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Higher-order conditions for local equilibria in a discontinuous Gale economic model*

by

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Abstract: The paper introduces the concept of a strict local equilibrium of order k in the Gale economic model. We obtain higher-order necessary and sufficient conditions for such equilibria without assuming continuity of the utility functions. These conditions are formulated in terms of generalized lower and upper directional derivatives, introduced by Studniarski (1986). A stability theorem for strict local equilibria of order k is also included.

Keywords: Gale model, equilibrium, higher-order conditions, generalized directional derivatives

1. Introduction

The aim of this paper is to obtain higher-order necessary and sufficient conditions for local equilibria in a simplified version of the Gale economic model (see Bula, 2003), in which the preferences of individual consumers and producers (or, more generally, agents) are described by the possibly discontinuous utility functions.

First of all, the paper presents a new definition of a strict local equilibrium of order k for the Gale model (see Definition 4). It may be regarded as a localized version of an equilibrium (Definition 3), which has some meaning by itself as it describes the situation, in which the consumers want to have only a little more of certain goods than they currently possess, but not infinitely more. Moreover, this definition allows for a simple characterization, which follows from the well-known characterization of strict local maximizers of order k for the general optimization problems. This characterization is formulated in terms

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of generalized higher-order lower and upper directional derivatives of utility functions. We also present a stability theorem for strict local equilibria of order k.

It is important that we allow the utility functions to be discontinuous. Discontinuous economic models are discussed in many papers (see in particular, Dasgupta and Maskin, 1986a,b; Bula, 2003; Nessah and Tian, 2016; Tian, 1992, 2015, 2016; and Tian and Zhou, 1992, and the references therein), where mainly existence theorems for equilibria are proved. There are also results on first order necessary conditions for various kinds of local Pareto-type optimal allocations in some models of welfare economics without continuity assumptions (for example, Mordukhovich, 2006, Theorems 8.5 and 8.8, and Bao and Mordukhovich, 2010, Theorem 4.3).

In our previous paper, Michalak and Studniarski (2014), we have applied the results of Rahmo and Studniarski (2012) to prove some higher-order necessary and sufficient conditions for locally Pareto optimal allocations in the Gale model, without assuming continuity of the utility functions. The optimality conditions obtained in this paper are similar to those of Michalak and Studniarski (2014) but not identical with them, because they involve a price vector, which is not used in Michalak and Studniarski (2014). There is also a difference in methods by which the respective optimality conditions are obtained. In Michalak and Studniarski (2014) we apply vector optimization and Pareto optimality, while in the present paper the problem of finding an equilibrium is reduced to a finite number of simple scalar optimization problems.

One may ask why we are concerned with strict local equilibria of order k if they do not have an evident economic interpretation (from the economic viewpoint it is only important whether or not we have a local equilibrium). The answer is that strict local equilibria of order $k = 1, 2, \ldots$ have useful stability properties: a sufficiently small change in the utility functions results in a model which has an equilibrium at a point close to the equilibrium of the original model (see Theorem 11). The number k controls the tolerance for possible changes of values of utility functions (see inequality (28)).

The paper is organized as follows. In Section 2 we review the results of Studniarski (1986), adapting them to the case of maximization problems. Section 3 contains the definitions of the Gale model, a feasible allocation and two kinds of equilibria. In Section 4 we apply the results of Studniarski (1986) to a special optimization problem, appearing in the Gale model. Section 5 contains our main results on characterization of strict local equilibria of order k for the Gale model. Finally, in Section 6 we present a stability theorem for strict local equilibria of order k.

2. Characterizations of strict local maximizers

In this section, we reformulate the main results of Studniarski (1986) so as to obtain higher-order conditions for local maximizers, instead of minimizers. These results can easily be obtained by substituting $-f$ for f in the original theorems.

Let E be a closed subset of $\mathbb{R}^n$, let $\bar{x} \in E$, and let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a function such that $|f(\bar{x})| < \infty$. We consider the following general optimization problem:

$$\max\{f(x) : x \in E\}. \tag{1}$$

Throughout the paper, we denote by $\|\cdot\|$ the Euclidean norm and by $B(\bar{x}, \varepsilon)$ the open ball with center $\bar{x}$ and radius ε.

