The Journal of Polish Society for Geometry and Engineering Graphics
Volume 14 (2004), 5 – 9

ON REDUCIBILITY OF THE INTERSECTION CURVE OF TWO SECOND-ORDER SURFACES

Andrzej BIELIŃSKI, Cecylia ŁAPIŃSKA

Warsaw University of Technology, Institute of Heating end Ventilation
20 Nowowiejska st., 00-653 Warsaw, Poland
e-mail: cecylia.lapinska@is.pw.edu.pl

Abstract. A generalization of the well known theorem about the division of the common curve of two quadrics in two parts which are tangent to a common sphere is given.

Key Words: conic, quadric, pole, polar, conjugate lines with respect to a quadric, perspective collineation, elation, geometric homology, harmonic involution.

1. Introduction

Almost each descriptive geometry manual which includes a chapter on second-order surfaces (also called quadratic surfaces or quadrics) gives the theorem about the division in two conics of the common curve of two quadrics which are tangent in two points. Usually a simple proof based on the Pascal's Theorem is also quoted (see for ex. [1]).

In exercises this basic theorem is often used, in particular in the case when two circular surfaces are circumscribed on the same sphere and their circles of tangency intersect in two points. If the circles of tangency have at most one common point the assumption of the above theorem is not satisfied and thus it cannot be applied there. However, even in this case the thesis of the theorem is true but there is no proof of that fact, which has been observed at the seminar in the Department of Descriptive Geometry of Warsaw Technical University. In the manual by E. Otto ([2], p.209) one can only find an exercise asking to prove that the intersection curve of two cones circumscribed on the same sphere consists of two conics also in the case when the line joining the vertices of the cones has at least one common point with the sphere. The author gives a hint stating that the sections of the cones by an appropriate plane coincide since they have common axis and foci.

In our paper we present a more general theorem which implies the reducibility of the intersection curve of two quadrics circumscribed on a sphere in the general case.

In the further part of the work definitions and terminology are conform to those included in the manual [2]. Unless the contrary is stated, we understand that we are dealing with nonsingular conics and quadrics.

2. Perspective Collineations from Sections of a Quadric onto Sections of a Cone Tangent to the Quadric

Let Φ be a quadric. Consider a conic $\boldsymbol{s}$ obtained by intersecting Φ with a plane ω, the point W which is the pole of the plane ω with respect to the quadric Φ and the cone Δ with vertex W tangent to Φ.

Observe that for every $P \in \omega$ the polar plane π_Δ of P with respect to Δ and the polar plane π_Φ of P with respect to Φ coincide because each of them contains the pole W of ω and the polar line p of P with respect to the conic $\boldsymbol{s}$. Therefore every line $\bar{l}$ conjugate to a line l lying in ω with respect to Φ is also conjugate to l with respect to Δ.

ISSN 1644-9363 / PLN 15.00

Let now a plane α intersect the quadric Φ in a conic s_Φ and the cone Δ in a conic s_Δ (Figure 1). Denote by k the edge of the planes α and ω ($k = \alpha \cap \omega$). The line $\bar{k}$ which is conjugate to k with respect to Φ and Δ cuts through the plane α in the point K (t.i. $K=\alpha \cap \bar{k}$).

With the above assumptions and notations we have the following statements.

Lemma 1: If the conic s_Φ is nonsingular, then there exists a perspective collineation which maps s_Φ onto s_Δ . The point K is the center and the line k the axis of the collineation.

Proof: It is known (see for ex. [2] p.163) that for any two nonsingular conics lying on a quadric there is a perspective collineation of three-dimensional space which transforms one of them into the other and the quadric onto itself. So we have a perspective collineation $f : X \mapsto X'$ which maps Φ onto Φ, the conic s_Φ onto s and the plane α onto the plane ω. Let γ be the fundamental plane of the collineation, t.i. $f(X) = X$ for every $X \in \gamma$. Evidently $k \subset \gamma$, so k is invariant, $f(k)=k$. Since perspective collineations preserve conjugate elements, we have also $f(\bar{k})=\bar{k}$.

