

THE POISSON INTEGRALS OF FUNCTIONS OF TWO VARIABLES FOR HERMITE AND LAGUERRE EXPANSIONS

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Abstract. In this paper we consider the Poisson integrals of functions of two variables for Hermite and Laguerre expansions in the spaces $L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$ and $L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$, respectively. We state some estimates of the rate of convergence of the Poisson integrals.

1. Introduction

In [1] Muckenhoupt considered the Poisson integral $A(f)$ of a function $f \in L^p(\mathbb{R}_+; z^\alpha \exp(-z))$, $1 \leq p \leq \infty$, $\alpha > -1$, $\mathbb{R}_+ = [0, \infty)$, defined by

$$A(f)(r, y) = A(f; r, y) = \int_0^\infty K(r, y, z) f(z) z^\alpha \exp(-z) dz,$$

where

$$\begin{aligned} K(r, y, z) &= \sum_{n=0}^{\infty} \frac{r^n n!}{\Gamma(n + \alpha + 1)} L_n^\alpha(y) L_n^\alpha(z) \\ &= \frac{(ryz)^{-\frac{\alpha}{2}}}{1-r} \exp\left(\frac{-r(y+z)}{1-r}\right) I_\alpha\left(\frac{2(ryz)^{\frac{1}{2}}}{1-r}\right), \quad 0 < r < 1, \end{aligned}$$

L_n^α is the n th Laguerre polynomial and I_α is the modified Bessel function.

Reference [1] also considered the Poisson integral of a function $f \in L^p(\mathbb{R}; \exp(-z^2))$ for Hermite expansions defined by

$$B(f; r, x) = \int_{-\infty}^{\infty} P(r, x, z) f(z) \exp(-z^2) dz, \quad 0 < r < 1,$$

with the Poisson kernel

$$P(r, x, z) = \sum_{n=0}^{\infty} \frac{r^n H_n(x) H_n(z)}{\sqrt{\pi} 2^n n!} = \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2 x^2 + 2rxz - r^2 z^2}{1-r^2}\right),$$

where H_n is the n th Hermite polynomial. Some approximation properties of these operators are given in [2].

Let us consider the operator $U(f)$:

$$\begin{aligned} U(f)(r, y_1, y_2) &= U(f; r, y_1, y_2) \\ &= \int_0^{\infty} \int_0^{\infty} K(r, y_1, z_1) K(r, y_2, z_2) f(z_1, z_2) (z_1 z_2)^{\alpha} \exp(-z_1 - z_2) dz_1 dz_2, \end{aligned}$$

where $f \in L^p(\mathbb{R}_+^2; (z_1 z_2)^{\alpha} \exp(-z_1 - z_2))$, $1 \leq p \leq \infty$, $\alpha > -1$.

This paper contains some properties of the above operator and of operator $W(f)$ defined by

$$\begin{aligned} W(f)(r, y_1, y_2) &= W(f; r, y_1, y_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(r, y_1, z_1) P(r, y_2, z_2) f(z_1, z_2) \exp(-z_1^2 - z_2^2) dz_1 dz_2, \end{aligned}$$

where $f \in L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$, $0 < r < 1$. The norm of a function f in $L^p(X^2; w(z_1, z_2))$ is given by

$$\|f\|_p = \begin{cases} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t_1, t_2)|^p w(t_1, t_2) dt_1 dt_2 \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{(t_1, t_2) \in \mathbb{R}^2} \text{ess } |f(t_1, t_2)|, & p = \infty, \end{cases}$$

where $X = \mathbb{R}_+$ and $w(z_1, z_2) = (z_1 z_2)^{\alpha} \exp(-z_1 - z_2)$, or $X = \mathbb{R}$ and $w(z_1, z_2) = \exp(-z_1^2 - z_2^2)$, respectively.

We state some estimates of the rate of convergence of the integrals $U(f)$ and $W(f)$ using the classical moduli of continuity.

