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APPROXIMATION OF THE SPHEROID OFFSET SURFACE AND THE TORUS OFFSET SURFACE

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Abstract. In this paper the issue of approximation of the spheroid offset surface *off*($S(u, v)$; s) at distance s by the spheroid surface $S_1(u, v)$ is considered. The problem of determining the appropriate parameter values for the spheroid offset surface *off*($S(u, v)$; s) is important due to the numerous practical applications of the spheroid as a mathematical model of the Earth. We present the algorithm which gives the appropriate parameter values for the spheroid surface $S(u, v)$ and its offset surface *off*($S(u, v)$; s).

Keywords: spheroid offset surfaces, torus, approximation, geoid

1 Introduction

An oblate spheroid (with semi-axes a=b and c) (cf. Fig. 1) and concentric with it spheroids (with semi-axes $a\pm s$, $a\pm s$ and $c\pm s$) are used as models for imaging and studying phenomena related to the surface of the Earth and factors which cause changes on (and under) its surface. An oblate spheroid is the next, after a geoid, the reference surface approximating the shape of the Earth. In contrast to the geoid, it can be described analytically. The oblate spheroid was introduced to geodesy to investigate and clearly describe the mathematical relationships between elements of the geodetic network projected onto the surface of the ellipsoid, calculate the coordinates of the points of the network so that maps could be made on the basis ofthe network (cf. [2]).

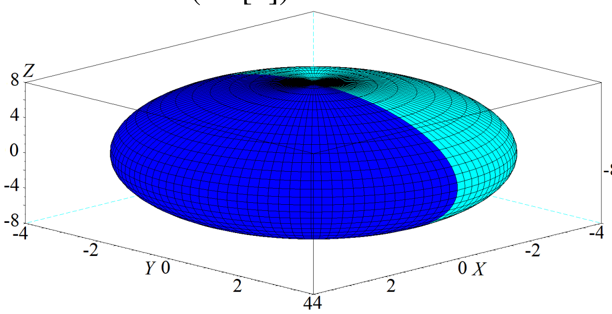

Figure 1: The oblate spheroid (a=b=4, c=8)

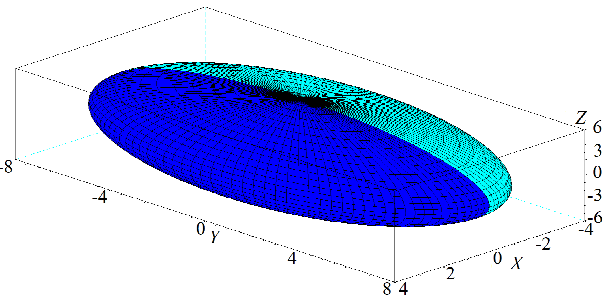

Figure 2: The three-axis ellipsoid (a=4, b=8, c=6)

In many cases, instead of concentric spheroids, it is more convenient to use the spheroid offset surfaces (especially when phenomena propagating perpendicular to the surface of the spheroid are investigated). The distance between any point P of the spheroid surface $S(u, v)$ and the point P_1 of the spheroid offset surface *off*($S(u, v)$; s)) (along the normal line at the point P) is constant. Concentric spheroids do not have this property. It is an important task to find the appropriate parameters for the ellipsoid offset surface *off*($S(u, v)$; s)) and for the ellipsoid $S_1(u, v)$ (which approximates the offset *off*($S(u, v)$; s))). The deviation between these surfaces must not exceed the fixed value k.

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The paper describes two problems: (a) approximation of the spheroid offset surface *off*($S(u, v)$; s)) by the spheroid $S_1(u, v)$ (sections 3, 4) and (b) approximation of the torus offset surface *off*($T(u, v)$; s)) by the torus $T_1(u, v)$ (Section 5).

Let us assume that P is any point on the surface S and l is the normal line to S at the point P. The point P_1 lies on the normal l at distance s from P. Q_1 is the intersection point of the normal line l with the surface S_1.

Section 2 contains mathematical facts necessary to describe research results. In section 3 useful formulas for coordinates of points P_1 and Q_1 (for any point P) for surfaces *off*($S(u, v)$; s)) and $S_1(u, v)$ were determined. Section 5 delivers analogous formulas for the torus offset surface *off*($T(u, v)$; s)). Section 4 contains the results of numerical analysis for the problem of the approximation of the spheroid offset surface *off*($S(u, v)$; s)) by the spheroid $S_1(u, v)$.

