Opuscula Math. 36, no. 5 (2016), 651–669
http://dx.doi.org/10.7494/OpMath.2016.36.5.651 Opuscula Mathematica

ON NONOSCILLATORY SOLUTIONS OF TWO DIMENSIONAL NONLINEAR DELAY DYNAMICAL SYSTEMS

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Communicated by Josef Diblík

Abstract. We study the classification schemes for nonoscillatory solutions of a class of nonlinear two dimensional systems of first order delay dynamic equations on time scales. Necessary and sufficient conditions are also given in order to show the existence and nonexistence of such solutions and some of our results are new for the discrete case. Examples will be given to illustrate some of our results.

Keywords: time scales, oscillation, two-dimensional dynamical system.

Mathematics Subject Classification: 34C10, 34K11.

1. INTRODUCTION

A number of oscillation and nonoscillation criteria has already been given for special cases of the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)), \\ y^{\Delta}(t) = -b(t)g(x(\tau(t))), \end{cases} \tag{1.1}$$

where $a, b \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, $\tau \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, [t_0,\infty)_{\mathbb{T}}\right)$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, f and g are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$, see [1, 10, 11]. Motivated by [12] in which $\tau(t) = t - \eta$, $\eta > 0$, our purpose is to obtain the existence and nonexistence of nonoscillatory solutions of (1.1). So according to our knowledge, not only we improve the results obtained in [12] but also some of our results are new for the discrete case. The theory of time scales, which is a closed subset of real numbers denoted by $\mathbb{T}$, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order not only to unify continuous and discrete analysis but also extend results for any time scale, see [2] and [3]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. We mean by $t \geq t_1$ that $t \in [t_1,\infty)_{\mathbb{T}} := [t_1,\infty) \cap \mathbb{T}$. We call (x, y) a *proper solution* if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$

for $t \geq t_0$. A solution (x, y) of (1.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

One can easily show that any nonoscillatory solution (x, y) of system (1.1) belongs to one of the following two classes:

$$M^+ := \{(x, y) \in M : \ xy > 0 \text{ eventually}\},$$
$$M^- := \{(x, y) \in M : \ xy < 0 \text{ eventually}\},$$

where M be the set of all nonoscillatory solutions of system (1.1).

For convenience, let us set

$$A(t) = \int_t^\infty a(s)\Delta s \quad \text{and} \quad B(t) = \int_t^\infty b(s)\Delta s.$$

The set up of this paper is as follows: In Section 1, we give essential lemmas which are used in proofs of our main results. In Section 2, we show the existence of nonoscillatory solutions of system (1.1) in some sub-classes of M^+ and M^- by using convergence/divergence of $A(t_0)$ and $B(t_0)$ for $t_0 \in \mathbb{T}$ and some other improper integrals. We also give examples in order to highlight our main results. In Section 3, we show the nonexistence of nonoscillatory solutions of system (1.1) in M^+ and M^-. Finally, we end up the paper by a conclusion.

It can be shown as in [1] that component functions x and y are themselves nonoscillatory if (x, y) is a nonoscillatory solution of system (1.1). In the following lemmas, we get oscillation and nonoscillation criteria of system (1.1). Since system (1.1) has been considered without a delay term in [11], we refer the reader to [11] for some of the proofs we skip here.

Lemma 1.1.

(a) *If $A(t_0) < \infty$ and $B(t_0) < \infty$, then system* (1.1) *is nonoscillatory.*
(b) *If $A(t_0) = \infty$ and $B(t_0) = \infty$, then system* (1.1) *is oscillatory.*

Proof. (a) Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. Choose $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\int_{t_1}^\infty a(t) f\left(1 + g(2)\int_t^\infty b(s)\Delta s\right)\Delta t < 1.$$

Let X be the space of all rd-continuous functions on $\mathbb{T}$ with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \Big\{x \in X : \quad 1 \leq x(\tau(t)) \leq 2, \quad \tau(t) \geq t_1\Big\}.$$

For any subset S of Ω, we have that $\inf S \in \Omega$ and $\sup S \in \Omega$. Define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = 1 + \int_{t_1}^t a(s) f\left(1 + \int_s^\infty b(u) g(x(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1.$$

By using the monotonicity and the fact that $x \in \Omega$, we have

$$1 \leq (Fx)(t) \leq 1 + \int_{t_1}^{t} a(s) f\left(1 + g(2) \int_{s}^{\infty} b(u)\Delta u\right) \Delta s \leq 2, \quad \tau(t) \geq t_1.$$

It is also easy to show that F is an increasing mapping. So by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $F\bar{x} = \bar{x}$. Then we have

$$\bar{x}^{\Delta}(t) = a(t) f\left(1 + \int_{t}^{\infty} b(u) g(x(\tau(u)))\Delta u\right).$$

Setting

$$\bar{y}(t) = 1 + \int_{t}^{\infty} b(u) g(x(\tau(u)))\Delta u$$

gives us

$$\bar{y}^{\Delta}(t) = -b(t) g(x(\tau(t))),$$

i.e., (x, y) is a nonoscillatory solution of (1.1). □

Lemma 1.2.