2. Auxiliary results

In this section we shall give some properties of the above operators which we shall apply to the proofs of the main theorems. We begin this section by recalling the following results of Toczec and Wachnicki [2].

Let $\varphi_{n,y}(z) = (z - y)^n$, $n \in \mathbb{N} = \{1, 2, \dots\}$, $y, z \in X$, where $X = \mathbb{R}_+$ or $X = \mathbb{R}$, respectively.

Lemma 1. For each $y \in \mathbb{R}_+$ we have

$$\begin{aligned} A(1; r, y) &= 1, \\ A(\varphi_{1,y}; r, y) &= (1-r)(1+\alpha-y), \\ A(\varphi_{2,y}; r, y) &= (1-r) \left\{ y^2(1-r) + 2(\alpha+2)ry - 2(\alpha+1)y \right. \\ &\quad \left. + (\alpha+2)(\alpha+1)(1-r) \right\}, \\ A(\varphi_{4,y}; r, y) &= (1-r)^2 \left\{ y^4(r-1)^2 - 4\alpha(r-1)^2y^3 - 4(4r^2-5r+1) \right. \\ &\quad \left. + 6(\alpha+4)(\alpha+3)r^2y^2 - 12(\alpha+3)(\alpha+2)ry^2 \right. \\ &\quad \left. + 6(\alpha+2)(\alpha+1)y^2 - 12(\alpha+3)(\alpha+2)(r-1)y \right. \\ &\quad \left. + (\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2 \right\}. \end{aligned}$$

Lemma 2. For each $y \in \mathbb{R}$

$$\begin{aligned} B(1; r, y) &= 1, \\ B(\varphi_{1,y}; r, y) &= -y(1-r), \\ B(\varphi_{2,y}; r, y) &= (1-r) \left\{ y^2(1-r) + \frac{1}{2}(r+1) \right\}, \\ B(\varphi_{4,y}; r, y) &= (1-r)^2 \left\{ (r-1)^2y^4 - 3(r^2-1)y^2 + \frac{3}{4}(r+1)^2 \right\} \end{aligned}$$

holds.

From the definitions of U and W we easily obtain:

Lemma 3. If $f_1, f_2 \in L^p(\mathbb{R}_+; z^\alpha \exp(-z))$, $1 \leq p \leq \infty$, $\alpha > -1$, then

$$U(f; r, y_1, y_2) = A(f_1; r, y_1) A(f_2; r, y_2)$$

for $(y_1, y_2) \in \mathbb{R}_+^2$, $0 < r < 1$, where $f(z_1, z_2) = f_1(z_1)f_2(z_2)$, $z_1, z_2 \in \mathbb{R}_+$.

Lemma 4. If $f_1, f_2 \in L^p(\mathbb{R}; \exp(-z^2))$, $1 \leq p \leq \infty$, then

$$W(f; r, y_1, y_2) = B(f_1; r, y_1) B(f_2; r, y_2)$$

for $(y_1, y_2) \in \mathbb{R}^2$, $0 < r < 1$, where $f(z_1, z_2) = f_1(z_1)f_2(z_2)$, $z_1, z_2 \in \mathbb{R}$.

Applying Lemmas 1 and 2, it is easy to prove the following two lemmas.

Lemma 5. For every $(y_1, y_2) \in \mathbb{R}_+^2$ it follows that

$$\begin{aligned}
U(1; r, y_1, y_2) &= 1, \\
U(\varphi_{1, y_i}; r, y_1, y_2) &= (1-r)(1+\alpha-y_i), \\
U(\varphi_{2, y_i}; r, y_1, y_2) &= (1-r) \{ y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \\
&\quad + (\alpha+2)(\alpha+1)(1-r) \}, \\
U(\varphi_{4, y_i}; r, y_1, y_2) &= (1-r)^2 \{ y_i^4(r-1)^2 - 4\alpha(r-1)^2 y_i^3 - 4(4r^2-5r+1) \\
&\quad + 6(\alpha+4)(\alpha+3)r^2 y_i^2 - 12(\alpha+3)(\alpha+2)ry_i^2 \\
&\quad + 6(\alpha+2)(\alpha+1)y_i^2 - 12(\alpha+3)(\alpha+2)(r-1)y_i \\
&\quad + (\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2 \}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Lemma 6. For every $(y_1, y_2) \in \mathbb{R}^2$ it follows that