Definition 1 *Let k be a positive integer. We say that $\bar{x}$ is a* strict local maximizer of order k *for problem* (1) *if there exist $\varepsilon > 0$ and $\beta > 0$ such that*

$$f(x) \leq f(\bar{x}) - \beta \|x - \bar{x}\|^k \quad \textit{for all} \;\; x \in E \cap B(\bar{x}, \varepsilon). \tag{2}$$

Let us introduce the following notation (for each $h \in \mathbb{R}^n$ and $k = 1, 2, \ldots$):

$$\begin{aligned} \underline{d}^k f(\bar{x}; h) &:= \liminf_{(t,v)\to(0^+,h)} \frac{f(\bar{x}+tv) - f(\bar{x})}{t^k} \\ &= \sup_{\delta>0} \left(\inf_{\substack{t\in(0,\delta)\\ v\in B(h,\delta)}} \frac{f(\bar{x}+tv) - f(\bar{x})}{t^k} \right), \end{aligned} \tag{3}$$

$$\begin{aligned} \overline{d}^k f(\bar{x}; h) &:= \limsup_{(t,v)\to(0^+,h)} \frac{f(\bar{x}+tv) - f(\bar{x})}{t^k} \\ &= \inf_{\delta>0} \left(\sup_{\substack{t\in(0,\delta)\\ v\in B(h,\delta)}} \frac{f(\bar{x}+tv) - f(\bar{x})}{t^k} \right). \end{aligned} \tag{4}$$

We shall write $\underline{d}f$ and $\overline{d}f$ instead of $\underline{d}^1 f$ and $\overline{d}^1 f$, respectively. We also denote by $K(E, \bar{x})$ the contingent cone to E at $\bar{x}$, that is,

$$\begin{aligned} K(E,\bar{x}) := \{&h \in \mathbb{R}^n : \exists h_\nu \to h,\, x_\nu \to \bar{x},\, t_\nu \in (0,+\infty) \\ &\text{with } x_\nu \in E \text{ and } h_\nu = (x_\nu - \bar{x})/t_\nu,\, \nu = 1,2,\ldots\}. \end{aligned} \tag{5}$$

Finally, we define

$$K_f(E,\bar{x}) := K(E,\bar{x}) \cap \left\{h \in \mathbb{R}^n : \overline{d}f(\bar{x};h) \geq 0\right\}, \tag{6}$$

$$f_E := f - \delta(\cdot|E) \;\; \text{where} \;\; \delta(x|E) := \begin{cases} 0 & \text{if} \;\; x \in E, \\ +\infty & \text{if} \;\; x \notin E. \end{cases} \tag{7}$$

Theorem 1 *(*see Studniarski, 1986, Theorem 2.1*) (i) If $k > 1$, then the following three conditions are equivalent:*

(a) *$\bar{x}$ is a strict local maximizer of order k for problem* (1)*;*

(b) *for all $h \in \mathbb{R}^n \backslash \{0\}$, we have*

$$\overline{d}^k f_E(\bar{x}; h) < 0; \tag{8}$$

(c) *inequality* (8) *holds for all* $h \in K_f(E,\bar{x})\backslash\{0\}$.

(ii) *If* $k = 1$, *then analogous equivalences are true with condition* (c) *replaced by the following one:*

(c′) *inequality* (8) *holds for all* $h \in K(E,\bar{x})\backslash\{0\}$.

Theorem 1 is difficult to apply in practice, because of the presence of the indicator function δ in inequality (8). Therefore, we shall also formulate two other results, providing necessary and sufficient conditions for strict local maximizers of order k for problem (1) (but not characterizations of such maximizers).

Theorem 2 (see Studniarski, 1986, Theorem 2.2) *If* $\bar{x}$ *is a strict local maximizer of order* $k \geq 1$ *for problem* (1), *then* $\underline{d}^k f(\bar{x};h) < 0$ *for all* $h \in K(E,\bar{x})\backslash\{0\}$.

Theorem 3 (see Studniarski, 1986, Corollary 2.1) (i) *If* $k > 1$ *and* $\overline{d}^k f(\bar{x};h) < 0$ *for all* $h \in K_f(E,\bar{x})\backslash\{0\}$, *then* $\bar{x}$ *is a strict local maximizer of order* k *for problem* (1).