(a) *If $A(t_0) < \infty$ and $B(t_0) = \infty$, then any nonoscillatory solution (x, y) of system* (1.1) *belongs to M^-, i.e., $M^+ = \emptyset$.*
(b) *If $A(t_0) = \infty$ and $B(t_0) < \infty$, then any nonoscillatory solution (x, y) of system* (1.1) *belongs to M^+, i.e., $M^- = \emptyset$.*

The following lemma shows the limit behaviors of the component functions x and y of solution (x, y) of system (1.1).

Lemma 1.3. *Let (x, y) be a nonoscillatory solution of system* (1.1).

(a) *If $A(t_0) < \infty$, then the component function x of (x, y) has a finite limit.*
(b) *If $A(t_0) = \infty$ or $B(t_0) < \infty$, then the component function y of (x, y) has a finite limit.*

2. EXISTENCE OF NONOSCILLATORY SOLUTIONS OF (1.1) IN M^+ AND M^-

In this section, we show the existence of nonoscillatory solutions of system (1.1) by considering convergence/divergence of $A(t_0)$ and $B(t_0)$. Since the system (1.1) is oscillatory for the case $A(t_0) = \infty$ and $B(t_0) = \infty$, we only consider the other three cases.

2.1. THE CASE $A(t_0) = \infty$ AND $B(t_0) < \infty$

Let (x, y) be a nonoscillatory solution of system (1.1) such that the component function x of the solution (x, y) is eventually positive. Then by the same discussion in [11], we have that any nonoscillatory solution of system (1.1) in M^+ belongs to one of the following sub-classes:

$$M^+_{B,0} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\},$$
$$M^+_{\infty,B} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^+_{\infty,0} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = 0\right\},$$

where $0 < c < \infty$ and $0 < d < \infty$.

Theorem 2.1. *$M^+_{B,0} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} a(t) f\left(k \int_t^{\infty} b(s)\Delta s\right)\Delta t < \infty \tag{2.1}$$

for some nonzero k.

Proof. Suppose that there exists a solution $(x, y) \in M^+_{B,0}$ such that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_0$, $x(t) \to c_1$ and $y(t) \to 0$ as $t \to \infty$. Since x is eventually increasing, there exist $t_1 \geq t_0$ and $c_2 > 0$ such that $c_2 \leq g(x(\tau(t)))$ for $t \geq t_1$. Integrating the second equation from t to ∞ gives us

$$y(t) = \int_t^{\infty} b(s) g(x(\tau(s)))\Delta s, \quad t \geq t_1. \tag{2.2}$$

Also by integrating the first equation from t_1 to t, using the monotonicty of g and (2.2), we have

$$x(t) \geq \int_{t_1}^{t} a(s) f\left(\int_s^{\infty} b(u) g(x(\tau(u)))\Delta u\right)\Delta s \geq \int_{t_1}^{t} a(s) f\left(c_2 \int_s^{\infty} b(u)\Delta u\right)\Delta s$$

Setting $c_2 = k$ and taking the limit as $t \to \infty$ prove the assertion. (For the case $x < 0$ eventually, the proof can be shown similarly with $k < 0$.)

Conversely, suppose that (2.1) holds for some $k > 0$. (For the case $k < 0$ can be shown similarly.) Then choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} a(t) f\left(k \int_t^{\infty} b(s)\Delta s\right)\Delta t < \frac{c_1}{2}, \quad t \geq t_1, \tag{2.3}$$

where $k = g(c_1)$. Let X be the space of all continuous and bounded functions on $[t_1, \infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$. Then X is a Banach space, see [4]. Let Ω be the subset of X such that

$$\Omega := \left\{x \in X : \quad \frac{c_1}{2} \leq x(\tau(t)) \leq c_1, \quad \tau(t) \geq t_1\right\},$$

and define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = c_1 - \int_t^\infty a(s) f\left(\int_s^\infty b(u)g(x(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1.$$

It is easy to see that Ω is bounded, convex and a closed subset of X. Now let us show F has the following properties. F maps into itself. Indeed, we have

$$c_1 \geq (Fx)(t) \geq c_1 - \int_t^\infty a(s) f\left(g(c_1)\int_s^\infty b(u)\Delta u\right)\Delta s \geq \frac{c_1}{2}, \quad \tau(t) \geq t_1,$$

by (2.3). In order to show that F is continuous on Ω, let x_n be a sequence in Ω such that $x_n \to x \in \Omega = \bar{\Omega}$. Then for $\tau(t) \geq t_1$, we have

$$\begin{aligned} &|(Fx_n)(t) - (Fx)(t)| \\ &\leq \int_{t_1}^\infty a(s)\left|\left[f\left(-\int_s^\infty b(u)g(x_n(\tau(u)))\Delta u\right) - f\left(-\int_s^\infty b(u)g(x(\tau(u)))\Delta u\right)\right]\right|\Delta s. \end{aligned}$$

Then the Lebesgue Dominated Convergence theorem and the continuity of g give $\|(Fx_n) - (Fx)\| \to 0$ as $n \to \infty$, i.e., F is continuous on Ω. Finally, we show that $F\Omega$ is precompact. Let $x \in \Omega$ and $s, t \geq t_1$. Without loss of generality assume $s > t$. Then we have

$$|(Fx)(s) - (Fx)(t)| \leq \int_s^t a(u) f\left(g(c_1)\int_u^\infty b(\lambda)\Delta\lambda\right)\Delta u < \epsilon, \quad \tau(t) \geq t_1,$$

by assumption, which implies that $F\Omega$ is relatively compact. Then by the Schauder Fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. So as $t \to \infty$, we have $\bar{x}(t) \to c_1 > 0$. Setting

$$\bar{y}(t) = \int_t^\infty b(u)g(\bar{x}(\tau(u)))\Delta u > 0, \quad \tau(t) \geq t_1,$$

gives that $\bar{y}(t) \to 0$ as $t \to \infty$, i.e., $M_{B,0}^{+} \neq \emptyset$. $\square$