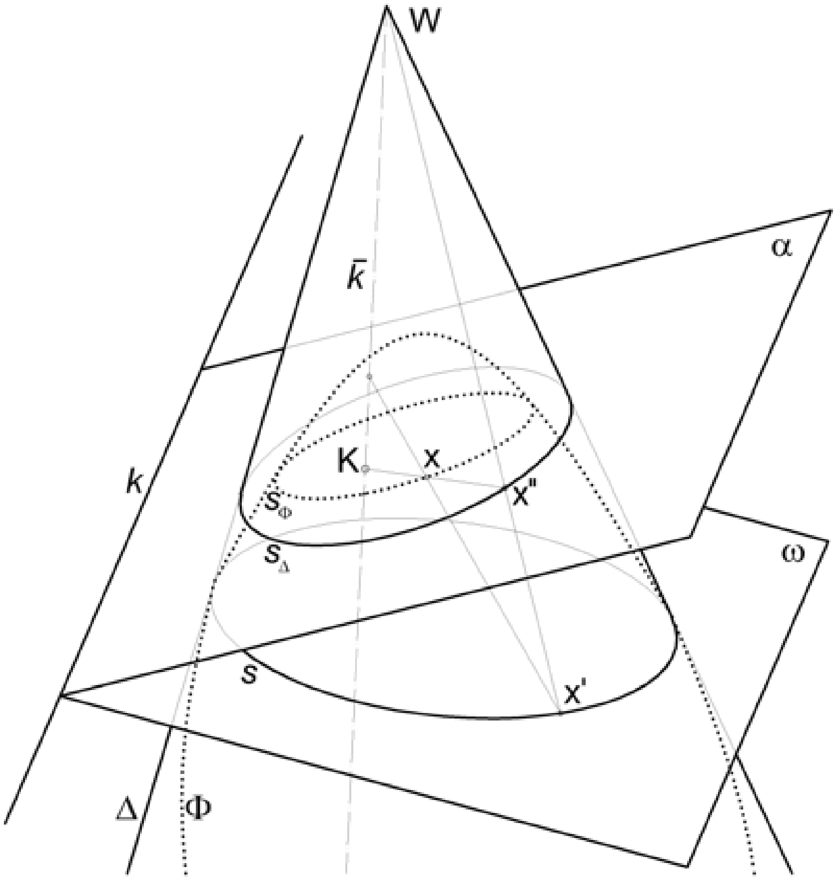

Figure 1

Take now a perspective collineation $g : X' \mapsto X''$ mapping ω onto α with the centre W and the fundamental plane γ considered above. Clearly, the cone Δ is mapping onto itself and the conic s is transformed into the conic section s_Δ . We have also $g(\bar{k})=\bar{k}$.

As products of two perspective collineations with the same fundamental plane are perspective collineations, we can state that the product h of f and g is a perspective collineation, $X'' = h(X) = g(f(X))$ .The image of the plane α under h is the plane α. We have also $h(\bar{k}) = g(f(\bar{k}) = g(\bar{k}) = \bar{k}$. Hence, the point $K = \alpha \cap \bar{k}$ is not changed under h and it is the centre of h. Now we can define a perspective collinetion from α to α as the restriction of h to the plane α whose centre is K and axis is the line k. Of course, the image of the conic $\boldsymbol{s}_{\Phi}$ under this collineation is $\boldsymbol{s}_{\Delta}$ .The proof is complete.

Remark that the point K is the pole of the line k with respect to both $\boldsymbol{s}_{\Phi}$ and $\boldsymbol{s}_{\Delta}$.