(ii) *If* $\overline{d} f(\bar{x};h) < 0$ *for all* $h \in K(E,\bar{x})\backslash\{0\}$, *then* $\bar{x}$ *is a strict local maximizer of order* 1 *for problem* (1).

3. The Gale model and the definitions of equilibria

We now describe a simplified version of the Gale model (Bula, 2003). Suppose we have n goods $G_1,...,G_n$ and m economic agents $A_1,...,A_m$. The set of goods includes all types of labor and services, as well as material commodities. The economic agents may be thought of as either consumers or producers.

The amount of goods $G_1,...,G_n$, supplied or consumed by an agent A_i in a certain fixed time interval is given by a vector

$$x_i = (x_{i,1},...,x_{i,n}) \in \mathbb{R}^n. \tag{9}$$

The j-th coordinate $x_{i,j}$ represents the amount of the good G_j and is positive (respectively, negative) if G_j is supplied (respectively, consumed). Such a vector is called a *commodity bundle* of A_i. The set C_i of all possible commodity bundles (9) is called the *commodity set* or *technology set* of the agent A_i, $i = 1,...,m$.

In the Gale model, it is assumed that the *balance inequalities* hold, i.e., the total amount of each good consumed by all agents must not exceed the total amount supplied:

$$\sum_{i=1}^{m} x_{i,j} \geq 0, \;\; j = 1,...,n. \tag{10}$$

Let us note that condition (10) may be written down in an equivalent vector form: $\sum_{i=1}^{m} x_i \geq 0$.

Definition 2 *A vector system* $\{x_1,...,x_m\}$ *is called*

(i) *a* feasible allocation *if* $x_i \in C_i$, $i = 1,...,m$, *and inequalities* (10) *hold;*

(ii) *a* strictly feasible allocation *if it is a feasible allocation and all inequalities* (10) *hold as strict inequalities:*

$$\sum_{i=1}^{m} x_{i,j} > 0, \quad j = 1, ..., n. \tag{11}$$

Remark 1 *Condition* (11) *means that, for each good* G_j*, there exists some excess of supply over demand.*

We assume that, for each agent A_i, there exists a *utility function*

$$f_i : C_i \to \mathbb{R}, \tag{12}$$

$i = 1, ..., m$. Each agent tends to maximize his utility function.

Definition 3 (Bula, 2003, Definition 2.3) *A price vector* $p^* \in \mathbb{R}^n$ *and a feasible allocation* $\{x_1^*, ..., x_m^*\}$ *are called an* equilibrium *if, for each* $i \in \{1, ..., m\}$, x_i^* *is a solution of the following optimization problem:*

$$\max\{f_i(x_i) : \langle p^*, x_i \rangle \geq 0, \; x_i \in C_i\}. \tag{13}$$

Clearly, for any equilibrium $(p^*, \{x_1^*, ..., x_m^*\})$, the following inequality holds:

$$\sum_{i=1}^{m} x_i^* \geq 0. \tag{14}$$

Definition 4 *Let* k *be a positive integer. A price vector* $p^* \in \mathbb{R}^n$ *and a feasible allocation* $\{x_1^*, ..., x_m^*\}$ *are called a* strict local equilibrium of order k *if* (14) *holds and, for each* $i \in \{1, ..., m\}$, x_i^* *is a strict local maximizer of order* k *for* (13).

Remark 2 *Obviously, every strict local equilibrium* $(p^*, \{x_1^*, ..., x_m^*\})$ *of order* k *is a* local equilibrium *in the sense that there exists* $\varepsilon > 0$ *such that* $f_i(x_i) \leq f_i(x_i^*)$ *for all* $i \in \{1, ..., m\}$ *and* $x_i \in C_i \cap B(x_i^*, \varepsilon)$ *satisfying* $\langle p^*, x_i \rangle \geq 0$. *But it is not necessarily an equilibrium, unless some convexity assumptions are imposed on* f_i *and* C_i. *Nevertheless, finding local equilibria can be helpful as they can be considered as possible "candidates" for equilibria.*