Example 2.2. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $m, n \geq 2$, $a(t) = \frac{1}{2t^{\frac{4}{5}}}$, $b(t) = \frac{3}{4t^2(8t-4)}$, $f(u) = u^{\frac{3}{5}}$, $k = 1$ and $g(u) = u$. First we need to show $A(t_0) = \infty$ and $B(t_0) < \infty$. Indeed,

$$\int_{t_0}^{t} a(s)\Delta s = \frac{1}{2} \sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} s^{\frac{1}{5}}.$$

So we have that

$$A(t_0) = \frac{1}{2} \lim_{n\to\infty} \sum_{m=2}^{n-1} (2^m)^{\frac{1}{5}} = \infty.$$

Since

$$\int_{t_0}^{t} b(s)\Delta s \leq \frac{3}{16} \sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} \frac{1}{s},$$

we have

$$B(t_0) \leq \frac{3}{16} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{2^m} < \infty$$

by the geometric series. Note that we have

$$\int_{t}^{T} b(s)\Delta s \leq \frac{3}{16} \sum_{s \in [t,T)_{2^{\mathbb{N}_0}}} \frac{1}{s}.$$

So this implies that

$$B(t) \leq \frac{3}{16} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{2^m} = \frac{3}{8} \lim_{n\to\infty} \left(\frac{1}{t} - \frac{1}{t2^n} \right) = \frac{3}{8t}.$$

Letting $k = 1$ and using the last inequality give us

$$\int_{t_0}^{T} a(t) f\left(k \int_{t}^{\infty} b(s)\Delta s \right) \Delta t \leq \int_{t_0}^{T} \frac{1}{2t^{\frac{4}{5}}} \left(\frac{3}{8t} \right)^{\frac{3}{5}} \Delta t = \left(\frac{3}{8} \right)^{\frac{3}{5}} \frac{1}{2} \sum_{t \in [1,T)_{2^{\mathbb{N}_0}}} \frac{1}{t^{\frac{2}{5}}}.$$

Therefore, we have that

$$\int_{t_0}^{\infty} a(t) f\left(k \int_{t}^{\infty} b(s)\Delta s \right) \Delta t \leq \left(\frac{3}{8} \right)^{\frac{3}{5}} \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^{\frac{2n}{5}}} < \infty$$

by the geometric series. One can also show that $(x, y) = \left(8 - \frac{1}{t}, \frac{1}{t^2}\right)$ is a nonoscillatory solution of

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{4}{5}}} \left(y(t)\right)^{\frac{3}{5}}, \\ \Delta_2 y(t) = -\dfrac{3}{4t^2(8t-4)} x\left(\dfrac{t}{4}\right) \end{cases} \tag{2.4}$$

such that $x(t) \to 8$ and $y(t) \to 0$ as $t \to \infty$, i.e., $M_{B,0}^{+} \neq \emptyset$ by Theorem 2.1.

When the case $A(t_0) = \infty$ and $B(t_0) < \infty$ holds, it can be shown that $M^+_{B,\infty} \neq \emptyset$ with $\tau(t) = t - \eta$ for $\eta \geq 0$, see [12].

2.2. THE CASE $A(t_0) < \infty$ AND $B(t_0) < \infty$

Since the component functions x and y have finite limits by Lemma 1.3, there can only exist two subclasses in M^+ by the same discussion in [11]:

$$M^+_{B,0} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c,\ \lim_{t\to\infty} |y(t)| = 0\right\},$$
$$M^+_{B,B} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c,\ \lim_{t\to\infty} |y(t)| = d\right\},$$

where $0 < c < \infty$ and $0 < d < \infty$. Because the existence of nonoscillatory solutions in $M^+_{B,0}$ is shown in the previous subsection, we only prove it for $M^+_{B,B}$.

Theorem 2.3. *$M^+_{B,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} a(s) f\left(d_1 + k \int_{s}^{\infty} b(u)\Delta u\right)\Delta s < \infty \tag{2.5}$$

for some $k \neq 0$ and $d_1 \neq 0$.

Proof. Suppose that there exists a nonoscillatory solution $(x, y) \in M^+_{B,B}$ such that $x > 0$ eventually, $x(t) \to c_1$ and $y(t) \to d_1$ as $t \to \infty$. (For the case $x < 0$ eventually, the proof can be shown similarly.) Since x is eventually positive and increasing, there exist a large $t_1 \geq t_0$ and $c_2 > 0$ such that $c_2 \leq x(\tau(t)) \leq c_1$ for $t \geq t_1$. Integrating the second equation from t to ∞ and the monotonicity of g give

$$y(t) \geq d_1 + g(c_2) \int_{t}^{\infty} b(s)\Delta s, \quad t \geq t_1. \tag{2.6}$$

Integrating also the first equation from t_1 to t and using the monotonicity of f yield us

$$x(t) \geq \int_{t_1}^{t} a(s) f\left(d_1 + g(c_2) \int_{s}^{\infty} b(\tau)\Delta\tau\right)\Delta s.$$

So as $t \to \infty$, the assertion follows for $k = g(c_2)$.