2 Mathematical formulas

Let $S(u, v) = (x(u, v), y(u, v), z(u, v))$ ($u \in [u_1, u_2]$, $v \in [v_1, v_2]$) be a smooth parametric surface in 3-dimensional space. $R_0 = (x_0, y_0, z_0)$ is a fixed point of the surface S, $\mathbf{n} = r_1(u_0, v_0) \times r_2(u_0, v_0)$ is a normal vector to the surface S at the point R_0. $R = (x, y, z)$ is any point of a tangent plane τ to the surface S at the point R_0. $R_0R = [x-x_0, y-y_0, z-z_0]$ is a vector. Then the equation of the tangent plane τ is of the form (cf. [6])

$$\tau : \mathrm{n} \circ R_0 R = 0 .$$

The symbol ° means the scalar product of vectors. The normal vector to the surface S at the point R_0 is defined as the vector product of vectors r_1 and r_2 (where r_1, r_2 are tangent vectors to the surface S at the point R_0) (cf. [7])

$$\mathrm{n} = r_1(u_0,v_0) \times r_2(u_0,v_0) =$$

$$= \left[\begin{vmatrix} \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \end{vmatrix}, \begin{vmatrix} \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \end{vmatrix}, \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \end{vmatrix} \right]$$

The length of the vector is expressed as (cf. [6])

$$|\mathrm{n}| = | r_1(u_0,v_0) \times r_2(u_0,v_0) | =$$

$$= \sqrt{ \begin{vmatrix} \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \end{vmatrix}^2 }$$

The unit normal vector to the surface S at the point R_0 is of the form (cf. [6])

$$\mathrm{n}_{\mathrm{ver}} = \frac{r_1(u_0,v_0) \times r_2(u_0,v_0)}{| r_1(u_0,v_0) \times r_2(u_0,v_0) |} .$$

Definition 1 (an offset surface S_d at distance d) (cf. [8], p. 341)
For a given smooth surface S, we define an offset S_d at distance d as follows. On each surface normal, we mark the two points that are at a constant distance d from the surface S. The set of all of these points forms the offset surface S_d. The offset surface $S_d(u, v)$ at distance d to S(u, v) is obtained as $S_d(u,v) = S(u,v) \pm d\,\mathrm{n}_{\mathrm{ver}}(u,v)$.

The interesting offset surfaces (offset curves) are described in [5], [6] and [3].

Lemma 1 (cf. [1]) If $\arccos(x) = \arccos(y)$ and x+y >0 then $x^2 + y^2 = 1$.

3 Spheroid

Let $S(u, v)$ and $S_1(u, v)$ be the concentric spheroid surfaces defined as follows (cf. [4], p. 238)

$$S(u, v)\; x = a\cos(u)\sin(v)\,,\;\; y = a\sin(u)\sin(v)\,,\;\; z = c\cos(v) \tag{1}$$

$$S_1(u, v)\;\; x = (a+s)\cos(u)\sin(v)\,,\;\; y = (a+s)\sin(u)\sin(v)\,,\;\; z = (c+s)\cos(v) \tag{2}$$

where $u \in [0, 2\pi]$, $v \in [0, \pi]$.

The normal vector to the surface $S(u, v)$ at the point P is of the form

$$\mathrm{n} = r_1(u_0, v_0) \times r_2(u_0, v_0) =$$

$$= \left[-ac\cos(u_0)\sin^2(v_0), -ac\sin(u_0)\sin^2(v_0), -a^2\sin(v_0)\cos(v_0)\right]$$

The unit normal vector to $S(u, v)$ at the point P is expressed as follows

$$\mathrm{n}_{\mathrm{ver}} = \frac{r_1(u_0, v_0) \times r_2(u_0, v_0)}{|\, r_1(u_0, v_0) \times r_2(u_0, v_0)\,|} =$$

$$= \frac{\left[-ac\cos(u_0)\sin(v_0), -ac\sin(u_0)\sin(v_0), -a^2\cos(v_0)\right]}{\sqrt{(ac\cos(u_0)\sin(v_0))^2 + (ac\sin(u_0)\sin(v_0))^2 + (a^2\cos(v_0))^2}}$$

