+K_A r\ln(r))\rho_0}{(1-\rho_0+K_A\rho_0\ln(\rho_0))r}\ ,\ \rho_0 \le r \le 1 \tag{25a}$$

where

$$K_A = \frac{K_2-K_1}{K_2-K_1\rho_0} \tag{25b}$$

Equation (15b) will be solved numerically using the finite difference method. The interval *[ρ0,1]* is divided into N equal subintervals. In every internal node the derivatives in Eq. (15b) are replaced with well-known difference equations based on the three nodes. A system containing N – 1 algebraic linear equations was obtained

$$:u_{i-1} - 2u_i + u_{i+1} + 0.5a_i\Delta r(u_{i+1} - u_{i-1}) - b_i(\Delta r)^2 u_i = c_i(\Delta r)^2\ , i = 1,2,\ldots,N-1 \ (26)$$

where

$$a_i = \frac{1}{A_1}\left(\frac{dA_1}{dr} + \frac{2A_1}{r}\right),\ r = \rho_i,\ i = 1,2,\ldots,N-1 \tag{27a}$$

$$b_i = \frac{2}{r^2 A_1}\left(A_2 + C - \frac{d(rB)}{dr}\right),\ r = \rho_i,\ i = 1,2,\ldots,N-1 \tag{27b}$$

$$c_i = \frac{1}{A_1}\left(\frac{d(\Lambda_1 T_{\mathrm{hom}})}{dr} + \frac{2(\Lambda_1-\Lambda_2)}{r}\, T_{\mathrm{hom}}\right),\ r = \rho_i,\ i = 1,2,\ldots,N-1 \tag{27c}$$

Equations (26) contain the unknown parameters $u_i$, $i$ = 0, 2, ..., $N$,, describing the values of the radial displacement u in the nodes ρi = ρ0 + iΔr, where Δr = (1 – ρ0)/N. The obtained system of equations should be supplemented by two equations obtained by substituting constitutive relations (6a) into the boundary conditions (5a). To improve the accuracy of the calculation of the derivative at a point r = ρ0, the well-known five-node difference equations were used:

$$\Delta r\frac{du_{\mathrm{hom}}(\rho_0)}{dr} = -\frac{25}{12}u_0 + 4u_1 - 3u_2 + \frac{4}{3}u_3 - \frac{1}{4}u_4 \tag{28}$$

The equation for the derivative at a point r = 1 is obtained from Eq. (28), substituting parameter Δr by the parameter –Δr, and index i (i = 0, 1, 2, 3 and 4) replacing by the index N – i.

## 6. NUMERICAL RESULTS AND DISCUSSION

The analysis of the received relations shows that the stresses in the homogenized model (second alternative approach to solving the problem) depend on the function η(r) and the six dimensionless parameters: K1/K2, E1/E2, α1/α2, ν1, ν2 and ρ0. However, if the tank is treated as a multi-layered solid (first approach to solving the problem), one should take into account the number of periodicity cells m.

In order to decrease the number of parameters and decrease the range of their changes, the following assumptions are used:

– the ratio between the internal and external radius of tank is 0.5, so ρ0 = 0.5;
– the function η(r) is described by the formula (14) (transversely isotropic gradient material) or the thickness of each spherical layer that is part of the tank is the same, so η = 0.5 (transversely isotropic homogeneous material);
– Poisson's ratios for both components of the periodicity cell are the same and ν1 = ν2 = 0.3;
– one of the components of a representative cell is a thermal insulator. The applied insulating materials are often characterized by a greater Young modulus but a smaller coefficient of linear thermal expansion. For this reason, the assumptions, that E2/E1 = α1/α2 = K1/K2 are taken into account;
– in the aim of an emphasis of possible differences between the solutions obtained by the two presented approaches, some relatively large values of the parameter E1/E2 (or E2/E1) are assumed. We assume that E1/E2 (or E2/E1) = 5 or 10.

Thermal stresses are related to the parameter E*α*T0, where E* = min(E1,E2), α* = max(α1,α2).

Figures 2 and 3 concern the problem, in whit the multi-layered tank with periodic structure is considered. Figure 2 shows the distributions of the radial stress along the tank thickness. Figure 3 presents the distributions of the circumferential stress. The continuous lines in Figs. 2 and 3 and the next figures describe the stresses in the tank obtained within the framework of the homogenization method (second approach to solving the problem). The squares (Fig. 2) or the rhombuses (Fig. 3) mark the numerical results obtained for the non-homogeneous multi-layered tank (first approach to solving the problem). The broken lines in Fig. 2 describe the stress distribution in the homogeneous tank with the parameters E* and α*.

