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Poles and zeros assignment by state feedbacks in positive linear systems

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Poles and zeros assignment problem by state feedbacks in positive continuous-time and discrete-time systems is analyzed. It is shown that in multi-input multi-output positive linear systems by state feedbacks the poles and zeros of the transfer matrices can be assigned in the desired positions. In the positive continuous-time linear systems the feedback gain matrix can be chosen as a monomial matrix so that the poles and zeros of the transfer matrices have the desired values if the input matrix B is monomial. In the positive discrete-time linear systems to solve the problem the matrix B can be chosen monomial if and only if in every row and every column of the $n \times n$ system matrix A the sum of $n-1$ its entries is less than one.

Key words: assignment, pole, zero, transfer matrix, linear, positive, system, state feedback

1. Introduction

A dynamical system is called positive if its state variables and outputs take nonnegative values for any nonnegative inputs and nonnegative initial conditions [2–5]. Models having positive behavior can be found in engineering, electrical circuits, economics, social sciences, biology medicine etc.

It is well-known [1–12] that if the pair $(A,\ B)$ of the linear systems is controllable then using the state feedback we may assign the poles of the closed-loop systems in the desired state positions.

In single-input single-output linear systems by the use of the state feedbacks we may modify the positions only of its poles [3, 7, 12]. In multi-inputs multi-outputs linear systems by the use of the state feedbacks we may also modify the positions of the zeros of their transfer matrices [3]. In this paper the state feedbacks in multi-inputs multi-outputs linear systems will be apply to assign the poles and zeros in the desired positions(regions).

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The paper is organized as follows. In section 2 the basic definitions and theorems concerning positive continuous-time and discrete-time linear systems are recalled. The main result of the paper for positive continuous-time linear systems are given in section 3 and for positive discrete-time linear systems in section 4. Concluding remarks are given in section 5.

The following notations will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Positive continuous-time and discrete-time linear systems

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 1 *[2–4] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geqslant 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geqslant 0$.*

Theorem 1 *[5, 6] The continuous-time linear system (1) is positive if and only if*

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{2}$$

Definition 2 *[5, 6] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable (shortly –stable) if*

$$\lim_{t\to\infty} x(t) = 0 \;\; \textit{for any} \;\; x(0) \in \Re_+^n. \tag{3}$$

Theorem 2 *[5, 6] The positive continuous-time linear system (1) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

All coefficient of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

There exists strictly positive vector $\lambda^T = [\,\lambda_1 \;\cdots\; \lambda_n]^T\,]$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0 \quad or \quad \lambda^T A < 0. \tag{5}$$

The transfer matrix of the system (1) is given by

$$T(z) = C[I_n z - A]^{-1} B\,. \tag{6}$$

Consider linear discrete-time system described by the equations

$$x_{i+1} = Ax_i + Bu_i\,, \quad i \in Z_+ = \{0,\ 1,\ \ldots\}, \tag{7a}$$

$$y_i = Cx_i\,, \tag{7b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 3 *[2–4] The system (7) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.*

Theorem 3 *[5] The system (7) is positive if and only if*

$$A \in \Re_+^{n\times n}, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}. \tag{8}$$

The transfer matrix of the system (7) is given by

$$T(z) = C[I_n z - A]^{-1} B. \tag{9}$$

Definition 4 *[5] The system (7) is called asymptotically stable (shortly stable) if the matrix A is a Schur matrix.*

Theorem 4 *[5] The positive system (7) is asymptotically stable if and only if*

1. *All coefficients of the polynomial*

$$p_A(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \tag{10a}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. *There exists strictly positive vector $\lambda^T = [\,\lambda_1 \ \cdots \ \lambda_n\,]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$A\lambda < \lambda \quad \text{or} \quad \lambda^T A < \lambda. \tag{10b}$$

3. Positive continuous-time linear systems

3.1. Problem formulation

Consider the continuous-time linear system shown in Fig. 1 with the positive linear system described by the equations (1) and the state feedback

$$u = v - Kx, \quad i = 0, 1, \ldots, \tag{11}$$

where $K \in \Re^{n\times n}$ and $v \in \Re^n$ is a new input.

