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On necessary and sufficient conditions for stability and quasistability in combinatorial multicriteria optimization*

by

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Abstract: We consider a multiple objective combinatorial optimization problem with an arbitrary vector-criterion. The necessary and sufficient conditions for stability and quasistability are obtained for large classes of problems with partial criteria possessing certain properties of regularity.

Keywords: sensitivity analysis, multiple criteria, combinatorial optimization, Pareto set, stability conditions

1. Introduction

Stability theory is one of the major parts in various fields of applied and pure mathematics. In optimization theory, we particularly deal with stability questions if the initial data set is given with no precision qualification. Under this assumption, it is very important to know what happens to the set of optimal solutions. We assume some initial realization of data to be fixed along with some associated set of optimal solutions, assumed to be found. We would like to clarify how the set would react to small modifications of the initial data set.

The main difficulty of solving discrete optimization problems is their combinatorial complexity. While studying stability of single objective discrete models, it is commonly observed that they may react unpredictably to small modifications of initial data. In the presence of multiple conflicting objectives, the problem complexity may only be increased (see, e.g., Ehrgott, 2000; Miettinen, 1999; Nogin, 2018).

For both single and multiple criteria cases, there is a lot of papers dedicated to different approaches that deal with uncertainty in discrete models (for a review, see, e.g., Emelichev and Kuzmin, 2006; Emelichev and Podkopaev,

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2010; Gordeev, 2015, and references therein). One of the specific approaches is known under the name of robust optimization. According to this approach, an additional objective is constructed and optimized in order to represent the possible worst-case data realization impact (see, e.g., Kasperski, 2008; Kouvelis, 1997, and references therein). Some other approaches are related to the so-called post-optimal or stability analysis, where optimal solution behavior is scrutinized as a response to initial data (problem parameters) changes. Numerous articles are devoted to analysis of conditions, which may guarantee a certain property of solution invariance to the problem parameter perturbations (see, e.g., Greenberg, 1998; Sotskov et al., 2010, and references therein). Some similarities between robust optimization and stability analysis are discussed in Nikulin (2014); Nikulin et al. (2013).

The present work continues the stability analysis investigations of different multicriteria discrete optimization problems with various partial criteria and optimality principles. The survey of recent results in the area can be found in Emelichev et al. (2012b); Emelichev and Kuzmin (2007); Emelichev and Podkopaev (2010); Gordeev (2015); Libura and Nikulin (2006); Nikulin et al. (2013). It is worth mentioning that similar research is done for scheduling theory problems (see, e.g., Gurevsky et al., 2012; Lai et al., 2004; Sotskov et al., 2004, 2009; Sotskov and Lai, 2012; Sotskov et al.,1998). Also, some well-known combinatorial problems with a specific structure, such as the maxcut problem, were at special focus in Kuzmin (2015), where it was proven that the problem of finding the radius for every type of stability is intractable unless $\mathcal{P} = \mathcal{NP}$. Some problems with nonlinear criteria have been also considered (see, e.g., Libura and Nikulin, 2004).

In this paper, we focus on studying a major question of what regularity properties should be imposed on partial criteria in order to guarantee certain properties of stability. Our approach aims to formulate regularity properties in the most general form. It allows us not only to generalize numerous results reported in literature earlier, but also to propose a universal strategy how for analyzing stability in various discrete optimization models.

2. Basic definitions and notations

We consider a generic class of vector discrete optimization problems, described as follows. Given a finite set of $m \geq 2$ different elements $\mathbb{N}_m = \{1, 2, \dots, m\}$, let x denote a subset of elements from $\mathbb{N}_m$. Let $\mathcal{X}$ be a collection of such subsets containing at least two subsets, i.e., $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N}_m)$, $|\mathcal{X}| \geq 2$, where $\mathcal{P}(\mathbb{N}_m)$ is the power set of $\mathbb{N}_m$.

The set $\mathcal{X}$ is called a *set of feasible solutions* and x denotes one feasible solution from the set $\mathcal{X}$. To each element $j \in \mathbb{N}_m$, we assign an n-dimensional vector of weights (or costs) $a(j) = (a_{1j}, a_{2j}, \dots, a_{nj})^\top \in \mathbb{R}^n$. Therefore, we deal with a matrix of weights $A = [a_{ij}] \in \mathbb{R}^{n\times m}$. Given A, we define a vector-function (vector-criterion):

$$f = f(x, A) = \big(f_1, f_2, \dots, f_n\big)^{\top} : \mathcal{X} \to \mathbb{R}^n.$$

For any fixed $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ with rows $A_i = (a_{i1}, a_{i2}, \dots, a_{im}) \in \mathbb{R}^m$, $i \in \mathbb{N}_n$, the partial criteria $f_i = f_i(x, A_i)$ are simultaneously being minimized on the set of feasible solutions:

$$\begin{aligned} &\min f_i(x, A_i) \\ &\text{s.t. } x \in \mathcal{X}. \end{aligned}$$

Then, every instance of a (n-criteria) *multicriteria combinatorial optimization problem* is uniquely defined by the triple $(\mathcal{X}, f, A)$, where, as it was mentioned above, $\mathcal{X}$ is the set of feasible solutions, $f = (f_1, f_2, \dots, f_n)$ is the objective vector-function and $A \in \mathbb{R}^{n\times m}$ is the matrix of weights. For this reason, it is natural to denote such instance as $(\mathcal{X}, f, A)$. Following notations in Emelichev et al. (2002, 2005); Emelichev and Podkopaev (2010), we will often refer to the instance as to the problem itself. We will also assume that all the partial criteria are mutually independent, i.e., for any given x, each function $f_i(x, A_i)$ depends on A_i and does not depend on any A_j if $j \neq i$.

Such formulation of multicriteria combinatorial optimization problem is quite general, which implies that many classical problems of graph theory such as minimum spanning tree, shortest path, assignment, maximum and minimum cut, traveling salesman etc. can be considered as its special cases. If, for instance, $\mathbb{N}_m$ is used to enumerate all edges in the graph, A is the matrix of associated vector weights, $\mathcal{X}$ describes a collection of all feasible spanning trees in the graph, and f is the vector-function calculating the vector of weights for each feasible spanning tree, then the problem becomes the well-known multiobjective minimum weight spanning tree problem. In case of Pareto optimality principle, which will be introduced below, this problem, depending on the type of criteria, may be intractable and $\mathcal{NP}$-hard (in case of linear criteria), see Ehrgott (2000), or polynomially solved (in case of k-MAX or bottleneck criteria), see Gorski et al. (2012).

In this article, we will consider objective vector-functions of very general classes, in particular, they may consist of the following well-known partial criteria:

$$f_i(x, A_i) = \sum_{j\in x} a_{ij}, \tag{SUM}$$

$$f_i(x, A_i) = \sqrt[p]{\sum_{j\in x} a_{ij}^p}, \quad p \in \mathbb{R},\ p > 0,\ p \neq 1, \tag{p-SUM}$$

$$f_i(x, A_i) = \sqrt[p]{\sum_{j\in x} |a_{ij}|^p}, \quad p \in \mathbb{R},\ p > 0, \tag{p-NORM}$$

$$f_i(x, A_i) = \max_{j \in x} a_{ij}, \tag{MAX}$$

$$f_i(x, A_i) = \min_{j \in x} a_{ij}, \tag{MIN}$$

$$f_i(x, A_i) = k\text{-}\max_{j \in x} a_{ij}, \quad k \in \mathbb{N}, \tag{k-MAX}$$

$$f_i(x, A_i) = k\text{-}\min_{j \in x} a_{ij}, \quad k \in \mathbb{N}, \tag{k-MIN}$$

where k-MAX and k-MIN functions return, correspondingly, the k^{th} largest and the k^{th} smallest weight coefficient among a_{ij}, $j \in x$ (see Gorski et al., 2012; Gorski and Ruzika, 2009).