Conversely, suppose (2.5) holds. Choose $t_1 \geq t_0$, $k > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^{\infty} a(s) f\left(d_1 + k \int_{s}^{\infty} b(u)\Delta u\right) \Delta s < d_1, \tag{2.7}$$

where $k = g(2d_1)$. (The case $k, d_1 < 0$ can be done similarly.) Let X be the Banach space of all continuous real valued functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad d_1 \leq x(\tau(t)) \leq 2d_1, \quad \tau(t) \geq t_1\}.$$

For any subset B of Ω, it is clear that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us define an operator $F : \Omega \to X$ as

$$(Fx)(t) = d_1 + \int_{t_1}^{t} a(s) f\left(d_1 + \int_{s}^{\infty} b(u) g(x(\tau(u)))\Delta u\right) \Delta s, \quad \tau(t) \geq t_1.$$

It is obvious that F is an increasing mapping into itself. Indeed, we have

$$d_1 \leq (Fx)(t) \leq d_1 + \int_{t_1}^{t} a(s) f\left(d_1 + g(2d_1) \int_{s}^{\infty} b(u)\Delta u\right) \Delta s \leq 2d_1, \quad \tau(t) \geq t_1.$$

Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. By setting

$$\bar{y}(t) = d_1 + \int_{t}^{\infty} b(u) g(x(\tau(u))), \quad \tau(t) \geq t_1,$$

we have

$$\bar{y}^{\Delta}(t) = -b(t) g(x(\tau(t))).$$

Therefore, we have $\bar{x}(t) \to \alpha$ and $\bar{y}(t) \to d_1$ as $t \to \infty$, where $0 < \alpha < \infty$, i.e., $M^{+}_{B,B} \neq \emptyset$. Note that a similar proof can be done for the case $k < 0$ and $d_1 < 0$ with $x < 0$. □

Example 2.4. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $n \geq 2$, $a(t) = \frac{2}{2t^{\frac{5}{3}}(3t+1)^{\frac{1}{3}}}$, $b(t) = \frac{1}{2t(6t-4)}$, $f(u) = u^{\frac{1}{3}}$ and $g(u) = u$. We first show $A(t_0) < \infty$ and $B(t_0) < \infty$.

$$\int_{t_0}^{t} a(s)\Delta s = \frac{1}{2} \sum_{s\in[4,t)_{2^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}(3s+1)^{\frac{1}{3}}}.$$

So we have that

$$A(t_0) = \frac{1}{2} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{(2^m)^{\frac{2}{3}}(3\cdot 2^m + 1)^{\frac{1}{3}}} < \infty$$

by the Ratio test. Similarly,

$$\int_{t_0}^{t} b(s)\Delta s = \frac{1}{2} \sum_{s\in[4,t)_{2^{\mathbb{N}_0}}} \frac{1}{6s-4}.$$

Hence, as $t \to \infty$, we have that

$$B(t_0) = \frac{1}{2} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{6 \cdot 2^m - 4} < \infty.$$

Since $A(t_0) < \infty$ and $B(t_0) < \infty$, it is easy to show that (2.5) holds. One can also show that $\left(6 - \frac{1}{t}, 3 + \frac{1}{t}\right)$ is a nonoscillatory solution of

$$\begin{cases} \Delta_2 x(t) = \frac{2}{2t^{\frac{5}{3}}(3t+1)^{\frac{1}{3}}} \left(3 + \frac{1}{t}\right)^{\frac{1}{3}}, \\ \Delta_2 y(t) = -\frac{1}{2t(6t-4)} \left(6 - \frac{4}{t}\right) \end{cases} \tag{2.8}$$

such that $x(t) \to 6$ and $y(t) \to 3$ as $t \to \infty$, i.e., $M^+_{B,B} \neq \emptyset$ by Theorem 2.3.

2.3. THE CASE $A(t_0) < \infty$ AND $B(t_0) = \infty$

By the similar argument in [11], we have that any nonoscillatory solution of system (1.1) in M^- belongs to one of the following sub-classes:

$$\begin{aligned} M^-_{0,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{B,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{0,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |y(t)| = \infty\right\}, \\ M^-_{B,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = \infty\right\}, \end{aligned}$$

where $0 < c < \infty$ and $0 < d < \infty$.

Theorem 2.5. *$M^-_{B,\infty} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} a(s) f\left(k \int_{t_0}^{s} b(u)\Delta u\right) \Delta s < \infty \tag{2.9}$$

for some $k \neq 0$, where f is an odd function.

