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THE CAUCHY PROBLEM FOR THE TIME-FRACTIONAL ADVECTION DIFFUSION EQUATION IN A LAYER

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Abstract

The time-fractional advection-diffusion equation with the Caputo time derivative is studied in a layer. The fundamental solution to the Cauchy problem is obtained using the integral transform technique. The logarithmic singularity term is separated from the solution. Expressions amenable for numerical treatment are obtained. The numerical results are illustrated graphically.

Introduction

The standard advection diffusion equation

$$\frac{\partial c}{\partial t} = a\,Dc - \mathbf{v} \cdot \operatorname{grad} c, \tag{1}$$

where:
a is the diffusivity coefficient, $\mathbf{v}$ is the given velocity vector, results from the balance equation for mass and the constitutive equation

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$$\mathbf{j} = -a \text{ grad } c + \mathbf{v}c \tag{2}$$

and has several physical interpretations in terms of Brownian motion, diffusion or heat conduction with additional force or with additional velocity field, transport processes in porous media, groundwater hydrology, diffusion of charge in the electric field on comb structures, etc. (FELLER 1971, KAVIANY 1995, NIELD and BEJAN 2006, RISKEN 1989, SCHEIDEGGER 1974, VAN KAMPEN 2007).

In the last few decades, equations with derivatives of fractional order have attracted considerable interest of researchers due to many applications in physics, geophysics, geology, rheology, engineering and bioengineering (GAFIYCHUK and DATSKO 2010, MAGIN 2006, MAINARDI 2010, METZLER and KLAFTER 2000, POVSTENKO 2015b, ROSSIKHIN and SHITIKOVA 1997, UCHAIKIN 2013, WEST et al. 2003).

The time-nonlocal generalizations of the constitutive equation for the matter flux (2) were studied in (POVSTENKO 2015a, 2015b). In the case of the „long-tail" power kernel, we have

$$\mathbf{j} = D_{RL}^{1-\alpha} \left[-a \text{grad } c + \mathbf{v}c\right], \tag{3}$$

where:
$D_{RL}^{\alpha}(t)$ is the Riemann-Liouville fractional derivative of the order α (GORENFLO and MAINARDI 1997, KILBAS et al. 2006, PODLUBNY 1999):

$$D_{RL}^{\alpha}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} c(\tau)\mathrm{d}\tau\right], \quad n-1 < \alpha < n. \tag{4}$$

In combination with the balance equation for mass, Eq. (3) leads to the time-fractional advection diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a D c - \mathbf{v}\cdot \text{grad } c, \tag{5}$$

with the Caputo fractional derivative of the order α (GORENFLO and MAINARDI 1997, KILBAS et al. 2006, PODLUBNY 1999):

$$\frac{\partial^\alpha c}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n c(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n. \tag{6}$$

In the literature there are only several papers in which the analytical solutions of Eq. (5) were obtained; equation with one spatial variable was considered in (HUANG and LIU 2005, LIU et al. 2003, POVSTENKO and KLEKOT 2014), while equation with two spatial variables was studied in (POVSTENKO 2014, 2015a, 2015b). A comprehensive survey of different approaches to solving the fractional advection diffusion equation as well as of the numerical methods used for its solving can be found in (POVSTENKO 2014). In the present paper, we study Eq. (5) in a layer $0 < x < l$, $-\infty < y < \infty$. The fundamental solution to the Cauchy problem is obtained using the integral transform technique. The logarithmic singularity term is separated from the solution. Expressions amenable for numerical treatment are derived. The numerical results are illustrated graphically.

The paper is organized as follows. In the succeeding section, the initial-boundary-value for the time-fractional advection diffusion equation is formulated and the new sought-for function is introduced to eliminate the gradient term from the equation. Next, the considered problem is solved using the Laplace and Fourier integral transforms. Concluding remarks are presented in the last section.

Statement of the problem

We investigate the time-fractional advection diffusion equation in a layer

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) - v\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y}, \quad x \in (0, l),\, y \in (-\infty, +\infty),\, t > 0; \quad (7)$$

with $a > 0$, $v > 0$, $\alpha \in (0,1)$.