(27c)

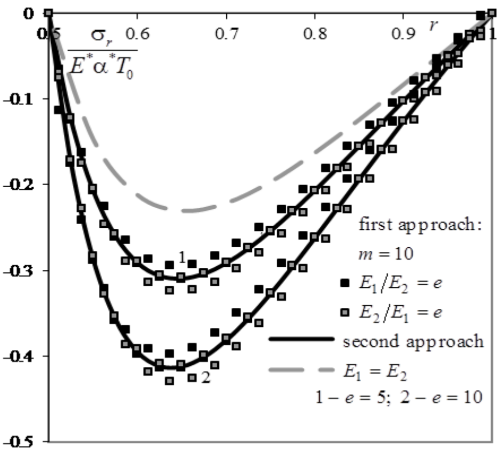


**Fig. 2.** Distribution of the radial stress along the tank thickness; the black lines are for E = max(E1, E2)/min(E1, E2) = e

The radial stress can be treated as a macro-characteristic, which in the second approach does not depend on the choice of the component of the periodicity cell. The distribution of radial stress in the first approach depends on the sequence of spherical layers. This relation in Fig. 2 is described by using black and grey squares. The black squares present the case, when the first component of the periodicity cell is the thermal insulator, and the grey squares for the first layer with larger thermal conductivity coefficient. From Fig. 2, it follows that the radial stress values for the homogenized model are between the adequate values described by black and grey squares. Calculations show that the difference between the locations of the black and grey squares decreases along with an increase of the number of periodicity cells.

An example of a micro-characteristic is the circumferential stress, the distribution of which along the tank thickness is shown

DOI 10.2478/ama-2024-0030 acta mechanica et automatica, vol.18 no.2 (2024)

in Fig. 3. When using the homogenization method, there is no information connected with the kind of spherical layer in the specified point of the tank. At each point we obtain two equations to calculate the circumferential stress. The equation with the index j (see the formula (6b)) allows to determine of the circumferential stress in the j-th (j = 1, 2) layer of the periodicity cell. Two continuous lines in Fig. 3 (black or grey) denoted by numbers 1 or 2 are appropriate for the values of the circumferential stress in layers of the first or second kind. If the circumferential stresses in the multilayered tank are calculated in odd-numbered layers, the adequate rhombuses are consistent with the continuous line denoted by the number 1. Otherwise, the corresponding rhombuses follow the continuous line marked with the number 2. This means that the proposed homogenization method allows for the calculation with a good accuracy not only of macro-parameters, but also of micro-parameters, whose values depend on the type of the considered component of the periodicity cell.

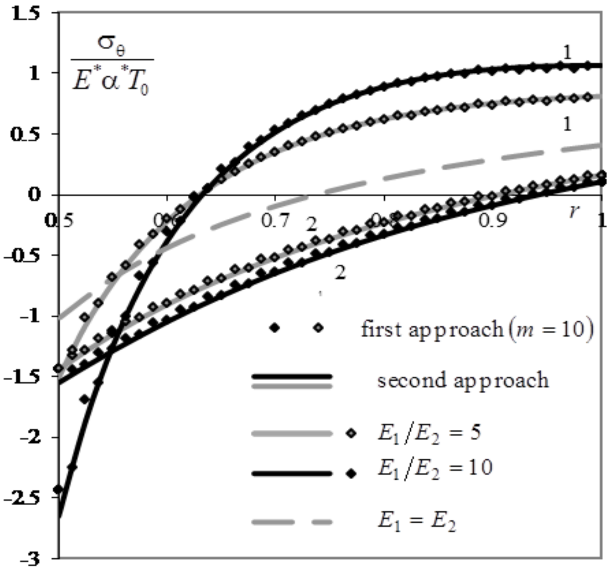


**Fig. 3.** Distribution of the circumferential stress along the tank thickness; the lines marked number 1 are for circumferential stress in layers with a higher Young modulus; the lines marked number 2 are for circumferential stress in layers with a smaller Young modulus