The equation of the spheroid offset surfaces is as follows

$$off\,(S(u,v); s) : [X, Y, Z] =$$

$$= [x, y, z] \pm \frac{s\left[-ac\cos(u)\sin(v), -ac\sin(u)\sin(v), -a^2\cos(v)\right]}{\sqrt{(ac\cos(u)\sin(v))^2 + (ac\sin(u)\sin(v))^2 + (a^2\cos(v))^2}} \tag{3}$$

Figure 3 shows the (green) fragment of the three-axis ellipsoid $S(u, v)$ (for a=3, b=4, c=6, u, $v \in [0, \pi/2]$) and the larger (blue) ellipsoid offset surface $off(S(u, v); s)$ for s=1. Figure 4 shows the (green) fragment of the three-axis ellipsoid $S(u, v)$ and the smaller (blue) ellipsoid offset surface $off(S(u, v); s)$. The offset (Fig. 3) and the fragment of the ellipsoid (Fg. 4) were cut out to show the ellipsoid (Fig. 3), the offset (Fig. 4).

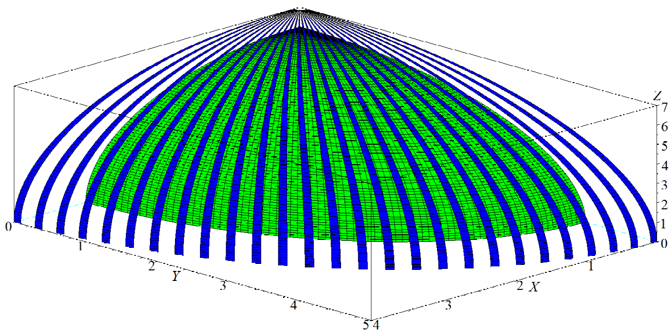

Figure 3: Fragment of the ellipsoid $S(u, v)$ (a=3, b=4, c=6, u, $v \in [0, \pi/2]$) and the offset surface $off(S(u, v); s)$

Let us determine the coordinates of the point P_1 lying on the normal l to the spheroid surface $S(u, v)$ (at the point P) and distant from P by the length s. We use the equation of the spheroid offset surfaces (cf. (3)). The coordinates of the point P_1 are expressed as follows

$$x_{P_1} = x_P + \frac{scx_P}{w},\;\; y_{P_1} = y_P + \frac{scy_P}{w},\;\; z_{P_1} = z_P + \frac{sa^2\cos(v_0)}{w}, \text{ where} \tag{4}$$

$$w = \sqrt{(cx_P)^2 + (cy_P)^2 + (a^2\cos(v_0))^2}\,,$$

$$x_P = a\cos(u_0)\sin(v_0)\,,\;\; y_P = a\sin(u_0)\sin(v_0)\,,\;\; z_P = c\cos(v_0)$$

The coordinates of the point Q_1 (the intersection of the normal line l to the surface $S(u, v)$ at the point P with the surface $S_1(u, v)$) were determined as follows. Let us write down the parametric equations of the normal line l to the spheroid surface $S(u, v)$ at the point P

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$$x = a\cos(u_0)\sin(v_0)(1 + hc\sin(v_0))$$
$$y = a\sin(u_0)\sin(v_0)(1 + hc\sin(v_0)),\ u_0 \in [0, 2\pi],\ v_0 \in [0, \pi],\ h \in R \quad (5)$$
$$z = \cos(v_0)(c + ha^2\sin(v_0)).$$

Let us assume that $u_0 \in [0, \pi/2]$, $v_0 \in [0, \pi/2]$ and set the parameter h giving the intersection points of the line l with the surface $S_1(u, v)$ (cf. (2)). The normal l and the axis Z lie on the same plane, therefore (after simplification) we obtain

$$\begin{cases} a\sin(v_0)(1 + hc\sin(v_0)) = (a + s)\sin(\varphi) \\ \cos(v_0)(c + ha^2\sin(v_0)) = (c + s)\cos(\varphi) \end{cases}.$$ From here

$$\varphi = \arcsin(a\sin(v_0)(1 + hc\sin(v_0))/(a + s)) \quad (6)$$

$\varphi = \arccos(\cos(v_0)(c + ha^2\sin(v_0))/(c + s))$. Hence and from lemma 1 we have