Equation (7) is considered under zero Dirichlet boundary conditions at the surfaces of a layer

$$c(0, y, t) = 0, \qquad y \in (-\infty, +\infty),\, t > 0; \quad (8)$$

$$c(l, y, t) = 0, \qquad y \in (-\infty, +\infty),\, t > 0; \quad (9)$$

and the initial condition

$$c(x, y, 0) = f(x, y), \qquad (x, y) \in (0, l) \times (-\infty, +\infty). \quad (10)$$

As usually, the zero condition at infinity is also assumed:

$$\lim_{y\to\pm\infty} c(x, y, t) = 0, \qquad x \in [0, l], t > 0. \tag{11}$$

The solution of the considered problem can be written as

$$c(x, y, t) = \int_{-\infty}^{\infty}\int_{0}^{l} f(\zeta, \sigma)\, G(x, y - \sigma, \zeta, t)\, \mathrm{d}\zeta \mathrm{d}\sigma, \tag{12}$$

where:
$G(x, y, \zeta, t)$ is the fundamental solution of (7)–(11) corresponding to the initial condition

$$c(x, y, 0) = G(x, y, \zeta, 0) = p_0\, \delta(x - \zeta)\, \delta(y), \qquad 0 < \zeta < l. \tag{13}$$

Here $\delta(x)$ is the Dirac delta function. In Eq. (13), we have introduced the constant multiplier p_0 to obtain the nondimensional quantity displayed in Figures (see Eq. (33)). The initial-boundary-value problem (7)–(11) is well-posed in the Banach space C of continuous functions vanishing at infinity, endowed with the sup norm (HANYGA 2002a, 2002b).

Now we introduce the new sought-for function $u(x, y, \zeta, t)$:

$$G(x, y, \zeta, t) = \exp\left[\frac{v(x + y)}{2a}\right] u(x, y, \zeta, t), \tag{14}$$

which allows us to eliminate the gradient terms from (7) and to reformulate the initial-boundary-value problem (7)–(11) as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{v^2}{2a}\, u, \qquad x \in (0,l), y \in (-\infty, +\infty), t > 0; \tag{15}$$

$$u(0, y, \zeta, t) = 0, \qquad y \in (-\infty, +\infty), t > 0; \tag{16}$$

$$u(l, y, \zeta, t) = 0, \qquad y \in (-\infty, +\infty), t > 0; \tag{17}$$

$$u(x, y, \zeta, t) = p_0 \exp\left(-\frac{v\zeta}{2a}\right)\delta(x - \zeta)\delta(y), \quad (x, y) \in (0, l) \times (-\infty, +\infty); \tag{18}$$

$$\lim_{y\to\pm\infty} u(x, y, \zeta, t) = 0, \qquad x \in [0, l], t > 0. \tag{19}$$

In Eq. (18), we have used the relation $f(x)\delta(x - \zeta) = f(\zeta)\delta(x - \zeta)$ understood in terms of distributions.

Solution of the problem

To solve the problem (15)–(19), the integral transform technique will be used. Recall that the Caputo fractional derivative for the Laplace transform rule requires the knowledge of the initial values of the function and its integer derivatives of the order $k = 1, 2, ..., n - 1$ (GORENFLO and MAINARDI 1997, KILBAS et al. 2006, PODLUBNY 1999):

$$\mathrm{L}\left\{\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \qquad n - 1 < \alpha < n, \tag{20}$$

where the asterisk denotes the Laplace transform, s is the transform variable.

The exponential Fourier transform with respect to the spatial coordinate y (denoted by the tilde) has the following form (SNEDDON 1972):

$$\mathrm{F}\{f(y)\} = \tilde{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{iy\eta} \mathrm{d}y, \tag{21}$$

$$\mathrm{F}^{-1}\{\tilde{f}(\eta)\} = f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\eta) e^{-iy\eta} \mathrm{d}\eta, \tag{22}$$

and

$$\mathrm{F}\left\{\frac{\mathrm{d}^{2} f(y)}{\mathrm{d}y^{2}}\right\} = -\eta^{2} \tilde{f}(\eta) \tag{23}$$

The finite sin-Fourier transform with respect to the spatial coordinate is marked by the hat and is expressed as (SNEDDON 1972).