**Tab. 1.** Dependence of the circumferential stress on the external surface of the tank with the periodic structure on the dimensionless parameter E = max(E1, E2)/min(E1, E2) and number of representative cells m

| | ***E*** | **"hom"** | | ***m* = 160 (%)** | ***m* = 80 (%)** | ***m* = 40 (%)** | ***m* = 20 (%)** |
|---|---|---|---|---|---|---|---|
| $\frac{\sigma_\theta(1)}{E^*\alpha^*T_0}$ | 5 | 0.1624, 0.8119 | $E_1/E_2$ = 5 | -0.257 | -0.515 | -1.031 | -2.071 |
| | | | $E_2/E_1$ = 5 | 0.257 | 0.513 | 1.023 | 2.040 |
| | 10 | 0.1066, 1.0660 | $E_1/E_2$ = 10 | -0.273 | -0.549 | -1.104 | -2.236 |
| | | | $E_2/E_1$ = 10 | 0.272 | 0.542 | 1.077 | 2.128 |

As follows from Fig. 3, the highest positive value of the circumferential stress is on the external tank surface. The calculated values of the circumferential stress in the point r = 1 are presented in Tab. 1. It is the greatest tensile stress. In the column marked "hom" are shown the values calculated using the homogenization method. The first (second) number describes the maximum tensile stress in the layer with the smaller (larger) Young's modulus. In order to compare the difference, which is caused by an application of the two proposed approaches to solving the problem, in the first approach the sequence of the layers in the periodicity cell is chosen in such manner, that the correct (with the smaller or larger Young's modulus) layer of the periodicity cell was located at the point r = 1. In the columns with m = 160, 80, 40 and 20 we present the relative deviations (given in percent's) obtained for the multilayered tank with the indicated number of periodicity cells. As can be seen from Tab. 1, the double increase in the layer number cause the double decrease in the difference between the analyzed stresses. It means, that for the adequate number of cells, the mathematical model of the problem can be based on the homogenization method.

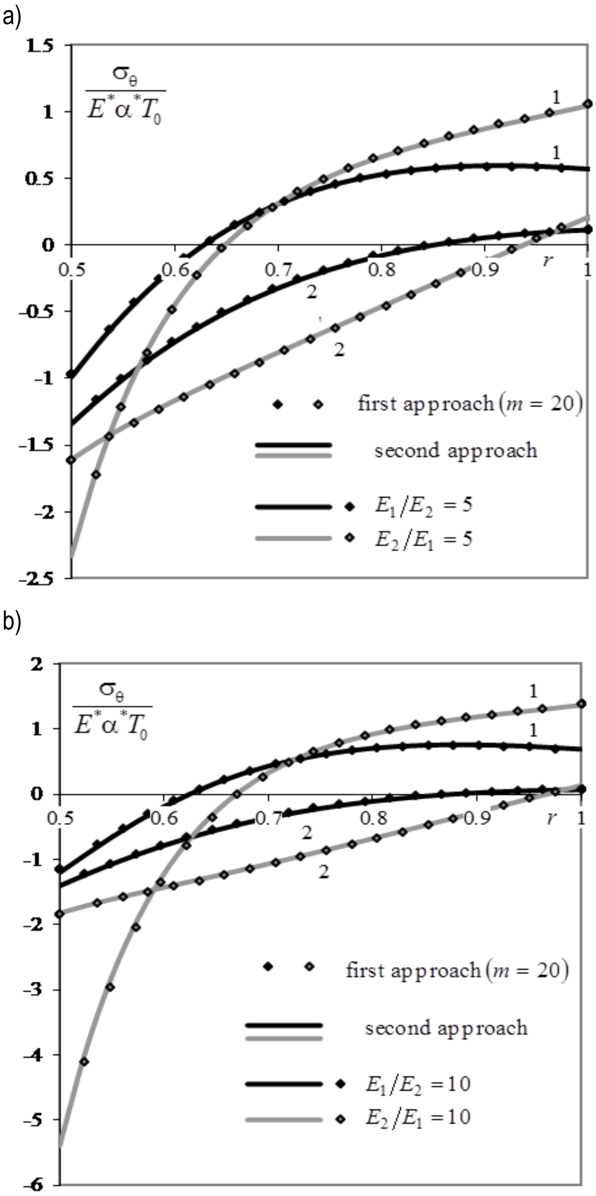


**Fig. 4.** Distribution of the circumferential stress along the tank thickness; the lines marked number 1 are for circumferential stress in layers with a higher Young modulus; the lines marked number 2 are for circumferential stress in layers with a smaller Young modulus