Since there does not exist a canonical ordering on the Euclidean vector space $\mathbb{R}^n$ when $n \geq 2$, we will use commonly accepted, in the theory of multi-objective optimization and multicriteria decision making, Edgeworth–Pareto concept of optimality. According to this concept, the set of non-dominated solutions constitutes the outcome we are interested in to get as a result of simultaneous minimization of several conflicting objectives f_i. Below, we define this set that is also generally known as a set of *efficient* (*Pareto optimal*) solutions, or simply the *Pareto set*.

For each pair of feasible solutions $x, x' \in \mathcal{X}$, we define a binary relation of dominance, i.e., we say that x is *dominated* by x' (in minimization sense) if

$$x \underset{f,A}{\succ} x' \iff \Big(\forall i \in \mathbb{N}_n \ \ (g_i(x, x', A_i) \geq 0)\Big) \wedge \Big(\exists k \in \mathbb{N}_n \ \ (g_k(x, x', A_k) > 0)\Big),$$

where

$$g_i(x, x', A_i) = f_i(x, A_i) - f_i(x', A_i), \quad i \in \mathbb{N}_n.$$

Hence, the Pareto set consists of all the feasible solutions $x \in \mathcal{X}$ that are *non-dominated*, i.e.,

$$P^n(f, A) = \left\{ x \in \mathcal{X} : \ \nexists x' \in \mathcal{X} \ \left(x \underset{f,A}{\succ} x' \right) \right\}.$$

Notice that sometimes other principles of optimality, different from the Pareto, e.g., lexicographic one, may be considered, see, e.g., Ehrgott (2000) or Emelichevet et al. (2010).

For the problem $(\mathcal{X}, f, A)$, we define the *Slater set* (the set of *weakly efficient* solutions)

$$Sl^n(f, A) = \big\{ x \in \mathcal{X} : \ \forall x' \in \mathcal{X} \setminus \{x\} \ \exists k \in \mathbb{N}_n \ \ (g_k(x, x', A_k) \leq 0) \big\},$$

and the *Smale set* (the set of *strictly efficient* solutions)

$$Sm^n(f, A) = \big\{ x \in \mathcal{X} : \ \forall x' \in \mathcal{X} \setminus \{x\} \ \exists k \in \mathbb{N}_n \ \ (g_k(x, x', A_k) < 0) \big\}.$$

It is well known that

$$Sm^n(f, A) \subseteq P^n(f, A) \subseteq Sl^n(f, A).$$

Since the set $\mathcal{X}$ is finite, the Pareto set $P^n(f, A)$ and the Slater set $Sl^n(f, A)$ are non-empty for any $f : \mathcal{X} \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$. The Smale set $Sm^n(f, A)$, in principle, can be empty, see Ehrgott (2000). Note also that strict efficiency is the multiple criteria analog of the unique optimal solutions for single objective problems.

The elements of the matrix A constitute initial data of the instance $(\mathcal{X}, f, A)$ of multicriteria combinatorial optimization problem. We consider the case when initial data is given with some uncertainty. Assume that instead of the matrix A we are given a matrix B which is taken from some neighbourhood of A. To specify the neighbourhood we will consider matrices as points in nm-dimensional real space endowed with the *Chebyshev norm* (the norm l_∞). Thus,

$$\|A\| = \big\|(a_{11}, a_{12}, \dots, a_{nm})\big\| = \max\big\{|a_{ij}| : \ (i, j) \in \mathbb{N}_n \times \mathbb{N}_m\big\}.$$

Henceforward, where it is convenient, matrices and their rows are called *points* in corresponding spaces. Notice that due to the well-known equivalence of any two norms in finite dimensional linear space, all the results specified here are valid for the Chebyshev norm as well as for any arbitrary norm in the space $\mathbb{R}^{n\times m}$.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Under *ε-neighbourhood* of a point $b \in \mathbb{R}^k$ we understand the set $\{y \in \mathbb{R}^k : \ \|y - b\| < \varepsilon\}$, which is denoted as $\Omega(\varepsilon, b)$. According to this definition, the ε-neighbourhood of a point (matrix) $A \in \mathbb{R}^{n\times m}$ is the set

$$\Omega(\varepsilon, A) = \big\{B \in \mathbb{R}^{n\times m} : \ \|B - A\| < \varepsilon\big\},$$

ε-neighbourhood of a point (vector) $A_i \in \mathbb{R}^m$ is the set

$$\Omega(\varepsilon, A_i) = \big\{y \in \mathbb{R}^m : \ \|y - A_i\| < \varepsilon\big\},$$

and, finally, ε-neighbourhood of a point (number) $a_{ij} \in \mathbb{R}$ is the set

$$\Omega(\varepsilon, a_{ij}) = \big\{z \in \mathbb{R} : \ |z - a_{ij}| < \varepsilon\big\}.$$

It is evident that

$$B \in \Omega(\varepsilon, A) \ \Leftrightarrow \ \forall i \in \mathbb{N}_n \quad \big(B_i \in \Omega(\varepsilon, A_i)\big)$$

and for any $i \in \mathbb{N}_n$

$$B_i \in \Omega(\varepsilon, A_i) \ \Leftrightarrow \ \forall j \in \mathbb{N}_m \quad \big(b_{ij} \in \Omega(\varepsilon, a_{ij})\big).$$

The problem $(\mathcal{X}, f, B)$, where $B \in \Omega(\varepsilon, A)$, is called a *perturbed problem.*

For any index $i \in \mathbb{N}_n$ and any solution $x \in \mathcal{X}$, we set

$$N_i(x, f_i, A_i) = \Big\{j \in \mathbb{N}_m : \forall\varepsilon > 0 \ \exists\delta \in (-\varepsilon, \varepsilon) \ \ \big(f_i(x, A_i) \neq f_i(x, A_i + \delta E_j)\big)\Big\},$$

where E_j is the j^{th} row of identity matrix E of dimension $m \times m$. In other words, $N_i(x, f_i, A_i)$ is the set of indices $j \in \mathbb{N}_m$ such that for any b_{ij} the function

$f_i(x, B_i)$ is not constant within any ε-neighbourhood of point A_i. Within the set $N_i(x, f_i, A_i)$ we distinguish two subsets, $N_i^+(x, f_i, A_i)$ and $N_i^-(x, f_i, A_i)$, defined as follows below:

$$N_i^+(x, f_i, A_i) = \Big\{ j \in \mathbb{N}_m : \forall \varepsilon > 0 \ \exists \delta \in (-\varepsilon, \varepsilon) \ \big(f_i(x, A_i) < f_i(x, A_i + \delta E_j) \big) \Big\},$$

$$N_i^-(x, f_i, A_i) = \Big\{ j \in \mathbb{N}_m : \forall \varepsilon > 0 \ \exists \delta \in (-\varepsilon, \varepsilon) \ \big(f_i(x, A_i) > f_i(x, A_i + \delta E_j) \big) \Big\}.$$

It is evident that

$$\emptyset \subseteq N_i^+(x, f_i, A_i) \cap N_i^-(x, f_i, A_i) \subseteq N_i^+(x, f_i, A_i) \cup N_i^-(x, f_i, A_i) = N_i(x, f_i, A_i).$$

Therefore, the set $N_i(x, f_i, A_i)$ is such a subset of indexes from $\mathbb{N}_m$ that even arbitrarily small changes of elements' weights may allow changing the value of the i^{th} objective of the feasible solution x. Analogously, $N_i^+(x, f_i, A_i)$ and $N_i^-(x, f_i, A_i)$ are the sets of indexes that for arbitrarily small changes allow, correspondingly, for an increase and a decrease of the i^{th} objective.