Lemma 2: Let two quadrics Φ_1 and Φ be tangent and let $\boldsymbol{s}$ be the conic of contact. If a plane α intersects the quadrics Φ_1 and Φ in the conics $\boldsymbol{s}_1$ and $\boldsymbol{s}_{\Phi}$ respectively, then there exists a perspective collineation mapping $\boldsymbol{s}_1$ onto $\boldsymbol{s}_{\Phi}$.

Proof: The quadrics Φ_1 and Φ have a common cone Δ tangent to them along the conic $\boldsymbol{s}$. Denote by $\boldsymbol{s}_{\Delta}$ the section of Δ by the plane α .By *Lemma 1* there exists a perspective collineation h_1 mapping $\boldsymbol{s}_1$ onto $\boldsymbol{s}_{\Delta}$ and a perspective collineation h mapping $\boldsymbol{s}_{\Phi}$ onto $\boldsymbol{s}_{\Delta}$. The point K and the line k are the common center and the common axis of the collineations h_1 and h. Hence the product f of the collineations h_1 and the inverse h^{-1} of h is a perspective collineation with the centre K and axis k which maps $\boldsymbol{s}_1$ onto $\boldsymbol{s}_{\Phi}$.

Lemma 3: Let Φ_1, Φ_2 and Φ be quadrics. Let Φ_1 and Φ_2 be tangent to Φ and their curves of contact lie in the planes α_1 and α_2 respectively. If a plane α including the edge of the planes α_1 and α_2 intersects the quadrics Φ_1 and Φ_2 in nonsingular conics $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ respectively, then there exists a perspective collineation which maps $\boldsymbol{s}_1$ onto $\boldsymbol{s}_2$.

Proof: Consider the line $k = \alpha_1 \cap \alpha_2$. The line $\bar{k}$ conjugate to k with respect to Φ is also conjugate to k with respect to Φ_1 and Φ_2 . Hence both the collineations f_1 and f_2 which exist by *Lemma 2* (f_1 mapping $\boldsymbol{s}_1$ onto $\boldsymbol{s}_{\Phi}$ and f_2 mapping $\boldsymbol{s}_2$ onto $\boldsymbol{s}_{\Phi}$) have the same centre K and the same axis k. The product of f_1 and the inverse of f_2 is a perspective collineation with the axis k and centre K and it maps $\boldsymbol{s}_1$ onto $\boldsymbol{s}_2$.

By an analogous reasoning we can also obtain the following lemma.

Lemma 4: If quadrics Φ_1 and Φ_2 are tangent to the same cone Δ with the conics of contact lying on the planes α_1 and α_2 respectively, then for every plane α including the edge of the planes α_1 and α_2 which intersects the quadrics Φ_1 and Φ_2 in nonsingular conics $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ respectively, there exists a perspective collineation which maps $\boldsymbol{s}_1$ onto $\boldsymbol{s}_2$.

3. Reducibility of Intersection Curve of Two Second-Order Non-degenerate Surfaces

Lemma 5: Let Φ_1, Φ_2 and Φ be quadrics such that Φ_1 is tangent to Φ and their conic of contact lies on a plane α_1 , Φ_2 is tangent to Φ and their conic of contact lies on a plane α_2. If there exists a common point P of the quadrics Φ_1 and Φ_2 which does not lie on the edge k of the planes α_1 and α_2 then the sections $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ of Φ_1 and Φ_2 respectively by the plane α including the point P and the line $k=\alpha_1 \cap \alpha_2$ coincide if they are nonsingular.