4. Higher-order optimality conditions for a special optimization problem

In this section, we derive optimality conditions for any of the problems (13). For simplicity, we consider one problem of this kind with the lower index i dropped. Let C be a nonempty closed convex subset of $\mathbb{R}^n$, let $f : C \to \mathbb{R}$ and $p^* \in \mathbb{R}^n \backslash \{0\}$. Our optimization problem is

$$\max\{f(x) : \langle p^*, x \rangle \geq 0, \; x \in C\}. \tag{15}$$

We denote by S the feasible set for problem (15):

$$S := \{x \in \mathbb{R}^n : \langle p^*, x\rangle \geq 0,\, x \in C\}. \tag{16}$$

In order to apply Theorems 1–3 to problem (15), we must extend the function f from C to the whole space $\mathbb{R}^n$. We introduce the following notation:

$$\bar{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in C, \\ -\infty & \text{if} \quad x \notin C. \end{cases} \tag{17}$$

Then $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$. It is easy to see that problem (15) is equivalent to the following one:

$$\max\{\bar{f}(x) : x \in S\}. \tag{18}$$

Theorem 4 *Let $\bar{x} \in S$. Define the following set of directions:*

$$D(\bar{x}) := \begin{cases} \{h \in \mathbb{R}^n : \langle p^*, h\rangle \geq 0\} & \textit{if} \quad \langle p^*, \bar{x}\rangle = 0, \\ \mathbb{R}^n & \textit{if} \quad \langle p^*, \bar{x}\rangle > 0. \end{cases} \tag{19}$$

(i) *If $k > 1$, then the following three conditions are equivalent:*
(a) *$\bar{x}$ is a strict local maximizer of order k for problem* (15)*;*
(b) *for all $h \in \mathbb{R}^n \backslash \{0\}$, we have*

$$\overline{d}^k \bar{f}_S(\bar{x}; h) < 0; \tag{20}$$

(c) *inequality* (20) *holds for all $h \in K(C, \bar{x}) \cap D(\bar{x}) \backslash \{0\}$ such that $\overline{d}\bar{f}(\bar{x}; h) \geq 0$.*

(ii) *If $k = 1$, then analogous equivalences are true with condition* (c) *replaced by the following one:*
(c$'$) *inequality* (20) *holds for all $h \in K(C, \bar{x}) \cap D(\bar{x}) \backslash \{0\}$.*

Proof. We will apply Theorem 1 with f and E replaced by $\bar{f}$ and S, respectively. Let $G := \{x \in \mathbb{R}^n : \langle p^*, x\rangle \geq 0\}$. Suppose, first, that $\langle p^*, \bar{x}\rangle = 0$. Then it is easy to verify that

$$K(S, \bar{x}) = K(C \cap G, \bar{x}) \subset K(C, \bar{x}) \cap K(G, \bar{x}) = K(C, \bar{x}) \cap D(\bar{x}). \tag{21}$$

For $k > 1$, it follows from Theorem 1 that conditions (a) and (b) of Theorem 4 are equivalent, and they are both equivalent to the following condition:

(c_1) inequality (20) holds for all $h \in K(S, \bar{x}) \backslash \{0\}$ such that $\overline{d}\bar{f}(\bar{x}; h) \geq 0$.

We see from relations (21) that the implications (b)$\Rightarrow$(c)$\Rightarrow$(c_1) are true. Therefore, (c) is equivalent to (a) and (b) as stated. The proof for $k = 1$ can be copied from this by deleting the condition $\overline{d}\bar{f}(\bar{x}; h) \geq 0$.

Now, suppose that $\langle p^*, \bar{x}\rangle > 0$. Then $\bar{x} \in \text{int}G$, and $K(S, \bar{x}) = K(C, \bar{x})$. Since $D(\bar{x}) = \mathbb{R}^n$, the proof in this case is trivial. ∎

The proofs of the following two theorems are analogous to that of Theorem 4. They follow from Theorems 2 and 3, respectively.

THEOREM 5 *Suppose that* (19) *holds. If* $\bar{x}$ *is a strict local maximizer of order* $k \geq 1$ *for problem* (15), *then* $\underline{d}^k \bar{f}(\bar{x}; h) < 0$ *for all* $h \in K(C, \bar{x}) \cap D(\bar{x}) \backslash \{0\}$.