Informally, the elements from $N_i(x, f_i, A_i)$ may be called *sensitive elements*, the elements from $N_i^+(x, f_i, A_i)$ – *sensitive for plus*, and, finally, the elements from $N_i^-(x, f_i, A_i)$ – *sensitive for minus*.

To exemplify this notation, let us consider (p-SUM) and (k-MAX) functions as the i^{th} partial criteria (objectives). The former case is quite obvious: all the elements from x are sensitive for plus and minus simultaneously, while the rest of elements are not sensitive. The latter case is more complicated. If there is only one k^{th} largest element in x, then this and only this element is sensitive for both plus and minus. Otherwise, if there are several k^{th} largest elements in x, then all of them and only them are sensitive for plus and none of the elements are sensitive for minus.

3. Stability conditions

Generalizing the traditional methodology of deducing stability conditions in multiple objective discrete optimization problem $(\mathcal{X}, f, A)$, see Emelichev et al. (2012a); Emelichev and Kuzmin (2008); Gurevsky et al. (2012); Nikulin et al. (2013), we specify necessary and sufficient conditions for stability for the problem. The vector-criterion of the problem is composed of arbitrary functions satisfying some regularity conditions that will be formulated later.

By analogy with Emelichev et al. (2002, 2012a) the problem $(\mathcal{X}, f, A)$ is called *stable* if

$$\exists \varepsilon > 0 \ \forall B \in \Omega(\varepsilon, A) \ \ \big(P^n(f, B) \subseteq P^n(f, A) \big),$$

i.e., if there exists an ε-neighbourhood in the space of initial problem data of $(\mathcal{X}, f, A)$ such that no new efficient solutions appear.

Let $J \subseteq \mathbb{N}_m$, $J \neq \emptyset$, and $b \in \mathbb{R}^m$. The function $h(y_1, y_2, \ldots, y_m) : \mathbb{R}^m \to \mathbb{R}$ is called *constant* on $\Omega(\varepsilon, b)$ with respect to y_j, $j \in J$, if for every collection of variables $y_j \in \Omega(\varepsilon, b_j)$, $j \in \mathbb{N}_m \setminus J$, there exists $c \in \mathbb{R}$ such that equality

$$h(y_1, y_2, \dots, y_m) = c$$

holds for any $y_j \in \Omega(\varepsilon, b_j)$, $j \in J$.

Definition 1 *Let $x \in \mathcal{X}$, $i \in \mathbb{N}_n$. The function $f_i(x, B_i)$ is called* α-regular at A_i *if there exists a positive number $\varepsilon = \varepsilon(i, x)$ with the following true conditions:*

(α.1) *the function $f_i(x, B_i)$ is nondecreasing on the set $\Omega(\varepsilon, A_i)$ w.r.t. b_{ij}, $j \in N_i(x, f_i, A_i)$;*

(α.2) *for each solution $x' \in \mathcal{X}$ the function $g_i(x, x', B_i)$ is constant on $\Omega(\varepsilon, A_i)$ w.r.t. b_{ij}, $j \in N_i^-(x, f_i, A_i) \cap N_i^-(x', f_i, A_i)$;*

(α.3) *the inclusion $N_i(x, f_i, B_i) \subseteq N_i(x, f_i, A_i)$ holds for any $B_i \in \Omega(\varepsilon, A_i)$.*

The vector function $f(x, B)$ is called α-regular at A *if $f_i(x, B_i)$ is α-regular at A_i for each $i \in \mathbb{N}_n$.*

Informally speaking, the α-regularity concept ensures that there exists such a neighbourhood of the initial point A, in which equal perturbations of each sensitive for minus element result in equal perturbations of solutions to which this element belongs. Moreover, within the neighbourhood, the appearance of new sensitive elements is prohibited.

For an arbitrary pair of solutions x and x' we define a binary relation $\underset{f,A}{\vdash}$ according to the rule:

$$x \underset{f,A}{\vdash} x' \;\Leftrightarrow\; \forall i \in \mathbb{N}_n \quad \big(g_i(x, x', A_i) = 0 \;\Rightarrow\; N_i^+(x', f_i, A_i) \subseteq N_i^+(x, f_i, A_i) \;\wedge\; N_i^-(x, f_i, A_i) \subseteq N_i^-(x', f_i, A_i)\big).$$

In addition, we introduce the following notation

$$\overline{P^n}(f, A) = \mathcal{X} \setminus P^n(f, A),$$

$$P^n(x, f, A) = \big\{x' \in P^n(f, A) : \; \forall i \in \mathbb{N}_m \quad (g_i(x, x', A_i) \geq 0)\big\}.$$

Obviously, $P^n(x, f, A) \neq \emptyset$ for any $x \in \mathcal{X}$. Moreover, $x \in P^n(x, f, A)$ if and only if $x \in P^n(f, A)$.

Now we are ready to formulate the first main result of this paper.

Theorem 1 *Suppose that the function $f = f(x, B)$ is α-regular at A for any $x \in Sl^n(f, A)$. If the problem $(\mathcal{X}, f, A)$ is stable, then*

$$\forall x \in Sl^n(f, A) \; \exists x' \in P^n(x, f, A) \quad \left(x \underset{f,A}{\vdash} x'\right). \tag{1}$$

The proof of Theorem 1 can be found in the Appendix.

Definition 2 *Let $x \in \mathcal{X}$, $i \in \mathbb{N}_n$. The function $f_i(x, B_i)$ is called* β-regular at A_i *if there exists a positive number $\varepsilon = \varepsilon(i, x)$ with the following valid conditions:*

(β.1) *the function* $f_i(x, B_i)$ *is continuous on the set* $\Omega(\varepsilon, A_i)$ *w.r.t.* b_{ij}, $j \in \mathbb{N}_m$*;*

(β.2) *for any* x' *such that* $N_i^+(x', f_i, A_i) \subseteq N_i^+(x, f_i, A_i)$ *and* $N_i^-(x, f_i, A_i) \subseteq N_i^-(x', f_i, A_i)$*, the function* $g_i(x, x', B_i)$ *is nonnegative at any* $B_i \in \Omega(\varepsilon, A_i)$*.*

The vector-function $f(x, B)$ *is called* β-regular at A *if its every component* $f_i(x, B_i)$*,* $i \in \mathbb{N}_n$*, is* β*-regular at* A_i*.*

In particular, the property of β-regularity guarantees that for each point $B \in \Omega(\varepsilon, A)$ the inequality $g(x, x', B) \geq 0$ derives from the binary relation $x \underset{f,A}{\vdash} x'$. Now we formulate the second main result of this paper.

Theorem 2 *Suppose that the condition* (1) *is fulfilled. Then the problem* $(\mathcal{X}, f, A)$ *is stable if for any* $x \in \mathcal{X}$ *the function* $f = f(x, B)$ *is* β*-regular at* A*.*

The proof of Theorem 2 can be found in the Appendix.

Consequently, for a certain class of vector functions f, Theorem 1 provides the necessary condition and Theorem 2 – the sufficient condition for stability of $(\mathcal{X}, f, A)$. We specify below some representatives of this class. It is easy to check that the vector function $f(x, A)$ composed of an arbitrary combination of the (SUM), (MAX), or (k-MAX) partial criteria is α- and β-regular at $A \in \mathbb{R}^{n\times m}$ for any $x \in \mathcal{X}$.