Proof. Suppose that there exists a nonoscillatory solution $(x,y) \in M^-_{B,\infty}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $t \geq t_1$, $x(t) \to c_2$ and $y(t) \to -\infty$ as $t \to \infty$, where $0 < c_2 < \infty$. Since x is monotonic and has a finite limit, there exist $t_2 \geq t_1$ and $c_3 > 0$ such that

$$c_2 \leq x(\tau(t)) \leq c_3 \quad \text{for} \quad t \geq t_2. \tag{2.10}$$

Integrating the first equation from t_2 to t gives us

$$c_2 \le x(t) = x(t_1) + \int_{t_1}^{t} a(s) f(y(s)) \Delta s \le c_3, \quad t \ge t_2.$$

So by taking the limit as $t \to \infty$, we have

$$\int_{t_2}^{\infty} a(s) |f(y(s))| \Delta s < \infty. \tag{2.11}$$

The monotonicity of g, (2.10) and integrating the second equation from t_2 to t yield us

$$y(t) \le y(t_2) - g(c_2) \int_{t_2}^{t} b(s)\Delta s \le -g(c_2) \int_{t_2}^{t} b(s) \Delta s.$$

Since $f(-u) = -f(u)$ for $u \neq 0$ and by the monotonicity of f, we have

$$|f(y(t))| \ge f\left(g(c_2) \int_{t_2}^{t} b(s)\Delta s \right), \quad t \ge t_2. \tag{2.12}$$

By (2.11) and (2.12), we have

$$\int_{t_2}^{t} a(s)|f(y(s))|\Delta s \ge \int_{t_2}^{t} a(s) f\left(g(c_2) \int_{t_2}^{s} b(u)\Delta u \right) \Delta s.$$

As $t \to \infty$, the assertion follows by setting $g(c_2) = k$. (The case $x < 0$ eventually can be proved similarly with $k < 0$.)

Conversely, without loss of generality suppose that (2.9) holds for some $k > 0$. (The case $k < 0$ can be done similarly.) Then we can choose $t_1 \ge t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} a(s) f\left(k \int_{t_1}^{s} b(u)\Delta u \right) \Delta s < d, \quad \tau(t) \ge t_1, \tag{2.13}$$

where $k = g(2d)$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup_{t \in [t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\le$. Define a subset Ω of X such that

$$\Omega := \{x \in X : \quad d \le x(\tau(t)) \le 2d, \quad \tau(t) \ge t_1\}. \tag{2.14}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \le)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = d + \int_{t}^{\infty} a(s) f\left(\int_{t_1}^{s} b(u) g(x(\tau(u)))\Delta u \right) \Delta s, \quad \tau(t) \ge t_1. \tag{2.15}$$

First we need to show that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping and since

$$d \leq (Fx)(t) = d + \int_t^\infty a(s) f\left(\int_{t_1}^s b(u) g(x(\tau(u)))\Delta u\right)\Delta s \leq 2d$$

by (2.13), it follows that $F : \Omega \to \Omega$. Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = d + \int_t^\infty a(s) f\left(\int_{t_1}^s b(u) g(\bar{x}(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1. \tag{2.16}$$

By taking the derivative of (2.16) and the fact that f is an odd function, we have

$$\bar{x}^\Delta(t) = a(t) f\left(-\int_{t_1}^t b(u) g(\bar{x}(\tau(u)))\Delta u\right), \quad \tau(t) \geq t_1.$$

Setting $\bar{y} = -\int_{t_1}^t b(u) g(\bar{x}(\tau(u)))\Delta u$ and using the monotonicity of g give

$$\bar{y}(t) \leq -g(d) \int_{t_1}^t b(u)\Delta u, \quad \tau(t) \geq t_1.$$

So we have that $\bar{x}(t) > 0$ and $\bar{y}(t) < 0$ for $t \geq t_1$, and $\bar{x}(t) \to d$ and $\bar{y}(t) \to -\infty$ as $t \to \infty$. This completes the proof. □

Example 2.6. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $m, n \geq 2$, $k = 1$, $a(t) = \frac{1}{2t^{\frac{7}{5}}(t^2+1)^{\frac{3}{5}}}$, $b(t) = \frac{2t^2-1}{2t^{\frac{9}{5}}(3t+4)^{\frac{1}{5}}}$, $f(u) = u^{\frac{3}{5}}$ and $g(u) = u^{\frac{1}{5}}$. One can easily show $A(t_0) < \infty$ and $B(t_0) = \infty$. So let us show (2.9) holds. First we have

$$\int_{t_0}^s b(u)\Delta u = \frac{1}{2} \sum_{u \in [4,s)_{2^{\mathbb{N}_0}}} \frac{2u^2-1}{u^{\frac{4}{5}}(3u+4)^{\frac{1}{5}}} \leq \sum_{u \in [1,s)_{2^{\mathbb{N}_0}}} u = s - 1.$$

Hence, we have

$$\begin{aligned}\int_{t_0}^\infty a(s) f\left(k \int_{t_0}^s b(u)\Delta u\right)\Delta s &\leq \int_{t_0}^T \frac{1}{2s^{\frac{7}{5}}(s^2+1)^{\frac{3}{5}}}(s-1)^{\frac{3}{5}}\Delta s \\ &= \frac{1}{2} \sum_{s \in [4,T)_{2^{\mathbb{N}_0}}} \frac{(s-1)^{\frac{3}{5}}}{s^{\frac{2}{5}}(s^2+1))^{\frac{3}{5}}} \leq \sum_{s \in [4,T)_{2^{\mathbb{N}_0}}} \frac{1}{s}.\end{aligned}$$

Since

$$\lim_{T\to\infty}\sum_{s\in[4,T)_{2^{\mathbb{N}_0}}}\frac{1}{s}=\sum_{m=2}^{\infty}\frac{1}{2^m}<\infty,$$

we have that (2.9) holds as $T\to\infty$. It can also be shown that $(3+\frac{1}{t},-t-\frac{1}{t})$ is a nonoscillatory solution of

$$\begin{cases}\Delta_2 x(t)=\frac{1}{2t^{\frac{7}{5}}(t^2+1)^{\frac{3}{5}}}(y(t))^{\frac{3}{5}},\\ \Delta_2 y(t)=-\frac{2t^2-1}{2t^{\frac{9}{5}}(3t+4)^{\frac{1}{5}}}\left(x\left(\frac{t}{4}\right)\right)^{\frac{1}{5}}\end{cases}\tag{2.17}$$

such that $x(t)\to 3$ and $y(t)\to-\infty$ as $t\to\infty$, i.e., $M^-_{B,\infty}\neq\emptyset$ by Theorem 2.5.