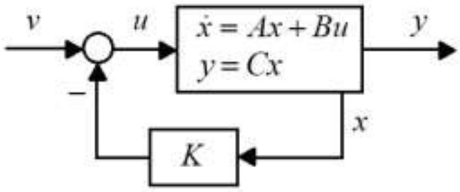

Figure 1: Feedback linear system

It is assumed that:

1) the matrix $A \in M_n$ is unstable,

2) the matrix $B \in \Re_+^{n\times n}$ is nonsingular, $\det B \neq 0$, $C \in \Re_+^{p\times n}$, $p > 1$.

Substituting (11) into (1a) we obtain

$$\dot{x} = A_c x + Bu, \tag{12a}$$

$$y = Cx, \tag{12b}$$

where

$$A_c = A - BK. \tag{13}$$

The transfer matrix of the closed-loop system (12) has the form

$$T_c(s) = C[I_n s - A_c]^{-1} B = \frac{N(s)}{d(s)}, \tag{14a}$$

where

$$d(s) = \det[I_n s - A_c] = s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0, \tag{14b}$$

$$N(s) = C[I_n s - A_c]_{ad} B = \begin{bmatrix} n_{11}(s) & \ldots & n_{1n}(s) \\ \ldots & \ldots & \ldots \\ n_{p1}(s) & \ldots & n_{pn}(s) \end{bmatrix} \tag{14c}$$

and $[I_n s - A_c]_{ad}$ is the adjoint matrix.

The zeros: $s_1, s_2, \ldots, s_n$ are the poles of the transfer matrix (14) and the zeros: $s_1^{ij}, s_2^{ij}, \ldots, s_{n-1}^{ij}$ of $n_{ij}(s)$, $i = 1, \ldots, p$, $j = 1, \ldots, n$ are its zeros.

The problem under the considerations can be stated as follows.

Given:

1) the matrices $A \in M_n$, $B \in \Re_+^{n\times n}$, $\det B \neq 0$, $C \in \Re_+^{p\times n}$, $p > 1$,

2) the desired poles of the transfer matrix (14a).

Find the gain matrix K such that the closed-loop transfer matrix (14) has the desired stable poles and stable zeros.

3.2. Problem solution

It is assumed that the desired closed-loop matrix $A_c \in M_n$ has the following diagonal form

$$A_c = \operatorname{diag}[-s_1, -s_1, \ldots, -s_n] \in M_n, \quad s_k > 0 \text{ for } k = 1, 2, \ldots, n. \tag{15}$$

Knowing the matrices A, A_c and the nonsingular matrix B using (13) and (15) we may find the feedback matrix

$$K = B^{-1}(A - A_c). \tag{16}$$

We shall show that the transfer matrix of the closed-loop system with the state feedback (11) has stable all its poles and zeros.

Substituting the matrix (15) into (14a) we obtain

$$\begin{aligned}
T_c(s) &= C[I_n s - A_c]^{-1}B \\
&= \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{p1} & \ldots & c_{pn} \end{bmatrix} \begin{bmatrix} s+s_1 & 0 & \ldots & 0 \\ 0 & s+s_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & s+s_n \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \\
&= \frac{1}{d(s)} \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{p1} & \ldots & c_{pn} \end{bmatrix} \\
&\times \begin{bmatrix} (s+s_2)\ldots(s+s_n) & 0 & \ldots & 0 \\ 0 & (s+s_1)(s+s_3)\ldots(s+s_n) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & (s+s_1)\ldots(s+s_{n-1}) \end{bmatrix} \\
&\times \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \\
&= \frac{1}{d(s)} \begin{bmatrix} n_{11}(s) & \ldots & n_{1n}(s) \\ \ldots & \ldots & \ldots \\ n_{p1}(s) & \ldots & n_{pn}(s) \end{bmatrix},
\end{aligned} \tag{17a}$$