THEOREM 6 *Suppose that* (19) *holds.*

(i) *If* $k > 1$ *and* $\overline{d}^k \bar{f}(\bar{x}; h) < 0$ *for all* $h \in K(C, \bar{x}) \cap D(\bar{x}) \backslash \{0\}$ *such that* $\overline{d} \bar{f}(\bar{x}; h) \geq 0$, *then* $\bar{x}$ *is a strict local maximizer of order* k *for problem* (15).

(ii) *If* $\overline{d} \bar{f}(\bar{x}; h) < 0$ *for all* $h \in K(C, \bar{x}) \cap D(\bar{x}) \backslash \{0\}$, *then* $\bar{x}$ *is a strict local maximizer of order* 1 *for problem* (15).

5. Higher-order conditions for strict local equilibria

Below, we reformulate Theorems 4,5 and 6 so as to obtain higher-order necessary and sufficient conditions for strict local equilibria in the Gale model. These modifications are quite obvious. We denote by S_i the set of feasible points for problem (13):

$$S_i := \{x_i \in \mathbb{R}^n : \langle p^*, x_i \rangle \geq 0,\ x_i \in C_i\}. \tag{22}$$

Moreover, we denote by $\bar{f}_i$ the extension of f_i to the whole space $\mathbb{R}^n$ by a formula analogous to (17).

THEOREM 7 *Suppose that* C_i *are nonempty, closed and convex.*

(i) If $k > 1$, *then the following three conditions are equivalent:*

(a) $(p^*, \{x_1^*, ..., x_m^*\})$ *is a strict local equilibrium of order* k;

(b) inequality (14) *holds and, for all* $i \in \{1, ..., m\}$ *and* $h \in \mathbb{R}^n \backslash \{0\}$, *we have*

$$\overline{d}^k (\bar{f}_i)_{S_i}(x_i^*; h) < 0; \tag{23}$$

(c) inequality (14) *holds and, for all* $i \in \{1, ..., m\}$ *and* $h \in K(C_i, x_i^*) \cap D(x_i^*) \backslash \{0\}$ *such that* $\overline{d} \bar{f}_i(x_i^*; h) \geq 0$, *we have* (23) *(here the set* $D(x_i^*)$ *is defined by a formula analogous to* (19)*).*

(ii) If $k = 1$, *then analogous equivalences are true with condition (c) replaced by the following one:*

(c') inequality (14) *holds and, for all* $i \in \{1, ..., m\}$ *and* $h \in K(C_i, x_i^*) \cap D(x_i^*) \backslash \{0\}$, *we have* (23).

THEOREM 8 *Suppose that* C_i *are nonempty, closed and convex. If* $(p^*, \{x_1^*, ..., x_m^*\})$ *is a strict local equilibrium of order* $k \geq 1$, *then inequality* (14) *holds and we have* $\underline{d}^k \bar{f}_i(x_i^*; h) < 0$ *for all* $i \in \{1, ..., m\}$ *and* $h \in K(C_i, x_i^*) \cap D(x_i^*) \backslash \{0\}$.

THEOREM 9 *Suppose that* C_i *are nonempty, closed and convex.*

(i) *If* $k > 1$, *inequality* (14) *holds and we have* $\overline{d}^k \bar{f}_i(x_i^*; h) < 0$ *for all* $i \in \{1, ..., m\}$ *and* $h \in K(C_i, x_i^*) \cap D(x_i^*) \backslash \{0\}$ *such that* $\overline{d} \bar{f}_i(x_i^*; h) \geq 0$, *then* $(p^*, \{x_1^*, ..., x_m^*\})$ *is a strict local equilibrium of order* k.

(ii) *If inequality* (14) *holds and we have* $\overline{d} \bar{f}_i(x_i^*; h) < 0$ *for all* $i \in \{1, ..., m\}$ *and* $h \in K(C_i, x_i^*) \cap D(x_i^*) \backslash \{0\}$, *then* $(p^*, \{x_1^*, ..., x_m^*\})$ *is a strict local equilibrium of order* 1.