$((a\sin(v_0)(1 + hc\sin(v_0)))/(a + s))^2 + ((\cos(v_0)(c + ha^2\sin(v_0)))/(c + s))^2 = 1$. Hence

$Ah^2 + Bh + C = 0$, where

$$A = a^2\sin^2(v_0)((c + s)^2c^2\sin^2(v_0) + (a + s)^2a^2\cos^2(v_0))$$
$$B = 2a^2c\sin(v_0)((c + s)^2\sin^2(v_0) + (a + s)^2\cos^2(v_0))$$
$$C = (c + s)^2a^2\sin^2(v_0) + (a + s)^2c^2\cos^2(v_0) - (a + s)^2(c + s)^2$$

For $\Delta = B^2 - 4AC > 0$, $h_1 = (-B + \sqrt{\Delta})/2A$, $h_2 = (-B - \sqrt{\Delta})/2A$.

$$x_{Q_1} = a\cos(u_0)\sin(v_0)(1 + h_1c\sin(v_0))$$
$$y_{Q_1} = a\sin(u_0)\sin(v_0)(1 + h_1c\sin(v_0)) \quad (7)$$
$$z_{Q_1} = \cos(v_0)(c + h_1a^2\sin(v_0)).$$

4 Parameter analysis for the spheroid offset surfaces

The approximation of the spheroid offset surface *off*($S(u, v)$; s) by the spheroid surface $S_1(u, v)$ is *satisfactory* if for any point P of the spheroid surface S we have $|\overline{P_1Q_1}| = d_{P_1Q_1} \le k$ for a fixed k (cf. [1], [3]).

We define the eccentricity as $e = \sqrt{1 - (c/a)^2}$. In this section such ranges of parameter values e, s were determined that the approximation of the spheroid offset surface *off*($S(u, v)$; s) by the spheroid surface $S_1(u, v)$ is satisfactory for k=1 (the acceptable deviation between the surfaces). Numerical analysis was carried out for the larger offset surfaces.

In order to check if the approximation of the offset surface *off*($S(u, v)$; s) by the surface $S_1(u, v)$ is satisfactory, one of the following problems should be solved.

(A) We have the semi-axis c, a length s and a deviation k. For the spheroid offset surface *off*($S(u, v)$; s) we need to find such eccentricity e_{Max} that for $e \in [0, e_{Max}]$ and any point P of the spheroid surface S the condition $d_{P_1Q_1}(e) \le k$ is met.

(B) We have the semi-axis c, eccentricity e and a deviation k. For the spheroid offset surface *off*($S(u, v)$; s) we need to find such a distance s_{Max} that for $s \in [1.5, s_{Max}]$ and any point P of the spheroid surface S the condition $d_{P_1Q_1}(s) \le k$ is met.

4.1 Parameters v_1, v_2

In this section, for the (larger) spheroid offset surface *off*($S(u, v)$; s) and the spheroid surface $S_1(u, v)$ the specific angles v_1 and v_2 were determined. For the established values c and s and the angle v_1 we calculate the value e_{Max}. For the established values c and e and the angle v_2 we calculate the value s_{Max}.

Test 1. The spheroid surfaces were considered for parameter values c=5, 10, …, 100 and $e=\sqrt{1-(c/a)^2}$. The length $s\in[4, 100]$ was tested (every 1). The eccentricity $e\in[0, 1)$ was taken (every 0.05). The permissible deviation is k=1. The angles v_1, $v_2\in[0, \pi/2)$ were taken (every $\pi/36$), the angle u=0°. The following facts have been verified for the larger spheroid offset surfaces. For the assumed values c and s, the value e_{Max} occurs for v_1=50° - 70° (see Table 1). For the assumed values c and e, the value s_{Max} occurs for v_2=50° - 65° (see Table 2).