3. NONEXISTENCE OF NONOSCILLATORY SOLUTIONS OF (1.1) IN M^+ AND M^-

The nonexistence of nonoscillatory solutions of system (1.1) in $M^+_{B,B}$, $M^+_{B,0}$ and $M^-_{B,\infty}$ directly follows from Theorems 2.1, 2.3 and 2.5, respectively. Hence, we only focus on $M^+_{\infty,B}$, $M^+_{\infty,0}$, $M^-_{0,B}$, $M^-_{B,B}$ and $M^-_{0,\infty}$.

3.1. THE CASE $A(t_0)=\infty$ AND $B(t_0)<\infty$

Theorem 3.1. *If*

$$\int_{t_0}^{\infty} b(s)g\left(c_1\int_{t_0}^{\tau(s)}a(u)\Delta u\right)\Delta s=\infty\tag{3.1}$$

for some nonzero c_1, then $M^+_{\infty,B}=\emptyset$.

Proof. Assume that there exists a solution $(x,y)\in M^+_{\infty,B}$ of (1.1) such that $x(t)>0$, $x(\tau(t))>0$, $y(t)>0$ for $t\geq t_0$, $x(t)\to\infty$ and $y(t)\to d_1$ as $t\to\infty$, where $0<d_1<\infty$. Since $y(t)>0$ and decreasing for $t\geq t_0$, there exists $t_1\geq t_0$ and $d_2>0$ such that $d_1\leq y(t)\leq d_2$ for $t\geq t_1$. Integrating the first equation from t_1 to $\tau(t)$ gives

$$x(\tau(t))\geq f(d_1)\int_{t_1}^{\tau(t)}a(s)\Delta s.\tag{3.2}$$

By integrating the second equation form t_1 to t and using (3.2) yield us

$$y(t_1)\geq\int_{t_1}^{t}b(s)g(x(\tau(s)))\Delta s\geq\int_{t_1}^{t}b(s)g\left(c_1\int_{t_1}^{\tau(s)}a(u)\Delta u\right)\Delta s,\quad t\geq t_1,$$

where $c_1=f(d_1)$. As $t\to\infty$, we have a contradiction to (3.1). The proof can be shown similarly when $x<0$ eventually with $c_1<0$. $\square$

Theorem 3.2. *If*

$$\int_{t_0}^{\infty} a(t) f\left(\int_{t}^{\infty} b(s) g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s\right)\Delta t < \infty \tag{3.3}$$

for some $c_1 \neq 0$*, then* $M^{+}_{\infty,0} = \emptyset$*.*

Proof. Proof is by contradiction. So assume that there exists a nonoscillatory solution in $M^{+}_{\infty,0}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$ for $t \geq t_0$, $x(t) \to \infty$ and $y(t) \to 0$ as $t \to \infty$. Integrating the second equation from t to ∞ gives

$$y(t) = \int_{t}^{\infty} b(s) g(x(\tau(s)))\Delta s. \tag{3.4}$$

Since y is eventually decreasing, there exist $t_1 \geq t_0$ and $d_1 > 0$ such that $f(y(t)) \leq d_1$ for $t \geq t_1$. Then by integrating the first equation from t_1 to t and the monotonicity of x and f, we have that

$$x(\tau(t)) \leq x(t) \leq x(t_1) + d_1 \int_{t_1}^{t} a(s)\Delta s \leq c_1 \int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1, \tag{3.5}$$

where $c_1 = 1 + \max\{x(t_1), d_1\}$. Integrating the first equation from t_1 to t, monotonicity of f and g, (3.4) and (3.5) give us

$$x(t) \leq x(t_1) + \int_{t_1}^{t} a(s) f\left(\int_{s}^{\infty} b(u) g\left(c_1 \int_{t_1}^{u} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s.$$

As $t \to \infty$, we have a contradiction to $x(t) \to \infty$. The proof can be done similarly when $x < 0$ eventually with $c_1 < 0$. □

3.2. THE CASE $A(t_0) < \infty$ AND $B(t_0) = \infty$

Theorem 3.3. *If*

$$\int_{t_0}^{\infty} b(t) g\left(c_1 \int_{t}^{\infty} a(s)\Delta s\right)\Delta t = \infty \tag{3.6}$$

for some $c_1 \neq 0$*, then* $M^{-}_{0,B} = \emptyset$*.*

Proof. Proof is by contradiction. So assume that there exists a solution $(x, y) \in M^{-}_{0,B}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \geq t_0$, $x(t) \to 0$ and $y(t) \to -d$ as $t \to \infty$, where $d > 0$. By integrating the first equation of system (1.1) and using the monotonicity of x, y and f, we have that there exist $c_1 > 0$ and $t_1 \geq t_0$ such that

$$x(\tau(t)) \geq x(t) \geq c_1 \int_{t}^{\infty} a(s)\Delta(s), \quad t \geq t_1. \tag{3.7}$$