Example 1 *Consider the case with one agent and two goods ($m = 1$, $n = 2$). Let $x = (x_1, x_2)$ be the amounts of the first and second good for the agent and let $x \in C = C_1 \times C_2 = [-10, 10] \times [-10, 10]$. We shall assume that the utility function of the agent has the form*

$$f(x) = \begin{cases} x_1 + x_2 & \text{for} \quad x_1^2 + x_2^2 \leq 25, \\ x_1 + x_2 - 1 & \text{for} \quad x_1^2 + x_2^2 > 25. \end{cases}$$

After exceeding a certain level of production, the agent loses some benefits, which he had, so his utility suddenly decreases. This is the reason why the utility function in our case is discontinuous.

Let $p^* = \left(\frac{3}{2}, 1\right)$ be a given price vector. We shall prove that the point $x^* = (x_1^*, x_2^*) = \left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$ is a strict local maximizer of order one for the problem

$$\max\{f(x) : x \in S\}, \tag{24}$$

where

$$S := \{x \in \mathbb{R}^2 : x \in G \cap C\}$$

and $G := \{x \in X : \langle p^*, x\rangle \geq 0\} = \{x \in X : \frac{3}{2}x_1 + x_2 \geq 0\}$. Now, take any direction $y \neq (0, 0)$. We compute (using (17) and the fact that $x^* \in \text{int}C$)

$$\bar{d}\bar{f}(x^*; y) = \limsup_{(t,v)\to(0^+,y)} \frac{f(x^* + tv) - f(x^*) - \delta(\bar{x} + tv|C)}{t} = \bar{d}f(x^*; y)$$

$$= \begin{cases} \limsup\limits_{(t,v)\to(0^+,y)} \frac{tv_1+tv_2-1}{t} & \text{for} \quad y_1 + y_2 > 0, \\ \limsup\limits_{(t,v)\to(0^+,y)} \frac{tv_1+tv_2}{t} & \text{for} \quad y_1 + y_2 \leq 0, \end{cases}$$

$$= \begin{cases} -\infty & \text{for} \quad y_1 + y_2 > 0, \\ y_1 + y_2 & \text{for} \quad y_1 + y_2 \leq 0. \end{cases}$$

This means that $\bar{d}\bar{f}(x^*; y) < 0$ for all $y \in K(G, x^*)\backslash\{(0, 0)\}$ (observe that $K(G, x^*) = G$). Hence, it follows from Theorem 6(ii) that $\bar{x}$ is a strict local maximizer of order 1 for problem (24). Since we have only one agent and $x^* \geq 0$, this means that the pair (p^*, x^*) is a strict local equilibrium of order 1.

6. A stability theorem

Hyers (1978) introduced the following notion of stability of minimum points: a relative minimum point $\bar{x}$ of a function f is called *stable* if all functions (of a suitable class) which are sufficiently close to f have relative minimum points within a prescribed distance from $\bar{x}$. Based on this idea, we present below a theorem on stability of strict local equilibria of order k (Theorem 11). This result shows, in particular, that our definition of a strict local equilibrium of order k (Definition 4) can be useful for applications, because it guarantees the stability of a local equilibrium in the sense of Hyers with respect to arbitrary perturbations of the utility functions.

The following theorem is a finite-dimensional version of Hyers (1985, Theorem 2.1), where minimization is replaced by maximization. Note that it does not require continuity of the function being maximized, but assumes upper semicontinuity of the perturbed function.

THEOREM 10 *Let* $\rho : [0, \infty) \to \mathbb{R}$ *be a strictly increasing function with* $\rho(0) = 0$. *Let* E *be a closed subset of* $\mathbb{R}^n$, *let* $\bar{x} \in E$, *and let* $f : E \to \mathbb{R}$ *be a function such that*

$$f(x) \leq f(\bar{x}) - \rho(\|x - \bar{x}\|) \quad \textit{for all} \;\; x \in E \cap \mathrm{cl}B(\bar{x}, \varepsilon). \tag{25}$$

For a given $\varepsilon > 0$, *let* $\tilde{f} : E \to \mathbb{R}$ *be any upper semicontinuous function satisfying the inequality*

$$\left|\tilde{f}(x) - f(x)\right| < \rho(\varepsilon)/2 \quad \textit{for all} \;\; x \in \mathrm{cl}B(\bar{x}, \varepsilon). \tag{26}$$

Then $\tilde{f}$ *has a maximum value on the set* $E \cap \mathrm{cl}B(\bar{x}, \varepsilon)$ *which is taken on at some point* $\tilde{x} \in B(\bar{x}, \varepsilon)$.