DOI 10.2478/ama-2024-0030

**Tab. 2.** Dependence of the circumferential stress on the surfaces of the tank made of the gradient material on the dimensionless parameter E1/E2 and number of representative cells m

| | | | "hom" | $m$ = 160 (%) | $m$ = 80 (%) | $m$ = 40 (%) | $m$ = 20 (%) |
|---|---|---|---|---|---|---|---|
| $\frac{\sigma_\theta(\rho_0)}{E^*\alpha^* T_0}$ | $E_1 > E_2$ | $E_1/E_2$ = 5 | -0.9911 | -0.267 | -0.526 | -1.021 | -1.921 |
| | | $E_1/E_2$ = 10 | -1.2029 | -0.597 | -1.181 | -2.296 | -4.335 |
| | $E_1 < E_2$ | $E_2/E_1$ = 5 | -1.6090 | 0.085 | 0.168 | 0.329 | 0.632 |
| | | $E_2/E_1$ = 10 | -1.8252 | 0.144 | 0.284 | 0.559 | 1.079 |
| $\frac{\sigma_\theta(1)}{E^*\alpha^* T_0}$ | $E_1 > E_2$ | $E_1/E_2$ = 5 | 0.1143 | -0.270 | -0.536 | -1.063 | -2.090 |
| | | $E_1/E_2$ = 10 | 0.0691 | -0.379 | -0.755 | -1.501 | -2.965 |
| | $E_1 < E_2$ | $E_2/E_1$ = 5 | 1.0471 | 0.137 | 0.272 | 0.538 | 1.047 |
| | | $E_2/E_1$ = 10 | 1.3679 | 0.105 | 0.210 | 0.413 | 0.794 |

Fig. 4 and Tab. 2 describe the distributions of the circumferential stress along the thickness of the tank made of the considered gradient material. It should be emphasized that the stress distributions calculated for parameters E1/E2 and E2/E1 $\neq$ E1/E2 differ significantly from each other. If E1/E2 > 1 (K1/K2 < 1), the thermal properties of the insulator are on the inner surface of the tank. If E1/E2 < 1 (K1/K2 > 1), they are on the external surface. The black lines and rhombuses in Fig. 4 obtained for the values of the parameter E1/E2 = e > 1, while gray lines and rhombuses describe the distribution of the circumferential stress for the values of the parameter E2/E1 = e > 1. As follows from Fig. 4, the extreme negative and positive values of the circumferential stress are mostly on the surfaces of the tank. These values are presented in Tab. 2. Similar to Tab. 1 in the column marked "hom" are shown the values calculated using the homogenization method. In the columns with m = 160, 80, 40 and 20 are shown the relative deviations (given in percent) obtained using the first approach for the solution of the problem. The results of the comparison of solutions obtained using the two described approaches are similar to those presented above. This means that the proposed homogenization method allows for performing calculations with good accuracy not only for problems concerning solids with a periodic structure, but also for problems concerning solids with a slowly graded structure.

## 7. CONCLUSIONS

In this paper, the central-symmetrical problem of thermoselasticity for the multi-layered spherical tank with periodical or slowly graded structure was solved. The solution of the boundary problem defined in the components of the considered inhomogeneous tank was compared with the solution of the problem in which the thermomechanical properties of the tank were described by the method of homogenization with microlocal parameters. If the tank has the periodic structure, the solution based on the homogenization method has the form of simple engineering relations. The good agreement between the obtained analytical solution and the solution of the boundary problem for the heterogeneous tank is an irrefutable argument for the reliability of both solutions. The algorithm for solving the boundary problem for the nonhomogeneous tank is the same for different ways of arranging the layers in the tank. This means that obtaining a good correspondence between the solutions in the case of the slowly graded structure also allows us to conclude that the method of homogenization with microlocal parameters adequately describes the thermomechanical properties of the spherical tank with the considered slowly graded structure. The proposed approach to the homogenization allows to correctly calculate not only the averaged characteristics in the representative cell (the macro-characteristics) but also the characteristics dependent on the choice of the component in the representative cell (the micro-characteristics).

The results obtained allow us to suggest that the proposed approach to the homogenization of a medium with a slowly graded structure will also be effective in other, more complex problems in which the consideration of inhomogeneity of the medium can be extremely laborious or sometimes simply impossible to accomplish.

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