By integrating the second equation from t_1 to t, using inequality (3.7) and the monotonicity of g, we have

$$y(t) = y(t_0) - \int_{t_0}^{t} b(s)g(x(\tau(s)))\Delta s \le - \int_{t_0}^{t} b(s)g\left(c_1 \int_{s}^{\infty} a(\tau)\Delta\tau\right)\Delta s.$$

So as $t \to \infty$, we have a contradiction to (3.6). For the case $x < 0$ eventually, the proof can be shown similarly with $c_1 < 0$. □

Theorem 3.4. *If*

$$\int_{t_0}^{\infty} b(t)g\left(c_1 - d_1 \int_{t}^{\infty} a(s)\Delta s\right) = \infty \tag{3.8}$$

for some $c_1 > 0$ *and* $d_1 < 0$ (*or* $c_1 < 0$ *and* $d_1 > 0$)*, then* $M^{-}_{B,B} = \emptyset$.

Proof. Proof is by contradiction. Hence, assume that there exists a nonoscillatory solution $(x, y) \in M^{-}_{B,B}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \ge t_0$, $\lim_{t\to\infty} x(t) = c_1 > 0$ and $\lim_{t\to\infty} y(t) = d_1 < 0$. Since y is decreasing, there exists $d_2 < 0$ and $t_1 \ge t_0$ such that $f(y(t)) \le d_2$ for $t \ge t_1$. Integrating the first equation from t to ∞ and the monotonicity of x yield us

$$x(\tau(t)) \ge x(t) = c_1 - \int_{t}^{\infty} a(s)f(y(s))\Delta s \ge c_1 - d_2 \int_{t}^{\infty} a(s)\Delta s, \quad t \ge t_1. \tag{3.9}$$

By integrating the second equation from t_1 to t and using (3.9), we have

$$y(t) \le - \int_{t_1}^{t} b(s)g(x(\tau(s)))\Delta s \le - \int_{t_1}^{t} b(s)g\left(c_1 - d_2 \int_{s}^{\infty} a(u)\Delta u\right)\Delta s,$$

where $d_2 = d_1 < 0$ and taking the limit of the last inequality as $t \to \infty$, we have a contradiction to (3.8). This completes the proof. Note that the case $x < 0$ eventually can be done similarly with $c_1 < 0$ and $d_1 > 0$. □

Theorem 3.5. *Suppose that* f *is an odd function. If*

$$\int_{t_0}^{\infty} a(s)f\left(\int_{t_1}^{s} b(u)g\left(c_1 \int_{u}^{\infty} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s = \infty \tag{3.10}$$

for some $c_1 \ne 0$*, then* $M^{-}_{0,\infty} = \emptyset$.

Proof. Proof is by contradiction. So assume that there exists a nonoscillatory solution $(x, y) \in M^{-}_{0,\infty}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \ge t_0$, $x(t) \to 0$ and

$y(t) \to -\infty$ as $t \to \infty$. Inequality (3.7) and the monotonicity of g yield us that there exists $c_1 > 0$ and $t_1 \geq t_0$ such that

$$g(x(\tau(t))) \geq g(x(t)) \geq g\left(c_1 \int_t^\infty a(s)\Delta s\right), \quad t \geq t_1. \tag{3.11}$$

Integrating the second equation of system (1.1) from t_1 to t and using (3.11) yield us

$$y(t) \leq -\int_{t_1}^t b(s) g\left(c_1 \int_s^\infty a(u)\Delta u\right)\Delta s, \quad t \geq t_1. \tag{3.12}$$

By integrating the first equation of system (1.1) from t_1 to t, (3.12) and the fact that f is an odd function, we have

$$x(t_1) \geq x(t_1) - x(t) \geq \int_{t_1}^t a(s)\left(\int_{t_1}^s b(u) g\left(c_1 \int_u^\infty a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s, \quad t \geq t_1.$$

Taking the limit of the last inequality as $t \to \infty$, we have a contradiction to (3.10). For the case $x < 0$, the proof can be shown similarly with $c_1 < 0$. □

4. CONCLUSION

In this section, we reconsider (1.1), where $\tau(t) = t$, namely,

$$\begin{cases} x^\Delta(t) = a(t) f(y(t)), \\ y^\Delta(t) = -b(t) g(x(t)), \end{cases} \tag{4.1}$$

and investigate the asymptotic properties of nonoscillatory solutions for (4.1). Since the existence and nonexistence of nonoscillatory solutions of (4.1) in M^- are considered in [11], we only focus on M^+. Notice that the results that are obtained for system (1.1) in Sections 2 and 3 also hold for system (4.1). Therefore, we only show the existence of nonoscillatory solutions for (4.1) in $M^+_{\infty,B}$ and $M^+_{\infty,0}$, that are not acquired for (1.1). In order to do that, we assume $A(t_0) = \infty$ and $B(t_0) < \infty$ throughout this section.

Theorem 4.1. *$M^+_{\infty,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^\infty b(s) g\left(c_1 \int_{t_0}^s a(u)\Delta u\right)\Delta s < \infty \tag{4.2}$$

for some $c_1 \neq 0$.