$$\begin{aligned}
W(1; r, y_1, y_2) &= 1, \\
W(\varphi_{1, y_i}; r, y_1, y_2) &= -y_i(1-r), \\
W(\varphi_{2, y_i}; r, y_1, y_2) &= (1-r) \left\{ y_i^2(1-r) + \frac{1}{2}(r+1) \right\}, \\
W(\varphi_{4, y_1} + \varphi_{4, y_2}; r, y_1, y_2) &= (1-r)^2 \left\{ (r-1)^2(y_1^4 + y_2^4) \right. \\
&\quad \left. - 3(r^2-1)(y_1^2 + y_2^2) + \frac{3}{2}(r+1)^2 \right\}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Using the Hölder inequality and Lemma 5, we obtain

Lemma 7. For $(y_1, y_2) \in \mathbb{R}_+^2$ and $0 < r < 1$ we have

$$\begin{aligned}
U(|\varphi_{1, y_i}|; r, y_1, y_2) &\leq (1-r)^{\frac{1}{2}} \{ y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \\
&\quad + (\alpha+2)(\alpha+1)(1-r) \}^{\frac{1}{2}}, \quad i = 1, 2.
\end{aligned}$$

Similarly we get

Lemma 8. For $0 < r < 1$ and $(y_1, y_2) \in \mathbb{R}^2$ we have

$$W(|\varphi_{1, y_i}|; r, y_1, y_2) \leq (1-r)^{\frac{1}{2}} \left\{ y_i^2(1-r) + \frac{1}{2}(r+1) \right\}^{\frac{1}{2}}, \quad i = 1, 2.$$

Applying the Riesz-Thorin theorem, it is easy to prove

Lemma 9. Let $f \in L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$, where $\alpha > -1$ and $1 \leq p \leq \infty$. Then $U(f; r, \cdot, \cdot) \in L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$ and

$$\|U(f; r, \cdot, \cdot)\|_p \leq \|f\|_p \quad (1)$$

for $0 < r < 1$.

In a similar fashion we can prove the following theorem for operator W .

Lemma 10. Let $f \in L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$, where $1 \leq p \leq \infty$. Then $W(f; r, \cdot, \cdot) \in L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$ and $\|W(f; r, \cdot, \cdot)\|_p \leq \|f\|_p$ for $0 < r < 1$.

3. The rate of convergence

In this section we present some estimates of the rate of convergence of the integrals U and W . We state this estimates using the classical modulus of continuity defined by

$$\omega(f; \delta_1, \delta_2) = \sup_{\substack{0 < h_1 \leq \delta_1 \\ 0 < h_2 \leq \delta_2}} \left\{ \sup_{(y_1, y_2) \in X^2} |f(y_1 + h_1, y_2 + h_2) - f(y_1, y_2)| \right\},$$

$\delta_1, \delta_2 > 0$, where $X = \mathbb{R}_+$ or $X = \mathbb{R}$, respectively.

Theorem 1. Let $f \in C(\mathbb{R}_+^2) \cap L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$, $1 \leq p \leq \infty$ and $\alpha > -1$. Then

$$|U(f; r, y_1, y_2) - f(y_1, y_2)| \leq 6\omega(f; \delta_1, \delta_2)$$

for $0 < r < 1$ and $(y_1, y_2) \in \mathbb{R}_+^2$, where

$$\begin{aligned} \delta_i &= (1-r)^{\frac{1}{2}} \{y_i^2(1-r) + 2(\alpha+2)ry_i \\ &\quad - 2(\alpha+1)y_i + (\alpha+2)(\alpha+1)(1-r)\}^{\frac{1}{2}}, \quad i = 1, 2. \end{aligned}$$