Proof: By *Lemma 3* there exists a perspective collineation f with the $k=\alpha_1 \cap \alpha_2$ and the centre $K = \bar{k} \cap \alpha$ which maps $\boldsymbol{s}_1$ onto $\boldsymbol{s}_2$. The point P belongs to $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, $P \notin k$ and $P \neq K$. If f is an elation we have necessarily $f(P)=P$. Hence in this case f is an identity and $\boldsymbol{s}_1=\boldsymbol{s}_2$ .If f is a geometric homology we cannot exclude that $f(P) = P_1 \neq P$. But K being the pole and k its polar with respect to $\boldsymbol{s}_2$, in this case because $f(P) \in \mathbf{s}_2$ the collineation f interchanges the points P and P_1, so it is an involution on $\boldsymbol{s}_2$. As $f(\boldsymbol{s}_1)=\boldsymbol{s}_2$ and f is one-to-one we have necessarily $\boldsymbol{s}_1=\boldsymbol{s}_2$, which completes the proof.

Let us remark now that the intersection curve of two second-order algebraic surfaces is a four-order curve. Hence if two quadrics have a common conic then the common points of these quadrics which does not belong to this conic form a second order curve or a conic. If the assumptions of *Lemma 4* are satisfied the quadrics Φ_1 and Φ_2 have a common nonsingular conic $\boldsymbol{s}_1=\boldsymbol{s}_2$ whose plane passes through the line $k=\alpha_1 \cap \alpha_2$. Thus the intersection curve of the quadrics Φ_1 and Φ_2 is reduced to two conics.

We summarise the preceding remarks in the following theorem.

Theorem 1: If quadrics Φ_1 and Φ_2 are tangent to a quadric Φ along conics $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ respectively, $\boldsymbol{s}_1$ lying on the plane α_1 and $\boldsymbol{s}_2$ on the plane α_2, and have a common point P, such that $P \notin \alpha_1 \cap \alpha_2$, then the intersection curve of the quadrics Φ_1 and Φ_2 consists of two conics[1] whose planes pass through the edge of the planes of the conics of contact.

Corollary 1: If two quadrics are circumscribed on the same sphere then their intersection curve divides into two conics.

Observe now that the surfaces Φ_1, Φ_2 and Φ in *Theorem 1* are not necessarily all non-singular quadrics. Two changes in the assumptions are possible. In fact, by applying *Lemma 3* it is not difficult to prove, by the same argumentation as that in the proof of *Lemma5,* that if the assumptions about Φ_1 and Φ_2 remain unchanged the thesis of *Theorem 1* holds even when Φ is a cone. On the other hand, by examining the argumentation of the proof of *Theorem 1* and those of the preceding lemmas it is easy to see that if Φ is a cone the thesis of *Theorem 1* is valid although we replace one or both of Φ_1 and Φ_2 by cones.

The above observations can be summarised in the following statements.

Theorem 2: If quadrics Φ_1 and Φ_2 are tangent to a cone Δ along conics $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ respectively, $\boldsymbol{s}_1$ lying on the plane α_1 and $\boldsymbol{s}_2$ lying on the plane α_2, and have a common point P, $P \notin \alpha_1 \cap \alpha_2$, then the intersection curve of the quadrics Φ_1 and Φ_2 consists of two conics[1] whose planes pass through the edge k of the planes of the conics of contact, $k=\alpha_1 \cap \alpha_2$.

Corollary 2: If two quadrics inscribed in the same cone are not disjoint their intersection curve consists of two conics[1].

Theorem 3: Let two second-ordered non-degenerated surfaces S_1 and S_2 be tangent to a quadric Φ along the conics of contact $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lying on the planes α_1 and α_2 respectively. If there exists a common point P, $P \notin \alpha_1 \cap \alpha_2$, then the intersection curve of the surfaces S_1 and S_2 consists of two conics[1] whose planes pass through the edge k of the planes of the conics of contact.

Corollary 3: Every two polar cones (i.e. consisting of tangents to quadric) of the same quadric intersect along conics[1].