Table 1. For given c and s the value e_{Max} (such that for $e\in[0, e_{Max}]$ $d_{P_1Q_1}$ (e)≤1) was obtained for the angle v_1

v	c=5	c=10	c=15	c=20	c=25	c=30	c=35	c=40	c=45	c=50
v=70°	$s\in[4,8]$	$s\in[4]$	$s\in[4]$	-	-	-	-	-	-	-
v=65°	$s\in[9,100]$	$s\in[5,14]$	$s\in[5,8]$	$s\in[4,7]$	$s\in[4,7]$	$s\in[4,6]$	$s\in[4,6]$	$s\in[4,6]$	$s\in[4,6]$	$s\in[4,6]$
v=60°	-	$s\in[15,100]$	$s\in[9,100]$	$s\in[8,47]$	$s\in[8,25]$	$s\in[7,20]$	$s\in[7,17]$	$s\in[7,16]$	$s\in[7,15]$	$s\in[7,14]$
v=55°	-	-	-	$s\in[48,100]$	$S\in[26,100]$	$s\in[21,100]$	$s\in[18,100]$	$s\in[17,100]$	$s\in[16,100]$	$s\in[15,100]$
	c=55	c=60	c=65	c=70	c=75	c=80	c=85	c=90	c=95	c=100
v=65°	$s\in[4,6]$	$s\in[4,6]$	$s\in[4,6]$	$s\in[4,5]$	$s\in[4,5]$	$s\in[4,5]$	$s\in[4,5]$	$s\in[4,5]$	$s\in[4,5]$	$s\in[4,5]$
v=60°	$s\in[7,14]$	$s\in[7,13]$	$s\in[7,13]$	$s\in[6,13]$	$s\in[6,12]$	$s\in[6,12]$	$s\in[6,12]$	$s\in[6,12]$	$s\in[6,12]$	$s\in[6,12]$
v=55°	$s\in[15,100]$	$s\in[14,100]$	$s\in[14,100]$	$s\in[14,100]$	$s\in[13,85]$	$s\in[13,75]$	$s\in[13,68]$	$s\in[13,63]$	$s\in[13,59]$	$s\in[13,56]$
v=50°	-	-	-	-	$S\in[86,100]$	$s\in[76,100]$	$s\in[69,100]$	$s\in[64,100]$	$s\in[60,100]$	$s\in[57,100]$

Table 2. For given c and e the value s_{Max} (such that for $s\in[1.5, s_{Max}]$ $d_{P_1Q_1}$ (s)≤1) was obtained for the angle v_2

v	c=5	c=10	c=15	c=20	c=25	c=30	c=35	c=40	c=45	c=50
[5°-85°]	$e\in[0,0.9]$	$e\in[0,0.85]$	$E\in[0,0.85]$	$e\in[0,0.8]$	$e\in[0,0.8]$	$e\in[0,0.8]$	$e\in[0,0.8]$	$e\in[0,0.75]$	$e\in[0,0.75]$	$e\in[0,0.75]$
v=55°	-	-	-	$e\in[0.85]$	$e\in[0.85]$	$e\in[0.85]$	$e\in[0.85]$	$e\in[0.8,0.85]$	$e\in[0.8,0.85]$	$e\in[0.8,0.9]$
v=60°	-	$e\in[0.9]$	$e\in[0.9]$	$e\in[0.9,0.95]$	$e\in[0.9,0.95]$	$e\in[0.9,0.95]$	$e\in[0.9,0.95]$	$e\in[0.9,0.95]$	$e\in[0.9,0.95]$	$e\in[0.95]$
v=65°	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	-	-	-	-	-	-	-
	c=55	c=60	c=65	c=70	c=75	c=80	c=85	c=90	c=95	c=100
[5°-85°]	$e\in[0,0.75]$	$e\in[0,0.75]$	$E\in[0,0.75]$	$e\in[0,0.75]$	$e\in[0,0.7]$	$e\in[0,0.7]$	$e\in[0,0.7]$	$e\in[0,0.7]$	$e\in[0,0.7]$	$e\in[0,0.7]$
v=50°	-	-	-	-	$e\in[0.75]$	$e\in[0.75]$	$e\in[0.75]$	$e\in[0.75]$	$e\in[0.75]$	$e\in[0.75]$
v=55°	$e\in[0.8,0.9]$	$e\in[0.8,0.9]$	$E\in[0.8,0.9]$	$e\in[0.8,0.9]$	$e\in[0.7,0.9]$	$e\in[0.8,0.9]$	$e\in[0.8,0.9]$	$e\in[0.8,0.9]$	$e\in[0.8,0.9]$	$e\in[0.8,0.9]$
v=60°	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$	$e\in[0.95]$

Example 1. Let us assume that c=35 and k=1. We need to find such length of the semi-axis a so that the approximation of the offset surface *off*($S(u, v)$; s) by the surface $S_1(u, v)$ is satisfactory. (a) s=10, (b) s=20.