Proof. The necessity directly follows from Theorem 3.1. So for suffiency, suppose that (4.2) holds. Choose $t_1 \geq t_0$, $c_1 > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^{\infty} b(s)g\left(c_1 \int_{t_1}^{s} b(u)\Delta u\right)\Delta s < d_1, \quad t \geq t_1, \tag{4.3}$$

where $c_1 = f(2d_1) > 0$. (The case $c_1 < 0$ can be done similarly.) Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm

$$\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \frac{|x(t)|}{\int_{t_1}^{t} a(s)\Delta s}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \left\{x \in X : f(d_1)\int_{t_1}^{t} a(s)\Delta s \leq x(t) \leq f(2d_1)\int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1\right\}. \tag{4.4}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} a(s)f\left(d_1 + \int_{t}^{\infty} b(u)g(x(u))\Delta u\right)\Delta s, \quad t \geq t_1. \tag{4.5}$$

First we need to show that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping. So let us show $F := \Omega \to \Omega$.

$$\begin{aligned} f(d_1)\int_{t_1}^{t} a(s)\Delta s &\leq (Fx)(t) \\ &\leq \int_{t_1}^{t} a(s)f\left(d_1 + \int_{s}^{\infty} b(u)g\left(f(2d_1)\int_{t_1}^{u} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s \\ &\leq f(2d_1)\int_{t_1}^{t} a(s)\Delta s \end{aligned}$$

by (4.3). Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = \int_{t_1}^{t} a(s)f\left(d_1 + \int_{s}^{\infty} b(u)g(\bar{x}(u))\Delta u\right)\Delta s, \quad t \geq t_1. \tag{4.6}$$

By taking the derivative of (4.6)

$$\bar{x}^{\Delta}(t) = a(t)f\left(d_1 + \int_{t}^{\infty} b(u)g(\bar{x}(u))\Delta u\right), \quad t \geq t_1.$$

Setting

$$\bar{y}(t) = d_1 + \int_t^{\infty} b(u)g(\bar{x}(u))\Delta u$$

and taking the limit as $t \to \infty$, we have that $\bar{x}(t) > 0$ and $\bar{y}(t) > 0$ for $t \geq t_1$, and $\bar{x}(t) \to \infty$ and $\bar{y}(t) \to d_1 > 0$ as $t \to \infty$, i.e., $M^+_{\infty,B} \neq \emptyset$. □

Theorem 4.2. *If*

$$\int_{t_0}^{\infty} a(t)f\left(k\int_t^{\infty} b(s)\Delta s\right)\Delta t = \infty \quad (-\infty)$$

and

$$\int_{t_0}^{\infty} b(t)g\left(l\int_{t_0}^{\infty} a(s)\Delta s\right)\Delta t < \infty$$

for any $k > 0$ and some $l > 0$ ($k < 0$ and $l < 0$), then $M^+_{\infty,0} \neq \emptyset$.

Proof. Choose $t_1 \geq t_0$ and $c_1 > 0$ such that

$$\int_{t_1}^{\infty} b(t)g\left(l\int_{t_0}^{t} a(s)\Delta s\right)\Delta t < \frac{c_1}{2}, \quad t \geq t_1, \tag{4.7}$$

where $l = f(c_1)$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \left\{y \in X : \quad g(1)\int_t^{\infty} b(s)\Delta s \leq y(t) \leq \frac{c_1}{2}, \quad t \geq t_1\right\}.$$

It is clear that $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ such that

$$(Fy)(t) = \int_t^{\infty} b(s)g\left(\int_{t_1}^{s} a(u)f(y(u))\Delta u\right)\Delta s.$$

It is clear that F is an increasing mapping. We also need to show that $F : \Omega \to \Omega$. By (4.7) and the monotonicity of g, we have

$$(Fy)(t) \leq \int_t^{\infty} b(s)g\left(l\int_{t_1}^{s} a(u)\Delta u\right)\Delta s \leq \frac{c_1}{2}$$

for $y \in \Omega$. Since

$$\int_{t_2}^{t} a(s) f\left(k \int_{s}^{\infty} b(u)\Delta u\right)\Delta s > 1$$

there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} a(s) f\left(k \int_{s}^{\infty} b(u)\Delta u\right)\Delta s > 1$$

for $t \geq t_2$ and any $k > 0$. So by setting $k = g(1)$, we have

$$(Fy)(t) \geq \int_{t}^{\infty} b(s) g\left(\int_{t_1}^{s} a(u) f\left(g(1)\int_{u}^{\infty} b(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s \geq g(1)\int_{t}^{\infty} a(s)\Delta s,$$

for $t \geq t_2$. Then by the Knaster fixed point theorem, there exists $\bar{y} \in \Omega$ such that $\bar{y} = F\bar{y}$. Then we have

$$\bar{y}^{\Delta}(t) = -b(t) g\left(\int_{t_1}^{t} a(u) f(\bar{y}(u))\Delta u\right).$$

Setting

$$\bar{x}(t) = \int_{t_1}^{t} a(u) f(\bar{x}(u))\Delta u$$

and taking the limit as $t \to \infty$ give us that $\bar{x} \to \infty$ and $\bar{y} \to 0$, i.e., $M^{+}_{\infty,0} \neq \emptyset$. The case $k < 0$ and $l < 0$ with $x < 0$ can be shown similarly. □

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Received: December 23, 2015.
Revised: March 9, 2016.
Accepted: March 15, 2016.