$$\mathrm{F}\{f(x)\} = \hat{f}(\xi_k) = \int_{0}^{L} f(x) \sin(x\xi_k) \mathrm{d}x, \tag{24}$$

$$\mathrm{F}^{-1}\{\hat{f}(\xi_k)\} = f(x) = \frac{2}{l}\sum_{k=1}^{\infty}\hat{f}(\xi_k)\sin(x\xi_k), \tag{25}$$

where

$$\xi_k = \frac{k\pi}{l}, \qquad k = 1,2,3,... \tag{26}$$

This transform is used in the case of the Dirichlet boundary conditions as for the second derivative of a function we have

$$\mathrm{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_k^2\hat{f}(\xi_k) + \xi_k\,[f(0) - (-1)^k f(l)]. \tag{27}$$

Applying the integral transforms to Eqs. (15)–(19), we get

$$\hat{\tilde{u}}^*(\xi_k,\eta,\zeta,s) = \frac{p_0}{\sqrt{2\pi}}\exp\left(-\frac{v\zeta}{2a}\right)\sin(\zeta\xi_k)\frac{s^{\alpha-1}}{s^\alpha + a(\xi_k^2+\eta^2) + \dfrac{v^2}{2a}} \tag{28}$$

The inverse integral transforms give

$$u(x,y,\zeta,t) = \frac{p_0}{\pi l}\exp\left(-\frac{v\zeta}{2a}\right)\sum_{k=1}^{\infty}\sin(x\xi_k)\sin(\zeta\xi_k)$$
$$\times\int_{-\infty}^{\infty}E_\alpha\left[-a\left(\xi_k^2+\eta^2+\frac{v^2}{2a^2}\right)t^\alpha\right]\cos(y\eta)\mathrm{d}\eta, \tag{29}$$

where $E_\alpha(z)$ is the Mittag-Leffler function in one parameter α (GORENFLO et al. 2014)

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ \alpha > 0,\ z\in C, \tag{30}$$

and the following formula

$$\mathrm{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+b}\right\} = E_\alpha(-bt^\alpha) \tag{31}$$

has been used.

Hence, for the fundamental solution $G(x,y,\zeta,t)$ we have the expression:

$$G(x,y,\zeta,t) = \frac{p_0}{\pi L} \exp\left[\frac{v(x+y-\zeta)}{2a}\right] \sum_{k=1}^{\infty} \sin(x\xi_k)\sin(\zeta\xi_k) \times \int_{-\infty}^{\infty} E_\alpha\left[-a\left(\xi_k^2 + \eta^2 + \frac{v^2}{2a^2}\right)t^\alpha\right] \cos(y\eta)\mathrm{d}\eta. \tag{32}$$

For $v = 0$, the fundamental solution (32) coincides with the corresponding solution to the time-fractional diffusion-wave equation obtained in (POVSTENKO 2015c).

In calculations we will use the following nondimensional quantities

$$\bar{c} = \frac{\sqrt{a}\, t^{\alpha/2} l}{p_0} G, \qquad \bar{x} = \frac{x}{l}, \qquad \bar{y} = \frac{y}{l}, \qquad \bar{\zeta} = \frac{\zeta}{l}, \qquad \bar{v} = \frac{lv}{a}, \qquad \tau = \frac{at^\alpha}{l^2}. \tag{33}$$

In the case of the standard advection diffusion equation corresponding to $\alpha = 1$, taking into account that $E_1(z) = e^z$ and evaluating the following integral (PRUDNIKOV et al. 1986a)

$$\int_0^{\infty} e^{-p^2x^2} \cos(qx)\mathrm{d}x = \frac{\sqrt{\pi}}{2p} \exp\left(-\frac{q^2}{4p^2}\right), \qquad p > 0, \tag{34}$$

we get

$$G(x,y,\zeta,t) = \frac{p_0}{\sqrt{\pi a t}\, l} \exp\left[\frac{v}{2a}\left(x - \zeta - \frac{vt}{2}\right)\right] \exp\left[-\frac{(y - vt)^2}{4at}\right] \times \sum_{k=1}^{\infty} \sin(x\xi_k)\sin(\zeta\xi_k)\exp(-at\xi_k^2). \tag{35}$$

The solution (35) is shown in Fig. 1 and Fig. 2 for different values of the drift parameter $\bar{v}$.