(a) Let s=10. We determine the values e_{Max} for v_1=60° (see Table 1 and 3), because c=35 and s=10∈[7, 17]. The approximation will be satisfactory if we choose a≤96.542 (Table 3).

(b) Let s=20. We determine the values e_{Max} for v_2=55° (see Table 1 and 3), because c=35 and s=20∈[18, 100]. The approximation will be satisfactory if we choose a≤75.316 (Table 3).

Table 3. The values e_{Max} (a_{Max}) (such that for $e\in[0, e_{Max}]$ $d_{P_1Q_1}$ (s)≤1) for given c, s and the angle v_1=60° / v_1=55°

The values e_{Max} (a_{Max}) for given s, c and the angle v_1=60°										
s=4	s=10	s=20	s=30	s=40	s=50	s=60	s=70	s=80	s=90	s=100
e_{Max}=0.983 (a_{Max}=191.1)	0.932 (96.542)	0.886 (75.443)	0.86 (68.648)	0.844 (65.215)	0.832 (63.128)	0.824 (61.72)	0.817 (60.704)	0.812 (59.937)	0.807 (59.335)	0.804 (58.851)
The values e_{Max} (a_{Max}) for given s, c and the angle v_1=55°										
s=4	s=10	s=20	s=30	s=40	s=50	s=60	s=70	s=80	s=90	s=100
e_{Max}=0.987 (a_{Max}=221.702)	0.934 (98.097)	0.885 (75.316)	0.858 (68.271)	0.841 (64.763)	0.829 (62.646)	0.82 (61.225)	0.814 (60.203)	0.808 (59.432)	0.804 (58.829)	0.8 (58.344)

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Example 2. Let us assume, that c=35 and k=1. We need to find such length s so that the approximation of the offset surface $off(S(u, v); s)$ by the surface $S_1(u, v)$ is satisfactory. (a) a=80 (e=0.899218411).

(a) Let a=80. We determine the values s_{Max} for v_2=60° (see Table 2 and 4). The approximation will be satisfactory if we choose $s \le s_{Max}$=16.357 (see Table 4).

Table 4. The values s_{Max} (such that for $s \in [1.5, s_{Max}]$ $d_{P_1Q_1}(s) \le 1$) for given c, e and the angle v_2=60°. The [+] symbol means: $d_{P_1Q_1}(s) \le 1$ for $s \in [4, s_{Max}]$

The values s_{Max} for given e, c and the angle v_2=60°										
e=0	e=0.1	e=0.2	e=0.3	e=0.4	e=0.5	e=0.6	e=0.7	e=0.8	e=0.899218	e=0.9
s_{Max}=[+]	[+]	[+]	[+]	[+]	[+]	[+]	[+]	[+]	16.357	16.168

5 Torus

Let $T(u, v)$ and $T_1(u, v)$ be the torus surfaces defined as follows

$$T(u, v) \quad x = \cos(u)(R + r\cos(v)),\; y = \sin(u)(R + r\cos(v)),\; z = r\sin(v) \tag{8}$$

$$T_1(u, v) \quad x = \cos(u)(R + (r+s)\cos(v)),\; y = \sin(u)(R + (r+s)\cos(v)),\; z = (r+s)\sin(v) \tag{9}$$

where $u \in [0, 2\pi]$, $v \in [0, 2\pi]$.

The normal vector to the torus $T(u, v)$ at the point P is of the form

$$\mathrm{n} = r_1(u_0, v_0) \times r_2(u_0, v_0) =$$

$$= \left[r\cos(u_0)\cos(v_0)(R + r\cos(v_0)), r\sin(u_0)\cos(v_0)(R + r\cos(v_0)), r\sin(v_0)(R + r\cos(v_0))\right]$$

The unit normal vector to $T(u, v)$ at the point P is expressed as follows

$$\mathrm{n}_{\mathrm{ver}} = \frac{r_1(u_0, v_0) \times r_2(u_0, v_0)}{| r_1(u_0, v_0) \times r_2(u_0, v_0) |} = [\cos(u_0)\cos(v_0), \sin(u_0)\cos(v_0), \sin(v_0)]$$

The equation of the torus offset surfaces is of the form

$$off(T(u,v); s) : [X, Y, Z] = [x, y, z] \pm s[\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)]$$

The torus surface $T(u, v)$ and its offset surfaces $off(T(u, v); s)$ are of the same type.