where

$$\begin{aligned}
d(s) &= (s+s_1)(s+s_2)\ldots(s+s_n), \\
n_{11}(s) &= b_{11}c_{11}(s+s_2)\ldots(s+s_n) + \ldots + b_{n1}c_{pn}(s+s_1)\ldots(s+s_{n-1}), \\
n_{1n}(s) &= b_{1n}c_{11}(s+s_2)\ldots(s+s_n) + \ldots + b_{nn}c_{1n}(s+s_1)\ldots(s+s_{n-1}), \\
n_{p1}(s) &= b_{11}c_{p1}(s+s_2)\ldots(s+s_n) + \ldots + b_{n1}c_{pn}(s+s_1)\ldots(s+s_{n-1}), \\
n_{pn}(s) &= b_{1n}c_{p1}(s+s_2)\ldots(s+s_n) + \ldots + b_{nn}c_{pn}(s+s_1)\ldots(s+s_{n-1}).
\end{aligned} \tag{17b}$$

From (17) it follows that all zeros of the transfer matrix (17a) are real and stable.

Therefore, the following theorem has been proved.

Theorem 5 *If the matrix $B \in \Re_+^{n\times n}$ is nonsingular then there exists the feedback matrix K and the closed-loop system matrix A_c such that the poles and zeros of the transfer matrix (14a) are stable.*

Example 1 Consider the positive linear system (1) with the matrices

$$A = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{18}$$

with the state feedback matrix

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}. \tag{19}$$

The Metzler matrix A given by (18) is unstable since its characteristic polynomial

$$\det[I_2 s - A] = \begin{vmatrix} s+1 & -3 \\ -2 & s+2 \end{vmatrix} = s^2 + 3s - 4 \tag{20}$$

has the zeros: $s_1 = 1$, $s_2 = -4$.

The matrix B given by (18) is nonsingular. Therefore, using (14) and (18) we obtain

$$K = B^{-1}(A - A_c) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

The transfer matrix of the closed-loop system has the form

$$T_c(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{s^2 + 9s + 14} \begin{bmatrix} 3s+11 & 2s+9 \end{bmatrix} \tag{22}$$

and its poles are: $s_1^1 = -2$, $s_1^2 = -7$. Note that zeros of (22) are stable.

3.3. Positive linear systems with monomial B matrices

A square matrix is called monomial if its every column and its every row has only one positive entry and the remaining entries are zero The inverse matrix of monomial matrix can be found by its transposition and replacing each element of the transposed matrix by its inverse. Therefore, the inverse matrix of monomial matrix is also monomial matrix [3, 5].

Consider the positive system (1a) with the matrices $A \in M_n$ unstable and $B \in \Re_+^{n\times n}$ monomial. For given asymptotically stable matrix $A_c \in M_n$ and monomial matrix B the feedback matrix K defined by (16) is also monomial matrix since the inverse B^{-1} of the matrix B is monomial and the matrix $A - A_c$ is diagonal with nonnegative diagonal entries. Therefore, the following theorem has been proved.

Theorem 6 *If the matrix* $B \in \Re_+^{n\times n}$ *is monomial then the feedback matrix* K *defined by (16) is also monomial matrix.*

Example 3.2 Consider the positive linear system (1) with the matrices

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}. \tag{23}$$

Note that the matrix A is unstable and the matrix B (given by (23)) is monomial. Let the desired matrix of the closed-loop system has the form

$$A_c = \begin{bmatrix} -5 & 1 & 2 \\ 2 & -4 & 1 \\ 2 & 2 & -5 \end{bmatrix}. \tag{24}$$

The using (16), (23) and (24) we obtain

$$\begin{aligned} K &= B^{-1}(A - A_c) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} -1 & 1 & 2 \\ 2 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix} - \begin{bmatrix} -5 & 1 & 2 \\ 2 & -4 & 1 \\ 2 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{25}$$