Proof. First we suppose that $f \in C^1(\mathbb{R}_+^2) \cap L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$,

$1 \leq p \leq \infty$, $\alpha > -1$. Let $f(z_1, z_2) - f(y_1, y_2) = \lambda_{y_1}(z_1, z_2) + \tau_{y_2}(z_1, z_2)$ for every $(z_1, z_2) \in \mathbb{R}_+^2$, where

$$\lambda_{y_1}(z_1, z_2) = \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u, z_2) du, \quad \tau_{y_2}(z_1, z_2) = \int_{y_2}^{z_2} \frac{\partial}{\partial v} f(y_1, v) dv.$$

Observe that

$$\begin{aligned} |\lambda_{y_1}(z_1, z_2)| &= \left| \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u, z_2) du \right| \leq \left| \int_{y_1}^{z_1} du \right| \cdot \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| \\ &= |z_1 - y_1| \cdot \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| \end{aligned}$$

and

$$|\tau_{y_2}(z_1, z_2)| \leq |z_2 - y_2| \cdot \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right|.$$

Using Lemma 7, we get

$$\begin{aligned} U(|\lambda_{y_1}|; r, y_1, y_2) &\leq U(|\varphi_{1, y_1}|; r, y_1, y_2) \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| \\ &\leq (1-r)^{\frac{1}{2}} \left\{ y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1 \right. \\ &\quad \left. + (\alpha+2)(\alpha+1)(1-r) \right\}^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right|, \end{aligned}$$

$$\begin{aligned} U(|\tau_{y_2}|; r, y_1, y_2) &\leq (1-r)^{\frac{1}{2}} \left\{ y_2^2(1-r) + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2 \right. \\ &\quad \left. + (\alpha+2)(\alpha+1)(1-r) \right\}^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right|. \end{aligned}$$

Hence we have

$$\begin{aligned} |U(f; r, y_1, y_2) - f(y_1, y_2)| &\leq U(|\lambda_{y_1}|; r, y_1, y_2) + U(|\tau_{y_2}|; r, y_1, y_2) \\ &\leq (1-r)^{\frac{1}{2}} \left\{ y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1 \right. \\ &\quad \left. + (\alpha+2)(\alpha+1)(1-r) \right\}^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| \\ &\quad + (1-r)^{\frac{1}{2}} \left\{ y_2^2(1-r) + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2 \right. \\ &\quad \left. + (\alpha+2)(\alpha+1)(1-r) \right\}^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right|. \end{aligned}$$

Let $f \in C(\mathbb{R}_+^2) \cap L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$, $1 \leq p \leq \infty$, $\alpha > -1$ and let f_{δ_1, δ_2} be the Steklov function of a function f of two variables given by the

formula

$$f_{\delta_1, \delta_2}(y_1, y_2) = \frac{1}{\delta_1 \delta_2} \int_0^{\delta_1} \int_0^{\delta_2} f(y_1 + u, y_2 + v) du dv \text{ for } (y_1, y_2) \in \mathbb{R}_+^2, \delta_1, \delta_2 > 0.$$

If $f \in C(\mathbb{R}_+^2) \cap L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$, then

$$f_{\delta_1, \delta_2} \in C^1(\mathbb{R}_+^2) \cap L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$$

for fixed $\delta_1, \delta_2 > 0$ and

$$\begin{aligned} \sup_{(y_1, y_2) \in \mathbb{R}_+^2} |f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2)| &\leq \omega(f; \delta_1, \delta_2), \\ \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial}{\partial y_1} f_{\delta_1, \delta_2}(y_1, y_2) \right| &\leq 2\delta_1^{-1} \omega(f; \delta_1, \delta_2), \\ \sup_{(y_1, y_2) \in \mathbb{R}_+^2} \left| \frac{\partial}{\partial y_2} f_{\delta_1, \delta_2}(y_1, y_2) \right| &\leq 2\delta_2^{-1} \omega(f; \delta_1, \delta_2) \end{aligned}$$

for all $\delta_1, \delta_2 > 0$.