REMARK 3 *The conclusion of Theorem 10 means that problem* (1) *with* f *replaced by* $\tilde{f}$ *has a local solution* $\tilde{x}$ *within the distance of* ε *from* $\bar{x}$.

We now consider the Gale model described in Section 3. Let us observe that, to apply Theorem 10, we do not need to use the extensions $\bar{f}_i$ of the utility functions f_i to the whole space $\mathbb{R}^n$.

THEOREM 11 *Let* $(p^*, \{x_1^*, ..., x_m^*\})$ *be a strict local equilibrium of order k, which implies, in particular, that there exist numbers* $\varepsilon > 0$ *and* $\beta > 0$ *such that*

$$f_i(x_i) \leq f_i(x_i^*) - \beta \left\|x_i - x_i^*\right\|^k \quad \textit{for all} \;\; x_i \in S_i \cap \mathrm{cl}B(x_i^*, \varepsilon),\; i \in \{1, ..., m\}, \tag{27}$$

where the sets S_i *are defined by* (22). *We assume that the sets* C_i, *appearing in* (22), *are closed, and* $f_i : C_i \to \mathbb{R}$, $i = 1, ..., m$, *are arbitrary functions. Let* $\tilde{f}_i : C_i \to \mathbb{R}$, $i = 1, ..., m$, *be any upper semicontinuous functions satisfying the inequalities*

$$\left|\tilde{f}_i(x_i) - f_i(x_i)\right| < \beta\varepsilon^k/2 \quad \textit{for all} \;\; x_i \in S_i \cap \mathrm{cl}B(x_i^*, \varepsilon). \tag{28}$$

Then each of the problems

$$\max\{\tilde{f}_i(x_i) : \langle p^*, x_i\rangle \geq 0,\; x_i \in C_i\} \tag{29}$$

has a local maximizer $\tilde{x}_i$ *which belongs to* $B(x_i^*, \varepsilon)$. *Moreover, if*

$$\sum_{i=1}^{m} \tilde{x}_i \geq 0 \tag{30}$$

holds, then $(p^*, \{\tilde{x}_1, \ldots, \tilde{x}_m\})$ *is a local equilibrium.*

Proof. The proof follows by applying Theorem 10 to each of the problems (13), with $\rho(t) := \beta t^k$ and $E := S_i$. ■

Remark 4 *The order k of a strict local equilibrium plays an important role in Theorem* 11 *as it determines the permitted distance between the values of f_i and $\tilde{f}_i$ in condition* (28). *In particular, if $\varepsilon < 1$, then $\varepsilon^{k'} < \varepsilon^k$ for $k' > k$, so the larger values of k lead to tighter bounds for perturbations of the utility functions.*

Remark 5 *The additional assumption* (30) *is necessary because, for every local equilibrium, the corresponding system of commodity bundles must be a feasible allocation. Unfortunately, this condition is not in line with the general stability principle of Hyers. One would prefer to have a local equilibrium for the perturbed model with $\tilde{f}_i$ sufficiently close to f_i without any extra requirement. The next result describes some particular situation where this is possible.*

Corollary 1 *Let the assumptions of Theorem* 11 *be satisfied. Suppose, in addition, that the allocation $\{x_1^*, ..., x_m^*\}$ is strictly feasible. Then, there exists $\delta > 0$ such that each of the problems* (29) *has a local maximizer $\tilde{x}_i$ which belongs to $B(x_i^*, \delta)$. Moreover, $(p^*, \{\tilde{x}_1, \ldots, \tilde{x}_m\})$ is a local equilibrium.*

Proof. Since $\{x_1^*, ..., x_m^*\}$ is strictly feasible, one can find $\delta \in (0, \varepsilon)$ such that inequalities (10) hold for all $x_i \in B(x_i^*, \delta)$, $i = 1, ..., m$. Now, we can apply Theorem 11 with ε replaced by δ. Then (30) follows from (10) and the inclusions $\tilde{x}_i \in B(x_i^*, \delta)$. ■

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