Clever proof. The surface $T(u, v)$ is defined above (cf. (8)). Let us write down the parametric equations of the normal line l to the torus surface $T(u, v)$ at the point P ($h \in R$)

$$\begin{cases} x = \cos(u_0)(R + r\cos(v_0)) + h\cos(u_0)\cos(v_0) \\ y = \sin(u_0)(R + r\cos(v_0)) + h\sin(u_0)\cos(v_0) \\ z = r\sin(v_0) + h\sin(v_0) \end{cases} \text{. Hence } \begin{cases} x = \cos(u_0)(R + (r+h)\cos(v_0)) \\ y = \sin(u_0)(R + (r+h)\cos(v_0)) \\ z = (r+h)\sin(v_0) \end{cases}$$

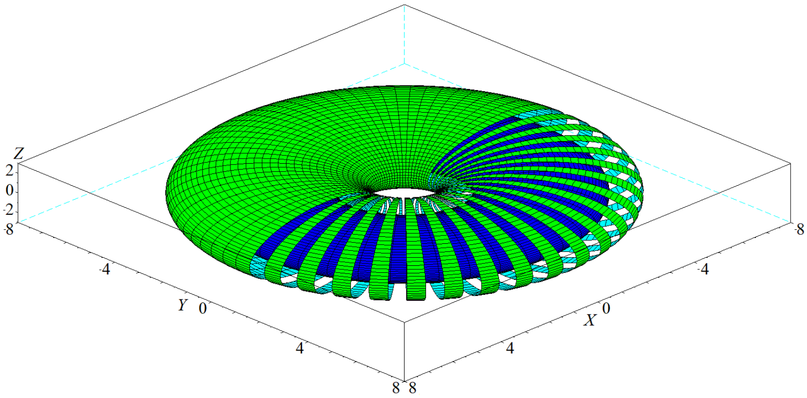
$u_0 \in [0, 2\pi]$, $v_0 \in [0, 2\pi]$. We obtained the equations of the surface $T_1(u, v)$ (cf. (9)) for h=s.

The coordinates of the point P_1 lying on the normal l to the torus surface $T(u, v)$ (at the point P) and distant from P by the length s:

$$x_{P_1} = x_{Q_1} = \cos(u_0)(R + (r+s)\cos(v_0))$$

$$y_{P_1} = x_{Q_1} = \sin(u_0)(R + (r+s)\cos(v_0))$$

$$z_{P_1} = x_{Q_1} = (r+s)\sin(v_0)$$



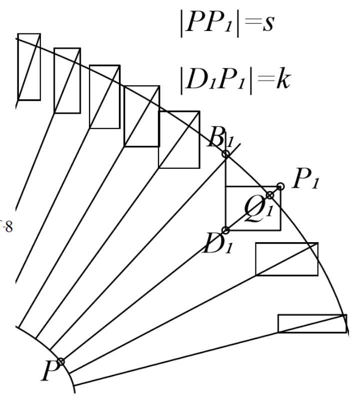


Figure 5: The torus surface $T(u, v)$ and its offset surface $off(T(u, v); s)$ Figure 6: The arrangement of the points

6 Algorithm

The algorithm presented below finds such parameter values for the spheroid offset surface $off(S(u, v); s))$ and for the spheroid $S_1(u, v)$ that the deviation between these surfaces does not exceed the fixed value k. The user gives parameter values (the semi-axis c, a length s and a deviation k) and receives such value e_{Max} that for each value $e \in [0, e_{Max}]$ the approximation of the offset surface $off(S(u, v); s)$ by the surface $S_1(u, v)$ is satisfactory.