Moreover, for $f \in L^p(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2))$ we can write

$$\begin{aligned} |U(f; r, y_1, y_2) - f(y_1, y_2)| \\ \leq |U(f - f_{\delta_1, \delta_2}; r, y_1, y_2)| + |U(f_{\delta_1, \delta_2}; r, y_1, y_2) - f_{\delta_1, \delta_2}(y_1, y_2)| \quad (2) \\ + |f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2)|, \end{aligned}$$

where $(y_1, y_2) \in \mathbb{R}_+^2, \delta_1, \delta_2 > 0$.

Now, from the first part of this proof we have

$$\begin{aligned} |U(f_{\delta_1, \delta_2}; r, y_1, y_2) - f_{\delta_1, \delta_2}(y_1, y_2)| \\ \leq 2\omega(f; \delta_1, \delta_2) \left\{ \delta_1^{-1}(1-r)^{\frac{1}{2}} \left[y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1 \right. \right. \\ \left. \left. + (\alpha+2)(\alpha+1)(1-r) \right]^{\frac{1}{2}} + \delta_2^{-1}(1-r)^{\frac{1}{2}} \left[y_2^2(1-r) + 2(\alpha+2)ry_2 \right. \right. \\ \left. \left. - 2(\alpha+1)y_2 + (\alpha+2)(\alpha+1)(1-r) \right]^{\frac{1}{2}} \right\}. \end{aligned}$$

Observe that

$$\begin{aligned} |U(f - f_{\delta_1, \delta_2}; r, y_1, y_2)| &\leq \\ &\int_0^\infty \int_0^\infty K(r, y_1, z_1) K(r, y_2, z_2) (z_1 z_2)^\alpha \exp(-z_1 - z_2) dz_1 dz_2 \\ &\times \sup_{(y_1, y_2) \in \mathbb{R}_+^2} |f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2)| \leq U(1; r, y_1, y_2) \omega(f; \delta_1, \delta_2) \leq \omega(f; \delta_1, \delta_2) \end{aligned}$$

and

$$|f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2)| \leq \omega(f; \delta_1, \delta_2).$$

From (2) we have

$$\begin{aligned} & |U(f; r, y_1, y_2) - f(y_1, y_2)| \\ & \leq \left\{ 1 + \delta_1^{-1}(1-r)^{\frac{1}{2}} [y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1 \right. \\ & \quad \left. + (\alpha+2)(\alpha+1)(1-r)]^{\frac{1}{2}} + \delta_2^{-1}(1-r)^{\frac{1}{2}} [y_2^2(1-r) \right. \\ & \quad \left. + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2 + (\alpha+2)(\alpha+1)(1-r)]^{\frac{1}{2}} \right\} 2\omega(f; \delta_1, \delta_2) \end{aligned}$$

for $0 < r < 1$, $\delta_1, \delta_2 > 0$ and all $(y_1, y_2) \in \mathbb{R}_+^2$. Setting

$$\delta_1 = (1-r)^{\frac{1}{2}} (y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1 + (\alpha+2)(\alpha+1)(1-r))^{\frac{1}{2}},$$

$$\delta_2 = (1-r)^{\frac{1}{2}} (y_2^2(1-r) + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2 + (\alpha+2)(\alpha+1)(1-r))^{\frac{1}{2}}$$

for fixed $(y_1, y_2) \in \mathbb{R}_+^2$, we get the required inequality.

In a similar fashion we obtain:

Theorem 2. *Let $f \in C(\mathbb{R}^2) \cap L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$, $1 \leq p \leq \infty$. Then*

$$|W(f; r, y_1, y_2) - f(y_1, y_2)| \leq 6\omega(f; \delta_1, \delta_2)$$

for $0 < r < 1$ and $(y_1, y_2) \in \mathbb{R}^2$, where

$$\delta_i = (1-r) \left\{ y_i^2(1-r) + \frac{1}{2}(r+1) \right\}, \quad i = 1, 2.$$

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