In this case the transfer matrix of the closed-loop system has the form

$$\begin{aligned} T_c(s) &= C[I_n s - A_c]^{-1} B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} s+5 & -1 & -2 \\ -2 & s+4 & -1 \\ -2 & -2 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \frac{N(s)}{d(s)}, \end{aligned} \tag{26a}$$

where

$$\begin{aligned} d(s) &= s^3 + 14s^2 + 57s + 54 = (s+s_1)(s+s_2)(s+s_3), \\ s_1 &= -6.645, \quad s_2 = -6, \quad s_3 = -1.354, \end{aligned} \tag{26b}$$

$$N(s) = \begin{bmatrix} n_{11}(s) & n_{12}(s) & n_{13}(s) \\ n_{21}(s) & n_{22}(s) & n_{23}(s) \end{bmatrix}, \tag{26c}$$

$$\begin{aligned} &n_{11}(s) = s^2 + 11s + 27, \quad n_{12}(s) = s^2 + 11s + 30, \quad n_{13}(s) = 6s + 42, \\ &n_{21}(s) = s^2 + 11s + 36, \quad n_{22}(s) = 6s + 36, \quad n_{23}(s) = 4s^2 + 44s + 108. \end{aligned}$$

Note that the zeros of the matrix (26c) are stable.

4. Positive discrete-time linear systems

In this section the results of section 3 will be extended to the positive discrete-time linear systems.

Consider the discrete-time linear system shown in Fig. 2 with the positive discrete-time linear system (7) and the state-feedback

$$u_i = v_i - Kx_i, \qquad i = 0, 1, \ldots, \tag{27}$$

where $K \in \Re^{n\times n}$ and $v_i \in \Re^i$ is a new input.

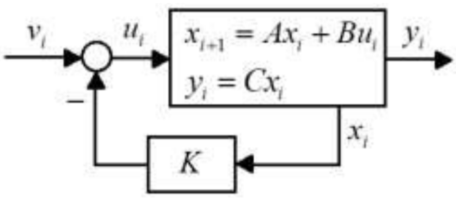

Figure 2: Feedback linear system

It is assumed that:

1) the matrix $A \in \Re^{n\times n}$ is unstable,
2) the matrix $B \in \Re_+^{n\times n}$ is nonsingular, $\det B \neq 0$, $C \in \Re_+^{p\times n}$, $p > 1$.

Substituting (27) into (7a) we obtain

$$x_{i+1} = A_c x_i + Bu_i, \qquad i = 0, 1, \ldots, \tag{28a}$$
$$y_i = Cx_i, \tag{28b}$$

where

$$A_c = A - BK. \tag{29}$$

The transfer matrix of the closed-loop system (28) has the form

$$T_c(z) = C[I_n z - A_c]^{-1} B = \frac{N(z)}{d(z)}, \tag{30a}$$

where

$$d(z) = \det[I_n z - A_c] = z^n + d_{n-1}z^{n-1} + \ldots + d_1 z + d_0, \tag{30b}$$

$$N(s) = C[I_n z - A_c]_{ad} B = \begin{bmatrix} n_{11}(z) & \ldots & n_{1n}(z) \\ \ldots & \ldots & \ldots \\ n_{p1}(z) & \ldots & n_{pn}(z) \end{bmatrix}. \tag{30c}$$

Theorem 7 *If the matrix $B \in \Re_+^{n\times n}$ is nonsingular then there exists the feedback matrix K and the closed-loop system matrix A_c such that the poles and zeros of the transfer matrix (30a) are stable and the zeros satisfy the condition*

$$z_1 \leqslant z_k^{ij} \leqslant z_n \quad \textit{for} \;\; k = 1,\ldots,n \;\; \textit{and} \;\; i = 1,\ldots,p, \;\; j = 1,\ldots,n. \tag{31}$$

Proof is similar to the proof of Theorem 5.