Many types of vector functions defined on $\mathcal{X}$ are β-regular at every point $A \in \mathbb{R}^{n\times m}$ for any $x \in \mathcal{X}$, but at the same time they are α-regular only for a few solutions x and points A. This happens, for example, when $f(t, A)$ is composed of an arbitrary combination of the (MIN), (k-MIN), (p-SUM), or (p-NORM) partial criteria. This can be explained by the fact that α-regularity property guarantees condition $P^n(f, B^0) \not\subseteq P^n(f, A)$ on matrix B^0 used in the proof of Theorem 1. In principle, if we use another perturbing matrix to guarantee the condition, then the limitations on f could be different.

It is known, see Emelichev et al. (2002), that the coincidence of the Pareto set $P^n(f, A)$ and the Slater set $Sl^n(f, A)$ gives necessary and simultaneously sufficient condition for stability of $(\mathcal{X}, f, A)$ with vector criterion of type (SUM). Using Theorem 1 and Theorem 2, this result can be easily generalized for the entire class of functions.

Corollary 1 *Let functions* f_i*,* $i \in \mathbb{N}_n$*, be continuous and* α*-regular at every point of* $\mathbb{R}^m$ *for any solution* $x \in \mathcal{X}$ *and suppose that for every index* $i \in \mathbb{N}_n$ *the following condition holds*

$$\forall B_i \in \mathbb{R}^m \quad \forall x \in \mathcal{X} \qquad \big(N_i^+(x, f_i, B_i) = N_i^-(x, f_i, B_i) = x\big). \tag{2}$$

Then

$$P^n(f, A) = Sl^n(f, A) \Longleftrightarrow (\mathcal{X}, f, A) \text{ is stable.}$$

As a matter of fact, due to (2) the implication (1) transforms into an equality $P^n(f, A) = Sl^n(f, A)$, and on account of its continuity the function f is β-regular at A. Therefore, based on Theorem 1 and Theorem 2 we conclude that Corollary 1 is true.

Corollary 2 *The single objective* ($n = 1$) *problem* $(\mathcal{X}, f, A)$ *is stable if* f *is* β*-regular at* $A \in \mathbb{R}^m$.

Actually, since $P^1(f, A) = Sl^1(f, A)$ and $x \underset{f,A}{\vdash} x$, then based on Theorem 2 the problem $(\mathcal{X}, f, A)$ is stable.

For any index $i \in \mathbb{N}_n$, we introduce the notation

$$N_i^+(x, A_i) = \mathrm{Argmax}\{a_{ij} : \ j \in x\},$$

$$N_i^-(x, A_i) = \mathrm{Argmin}\{a_{ij} : \ j \in x\}.$$

Corollary 3 Emelichev and Kuzmin (2008) *The problem* $(\mathcal{X}, f, A)$ *with partial criteria of type* (MAX) *is stable if and only if for any solution* $x \in Sl^n(f, A)$ *the following condition holds*

$$\exists x' \in P^n(x, f, A)\ \forall i \in \mathbb{N}_n \ \ \big(g_i(x, x', A_i) = 0 \ \Rightarrow \ \ N_i^+(x', A_i) \subseteq N_i^+(x, A_i)\big).$$

Indeed, we have

$$N_i^+(x, f_i, A_i) = N_i^+(x, A_i),$$

$$N_i^-(x, f_i, A_i) = \begin{cases} N_i^+(x, A_i) & \text{if } |N_i^+(x, A_i)| = 1, \\ \emptyset & \text{if } |N_i^+(x, A_i)| \geq 2. \end{cases}$$

Hence, the inclusions

$$N_i^+(x', f_i, A_i) \subseteq N_i^+(x, f_i, A_i),$$

and

$$N_i^-(x, f_i, A_i) \subseteq N_i^-(x', f_i, A_i)$$

are equivalent to the inclusions

$$N_i^+(x', A_i) \subseteq N_i^+(x, A_i).$$

Considering α- and β-regularity of the vector-function f at A we use Theorem 1 and Theorem 2 to see the correctness of Corollary 3.

Corollary 4 Emelichev et al. (2012a) *Suppose that*

$$\exists x' \in P^n(x, f, A)\ \forall i \in \mathbb{N}_n \ \ \big(g_i(x, x', A_i) = 0 \Rightarrow N_i^-(x, A_i) \subseteq N_i^-(x', A_i)\big).$$

Then $(\mathcal{X}, f, A)$ *with partial criteria of type* (MIN) *is stable.*

Since we have

$$N_i^-(x, f_i, A_i) = N_i^-(x, A_i),$$

$$N_i^+(x, f_i, A_i) = \begin{cases} N_i^-(x, A_i) & \text{if } |N_i^-(x, A_i)| = 1, \\ \emptyset & \text{if } |N_i^-(x, A_i)| \geq 2, \end{cases}$$

then the inclusions

$$N_i^+(x', f_i, A_i) \subseteq N_i^+(x, f_i, A_i),$$

$$N_i^-(x, f_i, A_i) \subseteq N_i^-(x', f_i, A_i)$$

are equivalent to

$$N_i^-(x, A_i) \subseteq N_i^-(x', A_i).$$

Taking into account β-regularity of f at A we use Theorem 2 to infer correctness of Corollary 4.

As it was mentioned above, $f = f(x, B)$ with partial criteria of type (MIN), in principle, is not α-regular at $A \in \mathbb{R}^{n\times m}$. Another stability criterion was obtained in Emelichev et al. (2012a), which we present here. Once $P^n(f, A) = \mathcal{X}$, the problem $(\mathcal{X}, f, A)$ is stable for any $A \in \mathbb{R}^{n\times m}$. The problem $(\mathcal{X}, f, A)$ with nonempty $\overline{P^n}(f, A)$ is called *non-trivial*.

Denote

$$V(x, I, A) = \prod_{i\in I} N_i^-(x, A_i), \qquad I \subseteq \mathbb{N}_n,$$

$$I(x, x') = \{i \in \mathbb{N}_n : \; g_i(x, x', A_i) = 0\}.$$

Let v_I be a projection of vector $v \in \mathbb{R}^n$ to the coordinate axes with numbers from the set $I \subseteq \mathbb{N}_n$.

Theorem 3 Emelichev et al. (2012a) *The vector non-trivial problem* $(\mathcal{X}, f, A)$, $n \geq 1$ *with partial criteria* (MIN) *is stable if and only if for any* $x \in Sl^n(f, A)$ *the following condition holds*

$$\forall v \in V(x, \mathbb{N}_n, A) \; \exists x^* \in P^n(x, f, A) \quad \big(v_{I(x,x^*)} \in V(x^*, I(x, x^*), A)\big).$$

4. Quasistability conditions

The property of quasistability describes another property of invariance of the Pareto set. This property is opposite to the property of stability and represents the case where every original optimum preserves optimality under admissible perturbations. In this section, we obtain necessary condition for quasistability of $(\mathcal{X}, f, A)$ with an arbitrary type of vector function f, and also describe certain classes of functions for which the same condition is not only necessary but also sufficient.

Following Emelichev et al. (2002, 2005) the problem $(\mathcal{X}, f, A)$ is called *quasistable* if

$$\exists \varepsilon > 0 \; \forall B \in \Omega(\varepsilon, A) \qquad \big(P^n(f, A) \subseteq P^n(f, B)\big),$$

i.e., if there exists ε-neighborhood in the space of initial data of $(\mathcal{X}, f, A)$ such that none of efficient solutions disappears.

For any x and x' define a binary relation $\underset{f,A}{\sim}$ as follows

$$x \underset{f,A}{\sim} x' \quad \Leftrightarrow \quad f(x,A) = f(x',A) \;\Rightarrow\; \forall i \in \mathbb{N}_n \; \big(N_i(x, f_i, A_i) = N_i(x', f_i, A_i)\big).$$

Theorem 4 *If $(\mathcal{X}, f, A)$ is quasistable, then the following formula holds*

$$\forall x, x' \in P^n(f,A) \qquad \left(x \underset{f,A}{\sim} x'\right). \tag{3}$$

The proof of Theorem 4 can be found in the Appendix.