Unfortunately, for $\alpha \neq 1$ Eq. (32) is not amenable for numerical treatment as it has singularity at the point $x = \zeta, y = 0$. The type of singularity is the same as in the case of an infinite plane, and for numerical calculations it is convenient to separate the singularity term from Eq. (32). For this purpose, we recall the fundamental solution (POVSTENKO 2014) to the Cauchy problem for the time-fractional advection diffusion equation in a plane:

$$\frac{\partial^\alpha G_\infty}{\partial t^\alpha} = a\left(\frac{\partial^2 G_\infty}{\partial x^2} + \frac{\partial^2 G_\infty}{\partial y^2}\right) - v\frac{\partial G_\infty}{\partial x} - v\frac{\partial G_\infty}{\partial y}, \quad x \in (-\infty, +\infty), y \in (-\infty, +\infty), t > 0; \tag{36}$$

$$G_\infty(x,y,\zeta,0) = p_0\, \delta(x-\zeta)\, \delta(y),\ x \in (-\infty, +\infty),\ y \in (-\infty, +\infty) \tag{37}$$

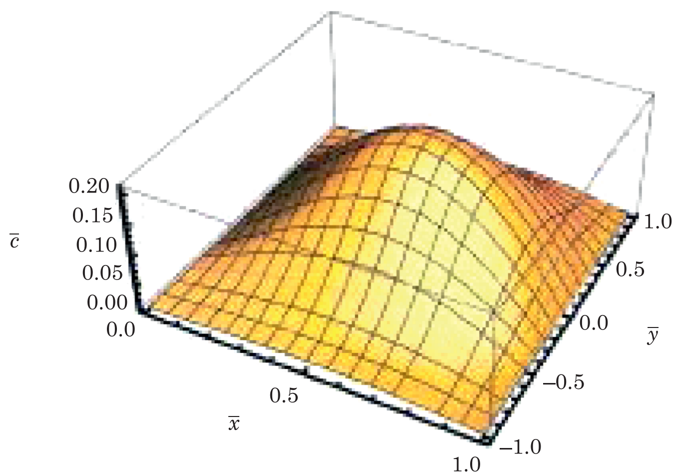

Fig. 1. Fundamental solution to the advection diffusion equation; $\alpha = 1$, $\tau = 0.1$, $\bar{\zeta} = 1/2$, $\bar{v} = 1$

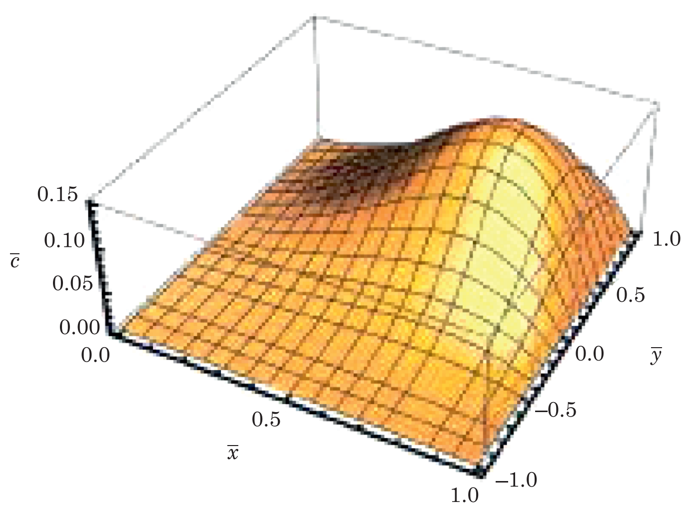

Fig. 2. Fundamental solution to the advection diffusion equation $\alpha = 1$, $\tau = 0.1$, $\bar{\zeta} = 1/2$, $\bar{v} = 5$