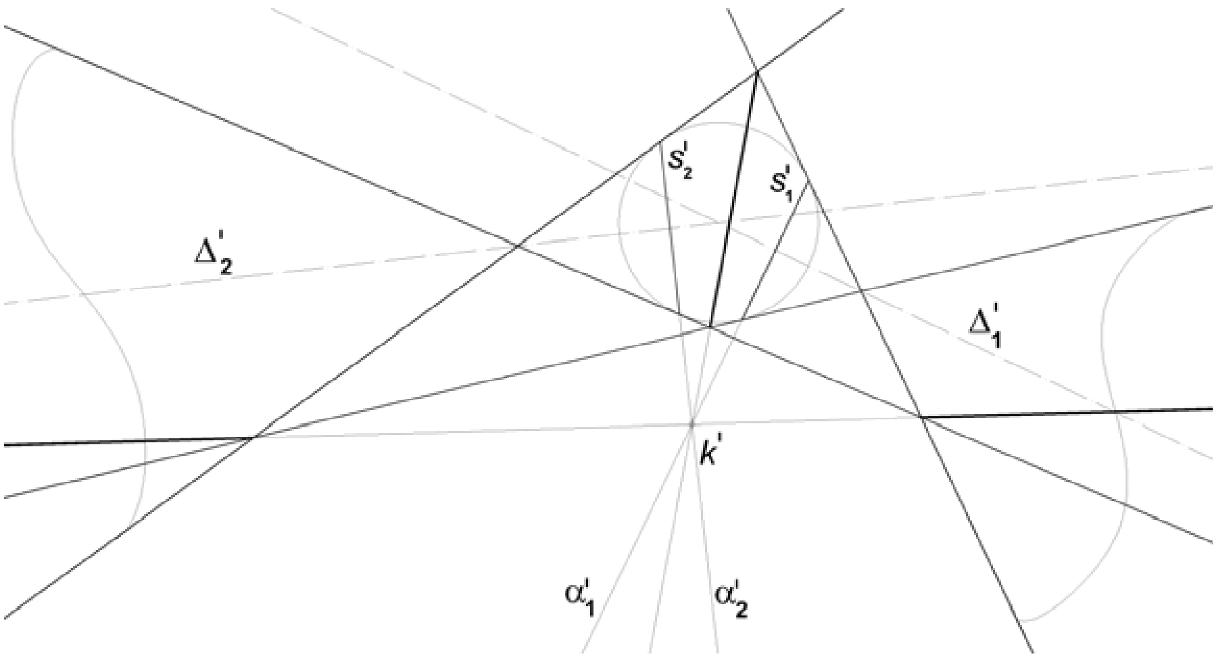

Figure 2

[1] may be singular

3. Example

We give a simple example to illustrate our considerations.

Figure 2 presents a projection on the symmetry plane of the curve of intersection of two cones Δ_1 and Δ_2 circumscribed on the same sphere. The intersection curve of these cones reduces to an ellipse and a hyperbola whose planes pass through the edge *k* of the planes α_1 and α_2 including the circles of contact of the cones and the sphere.

References

[1] Grochowski B.: *Geometria wykreślna z perspektywą stosowaną.* PWN, Warszawa 1988.
[2] Otto E.: *Geometria wykreślna.* PWN, Warszawa 1963.

O ROZPADZIE LINII PRZENIKANIA POWIERZCHNI DRUGIEGO STOPNIA

W pracy przedstawiono dowód twierdzenia o rozpadzie linii przenikania dwóch powierzchni drugiego stopnia stycznych do wspólnej kwadryki wzdłuż stożkowych. Idea dowodu polega na ustaleniu kolineacji środkowych zachodzących pomiędzy płaszczyznami stożkowych styczności i dowolną płaszczyzną, a następnie, korzystając z kolineacji pomiędzy przekrojami przenikających się powierzchni odpowiednio dobraną płaszczyzną, pokazanie, że przekroje te jednoczą się, uzyskując w ten sposób wspólną stożkową obu powierzchni. Sformułowano i udowodniono analogiczne twierdzenie dla dwóch kwadryk wpisanych w ten sam stożek.

Reviewer: Prof. Bogusław GROCHOWSKI, DSc

Received December 20, 2004