Definition 3 *Assume $x \in \mathcal{X}$, $i \in \mathbb{N}_n$. The function $f_i(x, B_i)$ is called* γ-regular at A_i *if there exists a positive number $\varepsilon = \varepsilon(i,x)$ satisfying the following conditions*

(γ.1) *the function $f_i(x, B_i)$ is continuous on $\Omega(\varepsilon, A_i)$ w.r.t. b_{ij}, $j \in \mathbb{N}_m$;*

(γ.2) *for any x' such that $N_i(x, f_i, A_i) = N_i(x', f_i, A_i)$, the function $g_i(x, x', B_i)$ is constant on $\Omega(\varepsilon, A_i)$.*

The vector function $f(x, B)$ is called γ-regular *at A if $f_i(x, B_i)$ is γ-regular at A_i for every $i \in \mathbb{N}_n$.*

In particular, the property of γ-regularity guarantees that for each point $B \in \Omega(\varepsilon, A)$ the equality $g(x, x', B) = 0$ derives from the binary relation $x \underset{f,A}{\sim} x'$. It is also worth mentioning that in spite of the coincidence of the properties (β.1) and (γ.1) neither β-regularity implies γ-regularity nor γ-regularity implies β-regularity. However, under certain additional conditions this may happen. For instance, β-regularity derives from γ-regularity when $N_i^+(x, f_i, A_i) = N_i^-(x, f_i, A_i) = N_i(x, f_i, A_i)$ for any $x \in \mathcal{X}$ and $i \in \mathbb{N}_n$.

Theorem 5 *Suppose that* (3) *holds. The problem $(\mathcal{X}, f, A)$ is quasistable if for any solution $x \in \mathcal{X}$ the function $f = f(x, B)$ is γ-regular at A.*

The proof of Theorem 5 can be found in the Appendix.

Notice that γ-regularity validity test for a certain type of vector functions can be done easily. Moreover, many well known types of vector criterion defined on $\mathcal{X}$ are γ-regular at every $A \in \mathbb{R}^{n\times m}$ for any $x \in \mathcal{X}$. For example, this is true for the vector-function $f(x, A)$ being an arbitrary combination of (SUM), (p-SUM), (p-NORM), (MAX), (MIN), (k-MAX), or (k-MIN) partial criteria.

Corollary 5 *Let functions f_i, $i \in \mathbb{N}_n$, be γ-regular at every point of $\mathbb{R}^m$ for any $x \in \mathcal{X}$ and let there be an index $s \in \mathbb{N}_n$ such that*

$$\forall B_s \in \mathbb{R}^m \; \forall x \in \mathcal{X} \quad (N_s(x, f_s, B_s) = t). \tag{4}$$

Then

$$P^n(f,A) = Sm^n(f,A) \implies (\mathcal{X}, f, A) \text{ is quasistable.}$$

Indeed, due to (4), formula (3) transforms into the equality $P^n(f, A) = Sm^n(f, A)$. Therefore, based on Theorems 4 and 5 we can conclude that Corollary 5 is true.

From Corollary 5, we derive the following known, see Emelichev and Podkopaev (2010), result. Let f be an arbitrary combination of (SUM), (MAX), or (MIN) partial criteria, with at least one criterion of (SUM)-type. Then the problem $(\mathcal{X}, f, A)$ is quasistable if and only if $P^n(f, A) = Sm^n(f, A)$.

Corollary 5 implies also the following fact.

Corollary 6 *Let continuous functions f_i, $i \in \mathbb{N}_n$, satisfy*

$$\forall B_i \in \mathbb{R}^m \quad \forall x \in \mathcal{X} \qquad (N_i(x, f_i, B_i) = t).$$

Then the problem $(\mathcal{X}, f, A)$, $n \geq 1$ is quasistable if and only if

$$Sm^n(f, A) = P^n(f, A).$$

From Corollary 6 we also derive the following statement.

Corollary 7 *Let f_i, $i \in \mathbb{N}_n$, be of type* (p-SUM) *or* (p-NORM). *Then the problem $(\mathcal{X}, f, A)$ is quasistable if and only if $P^n(f, A) = Sm^n(f, A)$.*

As before, we set

$$N_i^+(x, A_i) = \text{Argmax}\{a_{ij} : \ j \in x\},$$

$$N_i^-(x, A_i) = \text{Argmin}\{a_{ij} : \ j \in x\}.$$

Corollary 8 Emelichev and Kuzmin (2008) *The problem $(\mathcal{X}, f, A)$ with partial criteria* (MAX) *is quasistable if and only if*

$$f(x, A) = f(x', A) \implies \forall i \in \mathbb{N}_n \quad (N_i^+(x, A_i) = N_i^+(x', A_i))$$

for any $x, x' \in P^n(f, A)$.

Corollary 9 Emelichev et al. (2012a) *The problem $(\mathcal{X}, f, A)$ with partial criteria* (MIN) *is quasistable if and only if*

$$f(x, A) = f(x', A) \implies \forall i \in \mathbb{N}_n \quad (N_i^-(x, A_i) = N_i^-(x', A_i))$$

for any $x, x' \in P^n(f, A)$.

Notice that based on Theorems 4 and 5 we can obtain the majority of the results previously studied in the literature (for the survey see Emelichev et al., 2012; Emelichev and Podkopaev, 2010; Gurevsky et al., 2012). Using Theorem 4 and Theorem 5, we can establish quasistability criteria for many new problems.

We describe below certain families of such problems . The function $f_i(x, B_i)$, which is γ-regular at A_i, is called *peculiar* at A_i if for any $x' \in \mathcal{X}$ from $N_i(x, f_i, A_i) = N_i(x', f_i, A_i)$ the equality $f_i(x, A_i) = f_i(x', A_i)$ follows.

Proposition 1 *If function $f_i(x, B_i)$ is peculiar at A_i and function*

$$h : \mathbb{R} \to \mathbb{R}$$

is continuous and strictly monotone, then $h(f_i(x, B_i))$ is γ-regular at A_i.

Thus, if every component f_i of f is peculiar at A_i, then according to Proposition 1 as well as Theorem 4 and Theorem 5, the problem $(\mathcal{X}, f, A)$ is quasistable if and only if (3) holds.

The following statement is also true and follows from the above.

Proposition 2 *If $f_i(x, B_i)$ is γ-regular at A_i, then the function*

$$\xi f_i(x, B_i)$$

is γ-regular at A_i for any $\xi \in \mathbb{R}$.

The functions $f_i^1 = f_i^1(x, B_i)$ and $f_i^2 = f_i^2(x, B_i)$, which are γ-regular at A_i, are called *correlated* at A_i if for any $t' \in \mathcal{X}$

$$N_i(x, f_i^1 + f_i^2, A_i) = N_i(x', f_i^1 + f_i^2, A_i) \ \Rightarrow \\ N_i(x, f_i^1, A_i) = N_i(x', f_i^1, A_i) \ \wedge \ N_i(x, f_i^2, A_i) = N_i(x', f_i^2, A_i).$$

Proposition 3 *If functions $f_i^1(x, B_i)$ and $f_i^2(x, B_i)$ are correlated at A_i, then the function*

$$f_i(x, B_i) = f_i^1(x, B_i) + f_i^2(x, B_i)$$

is γ-regular at A_i.

Therefore, due to Proposition 2 and Proposition 3, as well as Theorem 4 and Theorem 5, the implication (3) is a necessary and sufficient condition for quasistability of $(\mathcal{X}, f, A)$ when every component of vector-function f is a linear combination of functions correlated at A_i, $i \in \mathbb{N}_n$.