Example 2 Consider the positive discrete-time linear system (7) with the matrices

$$A = \begin{bmatrix} 0.7 & 0.6 \\ 0.6 & 0.7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{32}$$

with the state feedback matrix

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}. \tag{33}$$

The matrix A given by (32) is unstable since sum of element of each row (column) is greater one.

The matrix B given by (32) is nonsingular. Therefore, using (16) and (32) for

$$A_c = \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & 0.5 \end{bmatrix} \tag{34}$$

we obtain

$$K = B^{-1}(A - A_c) = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0.7 & 0.6 \\ 0.6 & 0.7 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & 0.5 \end{bmatrix} \right) = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.05 \end{bmatrix}. \tag{35}$$

The transfer matrix of the closed-loop system has the form

$$\begin{aligned} T_c(z) &= C[I_2 z - A_c]^{-1} B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-0.5 & -0.3 \\ -0.4 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \\ &= \frac{1}{z^2 - z + 0.13} [1.5z{-}0.2 \quad z{-}0.2] \end{aligned} \tag{36}$$

and its poles are: $z_1 = -0.2$, $z_2 = -0.8$, zeros: $z_1^{11} = 0.13$, $z_1^{12} = 0.2$ are stable.

4.1. Positive discrete-time linear systems with monomial *B* matrices

Consider the positive discrete-time system (7a) with the matrices: $A \in \Re_+^{n\times n}$ unstable and $B \in \Re_+^{n\times n}$ monomial.

Let the following assumption be satisfied.

Assumption 1 *The unstable matrix* $A \in \Re_+^{n\times n}$ *can be stabilized by decreasing only one its entry in every row and in every column.*

In this case we have the following theorem.

Theorem 8 *If the assumption 1 is satisfied and the matrix B is monomial then the feedback matrix K defined by (16) is also monomial matrix.*

Proof. Note that if the assumptions are satisfied then the inverse matrix of the matrix B is monomial and the matrix $A - A_c$ is also monomial since in each its row and in each column only one entry is nonzero. Therefore, from (16) follows that the matrix K is also monomial. □

Example 3 Consider the positive linear system (7) with the matrices

$$A = \begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0.7 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}. \tag{37}$$

Note that the matrix A is unstable and satisfies the assumption 1 and the matrix B is monomial. Let the desired matrix of the closed-loop system A be stable and has the form

$$A_c = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.1 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}. \tag{38}$$

Using (16), (37) and (38) we obtain

$$K = B^{-1}(A - A_c) = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0.7 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.9 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.1 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}. \tag{39}$$

The transfer matrix of the closed-loop system has the form

$$T_c(z) = C[I_3 z - A_c]^{-1}B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} z-0.2 & -0.4 & -0.2 \\ -0.1 & z-0.3 & -0.4 \\ -0.2 & -0.1 & z-0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \frac{1}{z^3-0.8z^2+0.09z-0.02} \begin{bmatrix} 0.25z-2 & z^2-0.4z & z^2-0.3z+0.12 \\ z^2-0.45z+0.05 & 0.4z+0.05 & z^2+0.3z-0.1 \end{bmatrix}. \tag{40}$$

The stable poles of (40) are: $z_1 = 0.713$, $z_2 = 0.043 + j0.162$, $z_3 = 0.043 - j0.162$ and its stable zeros: $z_1^{11} = 8$, $z_1^{12} = 0$, $z_2^{12} = 0.4$, $z_1^{13} = 0.15 + j0.312$, $z_2^{13} = 0.15 - j0.312$, $z_1^{21} = 0.25$, $z_2^{21} = 0.2$, $z_1^{22} = -0.125$, $z_1^{23} = -0.5$, $z_2^{23} = 0.2$.

Now the necessary and sufficient conditions for the existence of monomial feedback matrix $K \in \Re_{+}^{n\times n}$ will be presented.