For the auxiliary function $u_\infty(x,y,\zeta,t)$ (14) after applying the Laplace transform with respect to the time t and the double exponential Fourier transform with respect to the spatial coordinates x and y we obtain

$$\tilde{\tilde{u}}^*_\infty(\xi,\eta,\zeta,s) = \frac{p_0}{2\pi}\exp\left(-\frac{v\zeta}{2a}\right)\exp(i\zeta\xi)\frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2) + \dfrac{v^2}{2a}} \tag{38}$$

and

$$\tilde{u}^*_\infty(x,\eta,\zeta,s) = \frac{p_0}{(2\pi)^{3/2}}\exp\left(-\frac{v\zeta}{2a}\right)\int_{-\infty}^{\infty}\frac{\cos[(x-\zeta)\xi]s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2) + \dfrac{v^2}{2a}}\,\mathrm{d}\xi, \tag{39}$$

$$\begin{aligned} u_\infty(x,\eta,\zeta,t) &= \frac{p_0}{(2\pi)^2}\exp\left(-\frac{v\zeta}{2a^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos[(x-\zeta)\xi]\cos(y\eta), \\ &\times E_\alpha\left[-a\left(\xi^2+\eta^2+\frac{v^2}{2a^2}\right)t^\alpha\right]\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{40}$$

After introducing the polar coordinates in the (ξ,η)-plane, we get (for details, see (POVSTENKO 2014)):

$$G_\infty(x,y,\zeta,t) = \frac{p_0}{2\pi}\exp\left[\frac{v(x+y-\zeta)}{2a}\right]\int_0^\infty E_\alpha\left[-\left(ar^2+\frac{v^2}{2a}\right)t^\alpha\right]J_0\left[\rho\sqrt{(x-\zeta)^2+y^2}\right]\rho\mathrm{d}\rho, \tag{41}$$

where $J_0(r)$ is the Bessel function.

For large values of the negative argument, the Mittag-Leffler function $E_\alpha(-x)$ has the asymptotic

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)x}, \tag{42}$$

and for $\alpha \neq 1$ the integral in (41) is divergent at the point $x = \zeta$, $y = 0$. To separate the singularity, we rewrite (41) as

$$G_\infty(x,y,\zeta,t)=\frac{p_0}{2\pi}\exp\left[\frac{v(x+y-\zeta)}{2a}\right]\left\langle\int_0^\infty\left\{E_\alpha\left[-\left(ar^2+\frac{v^2}{2a}\right)t^\alpha\right]-\frac{1}{\Gamma(1-\alpha)\left[1+\left(ar^2+\frac{v^2}{2a}\right)t^\alpha\right]}\right\}\right.$$

$$\left.\times J_0\left[\rho\sqrt{(x-\zeta)^2+y^2}\right]\rho\mathrm{d}\rho+\int_0^\infty\frac{J_0\left[\rho\sqrt{(x-\zeta)^2+y^2}\right]}{\Gamma(1-\alpha)\left[1+\left(a\rho^2+\frac{v^2}{2a}\right)t^\alpha\right]}\rho\mathrm{d}\rho\right\rangle. \tag{43}$$

The first integral in (43) has no singularity, the second one can be evaluated analytically taking that (PRUDNIKOV et al. 1986b)

$$\int_0^\infty\frac{x}{x^2+p^2}J_0(qx)\mathrm{d}x=K_0(pq),\quad p>0,\ q>0, \tag{44}$$

where $K_0(x)$ is the modified Bessel function having the logarithmic singularity at the origin. Hence

$$G_\infty(x,y,\zeta,t)=(\text{regular term})+$$

$$+\frac{p_0}{2\pi\Gamma(1-\alpha)at^\alpha}\exp\left[\frac{v(x+y-\zeta)}{2a}\right]K_0\left[\sqrt{1+\frac{v^2t^\alpha}{2a}}\sqrt{\frac{(x-\zeta)^2+y^2}{at^\alpha}}\right]. \tag{45}$$