As an example, consider $(\mathcal{X}, f, A)$ with partial criteria defined as

$$f_i(x, A_i) = \lambda \max_{j \in x} a_{ij} + \mu \min_{j \in x} a_{ij}, \qquad i \in \mathbb{N}_n, \tag{MAX+MIN}$$

where $\lambda, \mu \in \mathbb{R}$. It is easy to check that for any index $i \in \mathbb{N}_n$ the functions $\max\limits_{j \in x} b_{ij}$ and $\min\limits_{j \in x} b_{ij}$ are correlated for any $x \in \mathcal{X}$, $B_i \in \mathbb{R}^m$. Then, based on Proposition 2 and Proposition 3, every function $f_i(x, B_i)$, $i \in \mathbb{N}_n$ is γ-regular at A_i, i.e., f is γ-regular at A. From here, due to Theorem 4 and Theorem 5, we conclude that implication (3) gives a criterion for quasistability of $(\mathcal{X}, f, A)$.

Notice that when $\lambda \neq -\mu$ for any $i \in \mathbb{N}_n$, $x \in \mathcal{X}$, $B_i \in \mathbb{R}^m$, the following equality is valid

$$N_i(x, f_i, B_i) = N_i(x, B_i),$$

where

$$N_i(x, A_i) = N_i^+(x, A_i) \cup N_i^-(x, A_i).$$

Therefore, the following statement is true.

Corollary 10 *If $\lambda \neq -\mu$, then the necessary and sufficient condition for quasistability of $(\mathcal{X}, f, A)$, $n \geq 1$ with partial criteria of type* (MAX+MIN) *is the fulfillment of the following implication for any $x, x' \in P^n(f, A)$:*

$$f(x, A) = f(x', A) \quad \Rightarrow \quad \forall i \in \mathbb{N}_n \quad \big(N_i(x, A_i) = N_i(x', A_i)\big). \tag{5}$$

If $\lambda = -\mu$ for any $i \in \mathbb{N}_n$ and $B_i \in \mathbb{R}^m$, then the equality holds

$$N_i(x, f_i, B_i) = \begin{cases} N_i(x, B_i) & \text{if } |x| \geq 2, \\ \emptyset & \text{if } |x| = 1. \end{cases} \tag{6}$$

Moreover, if $\lambda = -\mu > 0$ and $|x| = 1$, then $f_i(x, B) = 0,\ i \in \mathbb{N}_m$, for any matrix $B \in \mathbb{R}^{n \times m}$, and therefore the inclusion holds

$$\mathcal{X}^* \subseteq P^n(f, A), \tag{7}$$

where

$$\mathcal{X}^* = \{x \in \mathcal{X} : \ |x| = 1\}.$$

Since the criterion for quasistability of $(\mathcal{X}, f, A)$ is the fulfillment of the implication (3), then using (6) and (7) it is easy to see the correctness of the corollaries formulated below.

Corollary 11 *Assume $\lambda = -\mu$ and either*

$$\mathcal{X}^* = \emptyset$$

or

$$\mathcal{X}^* \neq \mathcal{X} \quad \text{and} \quad \lambda < 0$$

holds. Then the problem $(\mathcal{X}, f, A)$ with partial criteria (MAX+MIN) *is quasistable if and only if for any $x, x' \in P^n(f, A)$ implication* (5) *is fulfilled.*

Corollary 12 *If $\lambda = -\mu$ and $\mathcal{X}^* = \mathcal{X}$, then the problem $(\mathcal{X}, f, A)$ with partial criteria* (MAX+MIN) *is quasistable for any $A \in \mathbb{R}^{n \times m}$.*

Corollary 13 *Assume $\lambda = -\mu > 0$ and $\emptyset \neq \mathcal{X}^* \neq \mathcal{X}$. The problem $(\mathcal{X}, f, A)$ with partial criteria* (MAX+MIN) *is stable if and only if*

$$P^n(f, A) = \mathcal{X}^*.$$

Notice that in the partial case of $\lambda = 1$, Corollary 11, Corollary 12 and Corollary 13 give the known conditions for quasistability of $(\mathcal{X}, f, A)$ with interval partial criteria, see Emelichev et al. (2012a).

5. Conclusion

In this paper, we considered a generic multiple criteria combinatorial optimization problem with arbitrary partial criteria under variations of its parameter weights. The uncertainty of initial data was modeled using a traditional concept of stability analysis, i.e., by perturbations of element weights. We scrutinized two main types of stability for a discrete optimization problem: stability itself, which implies that no new Pareto optimal solutions appear, and quasistability, which means that none of Pareto optimal solutions disappears. As a result, for large classes of multiple criteria combinatorial optimization problems, necessary and sufficient conditions of these types of stability were obtained. These conditions are based on the idea of identification of sensitive elements in feasible solutions, and are independent of the specific combinatorial problem considered.

In addition, we introduced some properties of regularity on partial criteria which could guarantee necessity and/or sufficiency of problem stability and quasistability. As corollaries a number of well-known and new results on stability of multiple objective problems with specific types of partial criteria were derived.

The generalized approach presented in the article allows for replacing the methodological tools of stability analysis with methods based on finding sensitive elements of solutions and testing regularity of partial criteria. The more precise methodological recommendations can be developed on this basis. This could be a potential subject for the further research in this area.

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7. Appendix

PROOF OF THEOREM 1. If $x \in P^n(f,A)$, then, due to the facts that $x \in P^n(x,f,A)$ and $x \underset{f,A}{\vdash} x$, formula (1) holds. The proof that (1) holds also for any $x \in Sl^n(f,A) \setminus P^n(f,A)$ is done by contradiction. Assume we have $x^0 \in Sl^n(f,A) \setminus P^n(f,A)$ such that for any $x \in P^n(x^0,f,A)$ the relation $x^0 \underset{f,A}{\vdash} x$ does not hold. Then (since $x^0 \in Sl^n(f,A)$) for every $x \in P^n(x^0,f,A)$ there exists $s = s(x) \in \mathbb{N}_n$ with $g_s(x^0,x,A_s) = 0$, such that

$$N_s^+(x,f_s,A_s) \not\subseteq N_s^+(x^0,f_s,A_s) \quad \text{or} \quad N_s^-(x^0,f_s,A_s) \not\subseteq N_s^-(x,f_s,A_s),$$

i.e., the set $\big(N_s^+(x,f_s,A_s) \setminus N_s^+(x^0,f_s,A_s)\big) \cup \big(N_s^-(x^0,f_s,A_s) \setminus N_s^-(x,f_s,A_s)\big)$ is nonempty. Let $p = p(x)$ be an index of that set. The following two cases are possible.

Case 1: $p \in N_s^-(x^0,f_s,A_s) \setminus N_s^-(x,f_s,A_s)$. Since there exists $\varepsilon = \varepsilon(t) > 0$ such that the property (α.1) holds. Taking into account the definitions of the sets $N_s^-(x^0,f_s,A_s)$ and $N_s^-(x,f_s,A_s)$ it is easy to see that for any positive number $\delta < \varepsilon$ the following relations are true

$$f_s(x^0,A_s - \delta E_p) < f_s(x^0,A_s), \quad f_s(x,A_s - \delta E_p) = f_s(x,A_s),$$

in other words $g_s(x^0, x, A_s - \delta E_p) < g_s(x^0, x, A_s)$.