Justification of the algorithm. The surfaces $S(u, v)$ and $S_1(u, v)$ are defined in section 3 (cf. (1), (2)). We limit the calculations to the fragment of the spheroid surface for $u \in [0, \pi/2]$, $v \in [0, \pi/2]$. The normal l and the axis Z lie on the same plane, so we can assume that $u=0°$. The points P, P_1, Q_1, D_1 and the normal line l are defined in Figure 6. The point B_1 belongs to the surface $S_1(u, v)$ for $u=0°$ and has the same coordinate x as the point D_1 (see Figure 6). The algorithm for consecutive angles $v \in [0, \pi/2]$ (taken every dt) calculates the value e_{Max} and then (for the value e_{Max}) computes the coordinates of the points P, P_1, D_1, Q_1 and the distance $d_{P_1Q_1}$.

```
void elipsoida::Algo3D(double dt, double s, double k){
//definitions of variables
i=1; v=dt;
while(v<M_PI/2){
  e=0; b1=0;
  while(e<0.99){
    a=sqrt((c*c)/(1-(e*e)));
    xP=a*sin(v);                 zP=c*cos(v);
    w=pow((c/a)*tan(v),2);
    xP1=xP+s*sqrt(w/(1+w));      zP1=zP+s/sqrt(1+w);
    xD1=xP+(s-k)*sqrt(w/(1+w)); zD1=zP+(s-k)/sqrt(1+w);
    fi=asin(xD1/(a+s));
    zB1=(c+s)*cos(fi);
    if(zB1>zD1){e+=0.000005; b1=1;}else{e-=0.000005; if(b1)break;}}
  a=sqrt((c*c)/(1-(e*e)));
  xP=a*sin(v);                   zP=c*cos(v);
  w=pow((c/a)*tan(v),2);
  xP1=xP+s*sqrt(w/(1+w));        zP1=zP+s/sqrt(1+w);
  xD1=xP+(s-k)*sqrt(w/(1+w));    zD1=zP+(s-k)/sqrt(1+w);
  xQ1=(a+s)*sin(fi);             zQ1=(c+s)*cos(fi);
  dpq1=sqrt(pow((xQ1-xP1),2)+pow((zQ1-zP1),2));
  cout << v << " " << e << " " << xQ1 << " " << zQ1 << " " << dpq1;
  i++; v=i*dt;}}
```

Results of the algorithm.

```
Data: c=45, dt=p/18, s=16, k=1
```

Result:

v	e	xQ1	zQ1	dpq1
10	0,99	55,765686	60,148874	0,164491
20	0,99	109,871472	57,625808	0,641438
30	0,969855	94,423220	53,824826	0,999993
40	0,927695	81,961875	48,786076	0,999978
50	0,901485	86,523090	42,256581	0,999955
60	0,89637	97,045264	34,397474	0,999973
70	0,913535	115,136695	25,393293	0,999998
80	0,952735	158,845361	15,344069	0,999909

Conclusions

The paper describes the problem of approximation of the spheroid offset surface *off*($S(u, v)$; s) at distance s by the spheroid $S_1(u, v)$. In section 3 useful formulas for coordinates of points P_1 and Q_1 (for any point P) were calculated. Section 4 presents the method of determining parameter values for the offset surface *off*($S(u, v)$; s) and for the surface $S_1(u, v)$ (which approximates it). If the parameter values are correctly selected, the distance between one surface and the other one does not exceed the given deviation k. Section 5 contains the justification of the fact that the torus surface $T(u, v)$ and its offset surface *off*($T(u, v)$; s) are of the same type. Section 6 contains the algorithm which finds such parameter values for the spheroid offset surface *off*($S(u, v)$; s)) and for the spheroid surface $S_1(u, v)$ that the deviation between these surfaces does not exceed the set value k.

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APROKSYMACJA POWIERZCHNI OFFSETOWEJ ELIPSOIDY I POWIERZCHNIA OFFSETOWA TORUSA

W niniejszej pracy rozważa się kwestię aproksymacji powierzchni offsetowej elipsoidy *off*($S(u, v)$; s) o odległości s przez elipsoidę $S_1(u, v)$. Umiejętność doboru odpowiednich wartości parametrów dla powierzchni elipsoidy i jej offsetu jest istotna ze względu na liczne zastosowania praktyczne elipsoidy spłaszczonej jako matematycznego modelu kuli ziemskiej. Prezentujemy algorytm, który zwraca odpowiednie wartości parametrów dla powierchni elipsoidy i jej offsetu.