Now we reformulate the considered problem for a layer $0 < x < l$, $-\infty < y\ \infty$. Let

$$u(x,y,\zeta,t)=u_\infty(x,y,\zeta,t)+g(x,y,\zeta,t), \tag{46}$$

and

$$G(x,y,\zeta,t)=G_\infty(x,y,\zeta,t)+\exp\left[\frac{v(x+y)}{2a}\right]g(x,y,\zeta,t) \tag{47}$$

where G_∞ is the solution of the corresponding Cauchy problem in an infinite plane. In such a way, we get

$$\frac{\partial^\alpha g}{\partial t^\alpha}=a\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}\right)-\frac{v^2}{2a}g,\ x\in(0,l),\ y\in(-\infty,+\infty),\ t>0; \tag{48}$$

$$g(0,y,\zeta,t)=-u_\infty(0,y,\zeta,t),\ y\in(-\infty,+\infty),\ t>0; \tag{49}$$

$$g(l,y,\zeta,t) = -u_\infty(l,y,\zeta,t),\ y \in (-\infty, +\infty),\ t > 0; \tag{50}$$

$$g(x,y,\zeta,0) = 0,\ x \in (0,l),\ y \in (-\infty, +\infty). \tag{51}$$

From (48)–(51), we have

$$\hat{\tilde{g}}^*(\xi_k,\eta,\zeta,s) = -\frac{a\xi_k[\tilde{u}_\infty^*(0,\eta,\zeta,s) - (-1)^k\tilde{u}_\infty^*(l,\eta,\zeta,s)]}{s^\alpha + a(\xi_k^2 + \eta^2) + \dfrac{v^2}{2a}} \tag{52}$$

The values $\tilde{u}_\infty^*(0,\eta,\zeta,s)$ and $\tilde{u}_\infty^*(l,\eta,\zeta,s)$ can be found from (39). Hence,

$$\hat{\tilde{g}}^*(\xi_k,\eta,\zeta,s) = -\frac{ap_0\xi_k}{(2\pi)^{3/2}}\exp\left(-\frac{v\zeta}{2a}\right)s^{\alpha-1}\int_{-\infty}^{\infty}\frac{\cos(\zeta\xi) - (-1)^k\cos[(l-\zeta)\xi]}{\Delta(\xi,\xi_k,\eta,s)}\,d\xi, \tag{53}$$

where

$$\Delta(\xi,\xi_k,\eta,s) = \left[s^\alpha + a(\xi_k^2 + \eta^2) + \frac{v^2}{2a}\right]\left[s^\alpha + a(\xi^2 + \eta^2) + \frac{v^2}{2a}\right]. \tag{54}$$

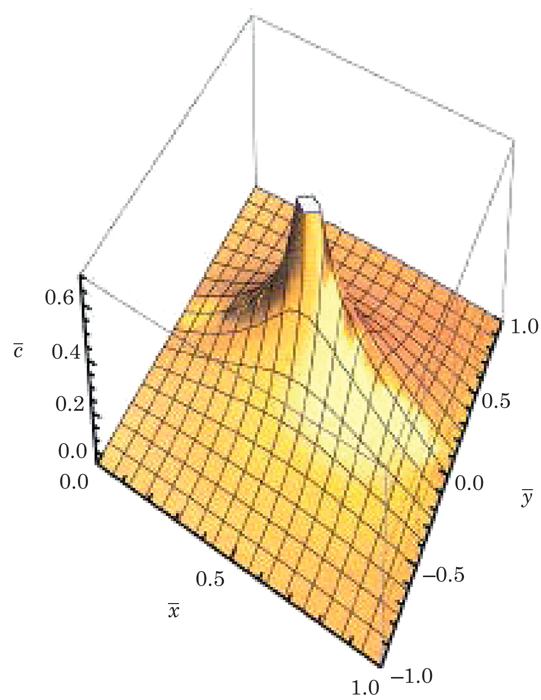

Fig. 3. Fundamental solution to the time-fractional advection diffusion equation; $\alpha = 0.5$, $\tau = 0.1$, $\bar{\zeta} = 1/2$, $\bar{v} = 1$