Case 2: $p \notin N_s^-(x^0, f_s, A_s) \setminus N_s^-(x, f_s, A_s)$. Similarly to the previous case we infer there exists $\varepsilon = \varepsilon(t) > 0$ such that for any positive number $\delta < \varepsilon$ the relations take place

$$f_s(x^0, A_s + \delta E_p) = f_s(x^0, A_s), \quad f_s(x, A_s + \delta E_p) > f_s(x, A_s),$$

i.e., $g_s(x^0, x, A_s + \delta E_p) < g_s(x^0, x, A_s)$.

Summarizing what has been proven for both cases, we conclude that

$$g_s(x^0, x, A_s + \theta E_p) < g_s(x^0, x, A_s), \tag{8}$$

where

$$\theta = \theta(\delta) = \begin{cases} -\delta & \text{if } p \in N_s^-(x^0, f_s, A_s) \setminus N_s^-(x, f_s, A_s), \\ \delta & \text{otherwise.} \end{cases}$$

Set $\varepsilon^0 = \min\{\varepsilon(x) : x \in P^n(x^0, f, A)\}$, $0 < \delta^0 < \varepsilon^0$ and define matrix $B^0 = [b_{ij}^0] \in \Omega(\varepsilon^0, A)$ according to the following

$$b_{ij}^0 = \begin{cases} a_{ij} - \delta^0 & \text{if } i \in \mathbb{N}_n,\ j \in N_i^-(x^0, f_i, A_i), \\ a_{ij} + \delta^0 & \text{if } i \in \mathbb{N}_n,\ j \in \mathbb{N}_m \setminus N_i^-(x^0, f_i, A_i). \end{cases}$$

Note that the matrix B^0 does not depend on x, however, for each $x \in P^n(x^0, f, A)$ and $p = p(x)$ we have

$$g_s(x^0, x, B_s^0) = g_s(x^0, x, A_s + \theta^0 E_p + y' + y''),$$

where $\theta^0 = \theta(\delta^0)$; $y', y'' \in \mathbb{R}^m$,

$$y_j' = \begin{cases} -\delta^0 & \text{if } j \in N_i^-(x^0, f_i, A_i) \cap N_i^-(x, f_i, A_i), \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j'' = \begin{cases} -\delta^0 & \text{if } j \in N_i^-(x^0, f_i, A_i) \setminus \big(N_i^-(x, f_i, A_i) \cup \{p\}\big), \\ \delta^0 & \text{if } j \in \mathbb{N}_m \setminus \big(N_i^-(x^0, f_i, A_i) \cup \{p\}\big), \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality we can assume that the vector A_i was changed (perturbed) successively in the following way: at first the vector θE_p was added, and then the vector y' joined and then, finally, the vector y'' was added. Due to (8), after the first perturbation, we get

$$g_s(x^0, x, A_s + \theta^0 E_p) < g_s(x^0, x, A_s).$$

On account of (α.2), after the second perturbation, we have

$$g_s(x^0, x, A_s + \theta^0 E_p + y') = g_s(x^0, x, A_s + \theta^0 E_p).$$

Lastly, due to (α.1) and (α.3) after the third perturbation, we find

$$g_s(x^0, x, A_s + \theta^0 E_p + y' + y'') \leq g_s(x^0, x, A_s + \theta^0 E_p).$$

Therefore, from the above we finally infer

$$g_s(x^0, x, A_s + \theta E_p + y' + y'') < g_s(x^0, x, A_s) = 0.$$

Thus, for any $x \in P^n(x^0, f, A)$, the following relation holds

$$x^0 \underset{f,B^0}{\succeq} x. \tag{9}$$

If $x^0 \in P^n(f, B^0)$, then $P^n(f, B^0) \not\subseteq P^n(f, A)$. If $x^0 \in \overline{P^n}(f, B^0)$, then because of the external stability of the Pareto set $P^n(f, B^0)$ there exists $x^* \in P^n(f, B^0)$ such that

$$x^0 \underset{f,B^0}{\succ} x^*.$$

Consequently, $x^* \in \overline{P^n}(f, A)$ due to (9), whence $P^n(f, B^0) \not\subseteq P^n(f, A)$, which means that the problem $(\mathcal{X}, f, A)$ is not stable.

Theorem 1 has been proven. □

Proof of Theorem 2. If $\overline{P^n}(f, A) = \emptyset$, the statement is self-evident. Henceforth assume $\overline{P^n}(f, A) \neq \emptyset$. Let $x \in \overline{P^n}(f, A)$. There are two possible cases.

Case 1: $x \in Sl^n(f, A)$. Then, there exists $x' \in P^n(f, A)$ with

$$x \underset{f,A}{\succ} x' \quad \text{and} \quad x \underset{f,A}{\vdash} x'.$$

Therefore, the set $\mathbb{N}_n$ is split into two disjoint subsets N^1 and N^2, determined via conditions

$$\begin{aligned}
&\forall i \in N^1 \qquad \big(g_i(x, x', A_i) > 0\big),\\
&\forall i \in N^2 \qquad \big(g_i(x, x', A_i) = 0,\\
&N_i^+(x', f_i, A_i) \subseteq N_i^+(x, f_i, A_i), \quad N_i^-(x, f_i, A_i) \subseteq N_i^-(x', f_i, A_i)\big).
\end{aligned}$$

From the above, using β-regularity at A of the functions $f(x, B)$ and $f(x', B)$ we conclude the existence of $\varepsilon > 0$, satisfying (due to property $(\beta.1)$) the condition

$$\forall i \in N^1 \;\; \forall B_i \in \Omega(\varepsilon, A_i) \qquad \big(g_i(x, x', B_i) > 0\big),$$

and (on the grounds that property $(\beta.2)$ holds) the condition

$$\forall i \in N^2 \;\; \forall B_i \in \Omega(\varepsilon, A_i) \qquad \big(g_i(x, x', B_i) \geq 0\big).$$

Summarizing, we get

$$\exists \varepsilon > 0 \;\; \forall B \in \Omega(\varepsilon, A) \qquad \big(x \underset{f,B}{\succ} x'\big). \tag{10}$$

Case 2: $x \in \mathcal{X} \setminus Sl^n(f, A)$. Then, there exists a solution $x' \in \mathcal{X} \setminus \{x\}$ such that

$$\forall i \in \mathbb{N}_n \qquad \big(g_i(x, x', A_i) > 0\big).$$

Therefore, using the property (β.1) of the functions $f(x, B)$ and $f(x', B)$ we conclude the existence of $\varepsilon > 0$ satisfying the condition

$$\forall i \in \mathbb{N}_n \ \ \forall B_i \in \Omega(\varepsilon, A_i) \qquad \big(g_i(x, x', B_i) > 0\big),$$

and hence (10) is valid.

Thus, it was shown that for any solution $t \in \overline{P^n}(f, A)$ there exist $x' \in \mathcal{X} \setminus \{x\}$ and $\varepsilon = \varepsilon(x) > 0$ such that $x \underset{f,B}{\succ} x'$ for all $B \in \Omega(\varepsilon, A)$. Setting $\varepsilon^* = \min\{\varepsilon(x) : x \in \overline{P^n}(f, A)\}$ we deduce

$$\exists \varepsilon^* > 0 \quad \forall B \in \Omega(\varepsilon^*, A) \ \ \forall x \in \overline{P^n}(f, A) \qquad \big(x \in \overline{P^n}(f, B)\big).$$

And hence the problem $(\mathcal{X}, f, A)$ is stable.