Theorem 9 *Let the matrix $A \in \Re_{+}^{n\times n}$ of the positive system (7) be unstable and the matrix $B \in \Re_{+}^{n\times n}$ be monomial. Then there exists a monomial matrix $K \in \Re_{+}^{n\times n}$ for the matrix (29) of the positive closed-loop system if and only if the sum of $n-1$ ($n \geqslant 3$) entries of each row and of each column of the matrix $A \in \Re_{+}^{n\times n}$ is less than 1.*

Proof. It is well-known [4] that the inverse matrix B^{-1} is monomial if and only if the matrix B is monomial. Note that if $n-1$ entries of each row and each column of the matrix A is less than 1 then to stabilize the closed-loop system it is possible to choose the matrix A so that the matrix $A - A_c$ is monomial. In this case from (16) it follows that the matrix K is monomial. □

Note that the positive system (7) with the matrices (37) satisfies the assumptions of Theorem 8 and the matrix K given by (39) is diagonal.

The following example shows that in general case when the matrix A does not satisfy the assumptions of Theorem 9 the closed-loop system can be stabilizes by a not-diagonal state feedback matrix K.

Example 4 Consider the positive linear system (7) with the matrices

$$A = \begin{bmatrix} 0.4 & 0.6 & 0.3 \\ 0.6 & 0.6 & 0.4 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{41}$$

Note the matrix A given by (41) is unstable and the matrix B is monomial.

Using (29), (41) and the closed-loop matrix

$$A_c = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}. \tag{42}$$

we obtain

$$K = B^{-1}(A - A_c) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0.4 & 0.6 & 0.3 \\ 0.6 & 0.6 & 0.4 \\ 0.2 & 0.4 & 0.7 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.1 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}. \tag{43}$$

The transfer matrix of the closed-loop system has the form

$$T_c(z) = C[I_3 z - A_c]^{-1}B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z-0.3 & -0.2 & -0.3 \\ -0.3 & z-0.2 & -0.2 \\ -0.2 & -0.4 & z-0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \frac{N(z)}{d(z)}, \tag{44}$$

where

$$d(z) = z^3 - 0.8z^2 + 0.01z - 0.008,$$
$$N(z) = \begin{bmatrix} z^2 - 0.2z - 0.05 & 0.5z + 0.03 & z^2 - 0.3z + 0.03 \\ 0.6z - 0.02 & z^2 - 0.3z + 0.06 & z^2 - 0.2z - 0.02 \end{bmatrix}.$$

Note that these considerations can be extended to the case of the output-feedback under the assumption that the matrices C of the systems (1) and (7) are nonsingular.

If $\det C \neq 0$ then knowing the matrices K and C from the equation

$$K = FC \tag{45}$$

we may find the desired output feedback matrix $F = KC^{-1}$.

5. Concluding remarks

Pole and zero assignment problem by state feedbacks in positive continuous-time and discrete-time has been considered. For the positive continuous-time linear systems it has been shown:

1. If the matrix $B \in \mathfrak{R}_+^{n\times n}$ is nonsingular then there exists the feedback matrix K such that the poles and zeros of the transfer matrix (14a) are stable (Theorem 5);

2. If the matrix B is monomial then the feedback matrix K defined by (13) is also monomial matrix (Theorem 6).

For the positive discrete-time linear systems it has been shown:

1. If the matrix B is nonsingular then there exists the feedback matrix K such that the poles and zeros of the transfer matrix (30a) are stable and the zeros satisfy the condition (31) (Theorem 7);

2. If the assumption 1 is satisfied and the matrix B is monomial then the feedback matrix K defined by (16) is also monomial matrix (Theorem 8);

3. If the matrix A of the positive system (7) is unstable and the matrix B is monomial then there exists a monomial matrix K of the positive closed-loop system if and only if the sum of $n-1$ ($n > 3$) entries of each row and of each column of the matrix A is less than 1 (Theorem 9).

The considerations have been illustrated by numerical examples. The considerations can be easily extended to positive linear systems with output feedbacks.

An open problem is an extension of these considerations to fractional positive continuous-time and discrete-time linear systems.

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