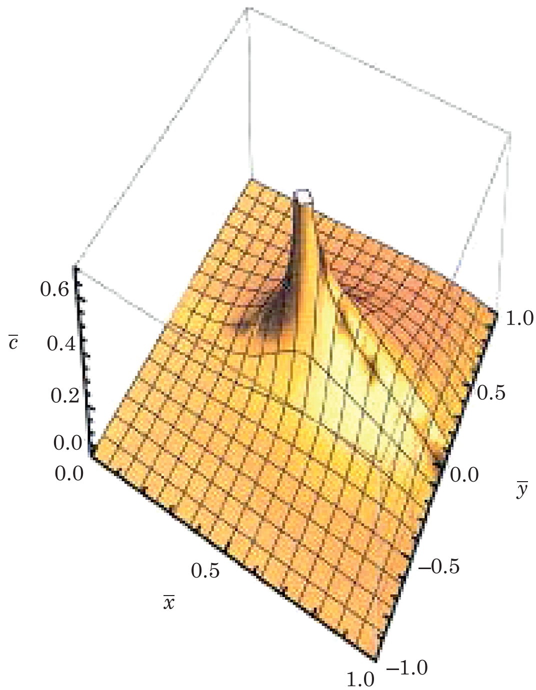

Fig. 4. Fundamental solution to the time-fractional advection diffusion equation; $\alpha = 0.5$, $\tau = 0.1$, $\bar{\zeta} = 1/2$, $\bar{v} = 5$

After decomposition of $1/\Delta(\xi,\xi_k,\eta,s)$ into partial fractions we obtain

$$\hat{\tilde{g}}^* (\xi_k,\eta,\zeta,s) = -\frac{p_0\xi_k}{(2\pi)^{3/2}} \exp\left(-\frac{v\zeta}{2a}\right)\int_{-\infty}^{\infty}\frac{\cos(\zeta\xi) - (-1)^k \cos[(l-\zeta)\xi]}{\xi^2 - \xi_k^2},$$

$$\times\left[\frac{s^{\alpha-1}}{s^\alpha + a(\xi_k^2 + \eta^2) + \dfrac{v^2}{2a}} - \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2 + \eta^2) + \dfrac{v^2}{2a}}\right]\mathrm{d}\xi. \tag{55}$$

The inversion of all the integral transforms gives the final result:

$$G(x,y,\zeta,t) = G_\infty(x,y,\zeta,t) + \frac{p_0}{2\pi^2 l} \exp\left[\frac{v(x + y - \zeta)}{2a}\right]$$

$$\times \sum_{k=1}^{\infty} \xi_k \sin(x\xi_k) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos(\zeta\xi) - (-1)^k \cos[(l - \zeta)\xi]}{\xi^2 - \xi_k^2} \cos(y\eta) \tag{56}$$

$$\times \left\{E_\alpha\left[-a\left(\xi^2 + \eta^2 + \frac{v^2}{2a^2}\right)t^\alpha\right] - E_\alpha\left[-a\left(\xi_k^2 + \eta^2 + \frac{v^2}{2a^2}\right)t^\alpha\right]\right\} \mathrm{d}\xi\mathrm{d}\eta.$$

Figs. 3 and 4 show the fundamental solution to the Cauchy problem to the time-fractional advection diffusion equation $\bar{c} = \sqrt{a t^{\alpha/2}} l p_0^{-1} G$ for $\alpha = 0.5$ and different values of the drift quantity $\bar{v}$.

Conclusions

We have considered the time-fractional advection diffusion equation with the Caputo fractional derivative in a domain $0 < x < l$, $-\infty < y < \infty$. The Laplace transform with respect to time t, the finite sin-Fourier transform with respect to the spatial coordinate x, and the exponential Fourier transform with respect to the spatial coordinate y have been used. The fundamental solution to the Cauchy problem has been obtained. The results of numerical calculations are displayed in Figures for different values of the nondimensional spatial variables $\bar{x}$ and $\bar{y}$, the drift parameter $\bar{v}$, and the order of the time-fractional derivative α. To evaluate the Mittag-Leffler function $E_\alpha(-x)$ we have used the algorithm suggested in (GORENFLO et al. 2002).

In the case of the standard diffusion equation with the order of time derivative $\alpha = 1$, the fundamental solution has no singularity, and the increase in the quantity $\bar{v}$ causes a drift of the maximum value of the solution and a decrease in this value (see Figs. 1 and 2). In the case of time-fractional advection diffusion equation with the order of time derivative $0 < \alpha < 1$, the fundamental solution to the Cauchy problem has no singularity only in the case of one spatial variable, whereas in the case of two spatial variables the fundamental solution has the logarithmic singularity, and the influence of the drift parameter $\bar{v}$ is less noticeable (Figs. 3 and 4). Increase in the drift parameter $\bar{v}$ only presses the plot to the singularity point.

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