Theorem 2 has been proven. □

Proof of Theorem 4. Conversely, let $(\mathcal{X}, f, A)$ be quasistable, but there exist solutions $x^0, x^* \in P^n(f, A)$ such that $x^0 \underset{f,A}{\sim} x^*$ does not hold. Then $f(x^0, A) = f(x^*, A)$ and there exists $s \in \mathbb{N}_n$ with $N_s(x^0, f_s, A_s) \neq N_s(x^*, f_s, A_s)$. Therefore, without loss of generality, assume $N_s(x^0, f_s, A_s) \setminus N_s(x^*, f_s, A_s)$ to be nonempty. Further, let $p \in N_s(x^0, f_s, A_s) \setminus N_s(x^*, f_s, A_s)$. Then according to the definition of sets $N_s(x^0, f_s, A_s)$ and $N_s(x^*, f_s, A_s)$ we get

$$\forall \varepsilon > 0 \ \exists \delta^0 = \delta^0(\varepsilon) \in (-\varepsilon, \varepsilon) \quad \big(f_s(x^0, A_s) \neq f_s(x^0, A_s + \delta^0 E_p)\big),$$

$$\exists \varepsilon^* > 0 \ \forall \delta \in (-\varepsilon^*, \varepsilon^*) \quad \big(f_s(x^*, A_s) = f_s(x^*, A_s + \delta E_p)\big).$$

From the above, upon setting

$$\theta = \begin{cases} \delta^0(\varepsilon) & \text{if } \varepsilon < \varepsilon^*, \\ \delta^0(\varepsilon^*) & \text{if } \varepsilon \geq \varepsilon^*, \end{cases}$$

it is easy to see that for any $\varepsilon > 0$ there exists $\theta \in (-\varepsilon, \varepsilon)$ such that

$$f_s(x^0, A_s) \neq f_s(x^0, A_s + \theta E_p) \quad \text{and} \quad f_s(x^*, A_s) = f_s(x^*, A_s + \theta E_p).$$

Define the elements of $B^0 = [b^0_{ij}] \in \Omega(\varepsilon, A)$ as

$$b^0_{ij} = \begin{cases} a_{ij} + \theta & \text{if } (i, j) = (s, p), \\ a_{ij} & \text{if } (i, j) \in \mathbb{N}_n \times \mathbb{N}_m \setminus \{(s, p)\}, \end{cases}$$

Taking into account that $f(x^0, A) = f(x^*, A)$ we get

$$g_s(x^0, x^*, B^0_s) = g_s(x^0, x^*, A_s + \theta E_p) = f_s(x^0, A_s + \theta E_p) - f_s(x^*, A_s + \theta E_p) \neq$$
$$\neq f_s(x^0, A_s) - f_s(x^*, A_s) = 0,$$

$$g_i(x^0, x^*, B_i^0) = g_i(x^0, x^*, A_i) = f_i(x^0, A_i) - f_i(x^*, A_i) = 0, \qquad i \in \mathbb{N}_n \setminus \{s\}.$$

So, we conclude that either $x^0 \underset{f,B^0}{\succ} x^*$ or $x^* \underset{f,B^0}{\succ} x^0$, and therefore $P^n(f, A) \not\subseteq P^n(f, B^0)$. Summarizing, we conclude that

$$\forall \varepsilon > 0 \; \exists B^0 \in \Omega(\varepsilon, A) \qquad \big(P^n(f, A) \not\subseteq P^n(f, B^0)\big).$$

The last contradicts the quasistability of $(\mathcal{X}, f, A)$.

Theorem 4 has been proven. $\square$

Proof of Theorem 5. Let $x \in P^n(f, A)$ and $x' \in \mathcal{X} \setminus \{x\}$. Then, due to γ-regularity of $f(x, B)$ and $f(x', B)$ there exists $\varepsilon > 0$ such that functions $f_i(x, B_i)$ and $f_i(x', B_i)$ satisfy (γ.1) and (γ.2), respectively, for any $i \in \mathbb{N}_n$. The two cases are possible

Case 1: $f(x, A) = f(x', A)$. Then, from (3) we get

$$\forall i \in \mathbb{N}_n \qquad \big(N_i(x, f_i, A_i) = N_i(x', f_i, A_i)\big).$$

Therefore, for every $i \in \mathbb{N}_n$, based on property (γ.2), we have

$$\forall B_i \in \Omega(\varepsilon, A_i) \qquad \big(g_i(x, x', B_i) = g_i(x, x', A_i)\big).$$

Thus,

$$\exists \varepsilon > 0 \; \forall B \in \Omega(\varepsilon, A) \qquad \big(x \underset{f,B}{\not\succ} x'\big). \tag{11}$$

Case 2: $f(x, A) \neq f(x', A)$. Then, there exists $s \in \mathbb{N}_n$, with $g_s(x, x', A_s) < 0$. Therefore, using (γ.1), it is easy to see that there exists a positive number ε such that for any matrix $B \in \Omega(\varepsilon, A)$ the inequality $g_s(x, x', B_s) < 0$ holds. Thus, in this case formula (11) is true.

Summarizing what has been proven for both cases, we conclude that any efficient solution x remains efficient in the perturbed problem $(\mathcal{X}, f, B)$ for any $B \in \Omega(\varepsilon, A)$, and hence $(\mathcal{X}, f, A)$ is quasistable.

Theorem 5 has been proven. $\square$

Proof of Proposition 1. For brevity sake, $h(f_i(x, B_i))$ is denoted by h. Since $f_i(x, B_i)$ is γ-regular at A_i, then there exists $\varepsilon > 0$ with valid conditions (γ.1) and (γ.2). Let us check that the same conditions are true for h for the same ε.

Validity of (γ.1) for h is obvious, since it is continuous, and $f_i(x, B_i)$ satisfies (γ.1). Further, we validate (γ.2) for h. Let x and x' be such that

$$N_i(x, f_i, A_i) = N_i(x', f_i, A_i). \tag{12}$$

Then, due to the strict monotonicity of h for any $B_i \in \Omega(\varepsilon, A_i)$, we have

$$N_i(x, h, B_i) = N_i(x, f_i, B_i) = N_i(x', f_i, A_i) = N_i(x', h, A_i). \tag{13}$$

On the other hand, since $f_i(x, B_i)$ satisfies (γ.2) and it is peculiar at A_i from (12) it follows that

$$\forall B_i \in \Omega(\varepsilon, A_i) \quad \big(g_i(x, x', B_i) = 0\big).$$

Thus, due to the strict monotonicity of h, we get

$$\forall B_i \in \Omega(\varepsilon, A_i) \quad \big(h(f_i(x, B_i)) = h(f_i(x', B_i))\big).$$

From the last and (13) we conclude that h satisfies (γ.2).

Proposition 1 has been proven. □

Proof of Proposition 3. Denote $f_i^1(x, B_i)$, $f_i^2(x, B_i)$ and $f_i(x, B_i)$ as f_i^1, f_i^2 and f_i, respectively. Since f_i^1 and f_i^2 are γ-regular at A_i, then there exists $\varepsilon > 0$ such that for each f_i^1 and f_i^2 the conditions (γ.1) and (γ.2) hold. Therefore, it is clear that f_i satisfies (γ.1).

Further, we will show that f_i satisfies (γ.2). Let x' be such that the equality $N_i(x, f_i, A_i) = N_i(x', f_i, A_i)$ remains. Then, due to the fact that functions f_i^1 and f_i^2 are correlated at point A_i, we have the following equalities: $N_i(x, f_i^1, A_i) = N_i(x', f_i^1, A_i)$ and $N_i(x, f_i^2, A_i) = N_i(x', f_i^2, A_i)$. Consequently, based on property (γ.2), valid for functions f_i^1 and f_i^2 on $\Omega(\varepsilon, A_i)$, the functions $f_i^1(x, B_i) - f_i^1(x', B_i)$ and $f_i^2(x, B_i) - f_i^2(x', B_i)$ are constant. Thus, $g_i(x, x', B_i)$ is also constant on $\Omega(\varepsilon, A_i)$. Hence, f_i satisfies condition (γ.2).

Proposition 3 has been proven. □