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GENERALIZED POWERS AND MEASURES

Zbigniew Burdak, Marek Kosiek, Patryk Pagacz, Krzysztof Rudol, and Marek Słociński

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Abstract. Using the winding of measures on torus in "rational directions" special classes of unitary operators and pairs of isometries are defined. This provides nontrivial examples of generalized powers. Operators related to winding Szegö-singular measures are shown to have specific properties of their invariant subspaces.

Keywords: representing measures, Szegö-singular measure, compatible pair of isometries, spectral measure.

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1. INTRODUCTION

In the "one-dimensional" case of the disc algebra $A(\mathbb{D})$ the normalized Lebesgue measure $\mathbf{m}$ is the unique representing measure on the unit circle $\mathbb{T}$ for the evaluation at 0 functional. The structure of "two-dimensional" representing measures for $A(\mathbb{D}^2)$ is quite complicated. Its better understanding plays essential role in studying Banach algebras of analytic functions of many variables.

In order to understand the rich family of all representing measures on the torus $\mathbb{T}^2$ a special class of measures $J_{pq}(\nu)$ was defined in [6]. Here by a *representing measure* we mean a positive Borel measure μ such that $u(0) = \int_{\mathbb{T}^2} u(z)d\mu(z)$ for all functions u analytic on the bidisc $\mathbb{D}^2$ and continuous on its closure. By $\mathcal{B}(\mathbb{T}^2)$ (respectively by $\mathcal{B}(\mathbb{T})$) we denote the corresponding sigma – algebra of Borel sets.

For any pair of relatively prime positive integers (p, q) a process of "winding of $\mathbf{m}$ on the torus" yields a one-parameter family of measures η^w_{pq}, where $w \in \mathbb{T}$ (i.e. $w \in \mathbb{C}, |w| = 1$). Now this family is integrated with respect to a given probabilistic Borel measure ν on $\mathbb{T}$, yielding a representing measure $\mu = J_{pq}(\nu)$ described below. On the other hand, representing measures satisfying certain additional conditions are shown to be decomposable as infinite sums of measures of this type.

In this short note we study a related class of pairs (U_1, U_2) of unitary operators on the space $L^2(\mu)$ for such a measure $\mu = J_{pq}(\nu)$. Their restriction to a certain invariant subspace $\mathcal{H}$ is shown to be *a pair of generalized powers* (and *compatible isometries*) in the sense of [1, 5]. On the other hand, compatible isometries are important due to their relation with stochastic processes. More precisely, a two-dimensional stochastic processes with a so-called *half-plane past* is compatible as a pair of isometries. The considered measures $J_{pq}(\nu)$ provide examples of generalized powers and of two-dimensional stochastic processes as well. The examples are far more interesting than those previously known which were uncorrelated pairs (a white noise) and pairs of powers of the same unilateral shift. (The latter example was very close to the one dimensional case.)

We also consider the measures $J_{pq}(\nu)$ related to a special class of Szegö-singular measures ν, obtaining some additional properties in this case.

2. "WINDING" MEASURES ON THE TORUS

Throughout this paper (p, q) will be a fixed pair of relatively prime positive integers. For a complex number $w \in \mathbb{T}$ let $w^{-1/p}$ be the inverse to the principal value of its p-th root. Then we have a family of measurable mappings $\varphi_{pq}^w : \mathbb{T} \to \mathbb{T}^2$ defined in [6] for $z \in \mathbb{T}$ by

$$\varphi_{pq}^w(z) := (z^p, w^{-1/p} z^q). \tag{2.1}$$

These mappings pushforward the normalized Lebesgue measure $\mathbf{m}$ yielding the family of Borel measures $\eta_{pq}^w := \varphi_{pq}^w(\mathbf{m})$ on $\mathbb{T}^2$, so that

$$\int_{\mathbb{T}^2} u\, d\eta_{pq}^w = \int_{\mathbb{T}} u \circ \varphi_{pq}^w \, d\mathbf{m}, \quad u \in C(\mathbb{T}^2). \tag{2.2}$$

Finally, for a probabilistic Borel measure ν on $\mathbb{T}$ let

$$\mu := J_{pq}(\nu) = \int \eta_{pq}^w \, d\nu(w).$$

This means that μ is a probabilistic measure on the torus $\mathbb{T}^2$ such that

$$\int u\, d\mu = \int_{\mathbb{T}} \left(\int_{\mathbb{T}^2} u\, d\eta_{pq}^w \right) d\nu(w), \quad u \in C(\mathbb{T}^2). \tag{2.3}$$

For any $k, l \in \mathbb{Z}$, we have [6, Proposition 2.5]:

Proposition 2.1. *The Fourier coefficient $\hat{\mu}(k, l) := \int z_1^{-k} z_2^{-l} \, d\mu(z_1, z_2)$ equals zero unless $k = aq$, $l = -ap$ for some integer $a \in \mathbb{Z}$. In the latter case $\hat{\mu}(k, l) = \hat{\nu}(a)$.*

Let U_i $(i = 1, 2)$ denote the operator of multiplication by the variable z_i on $L^2(\mu)$, where $\mu = J_{pq}(\nu)$. Since $|z_i| = 1$ on $\mathbb{T}^2$, the pair (U_1, U_2) consists of unitary operators, which clearly commute. Let $E : \mathcal{B}(\mathbb{T}^2) \to \mathcal{L}(L^2(\mu))$ be its common spectral measure. So for any Borel set $\sigma \in \mathcal{B}(\mathbb{T}^2)$ we have

$$\langle E(\sigma)f, g\rangle = \int_\sigma f\bar{g}\, d\mu, \quad f, g \in L^2(\mu). \tag{2.4}$$

Moreover,

$$\langle U_i f, g\rangle = \Big\langle \Big(\int z_i\, dE\Big) f, g\Big\rangle = \int z_i\, d\langle E(\cdot)f, g\rangle = \int z_i f\bar{g}\, d\mu, \quad f, g \in L^2(\mu). \tag{2.5}$$

Put

$$\varphi_{pq}(z, w) := \varphi^w_{pq}(z).$$

By (2.2), (2.3) and by Fubini's Theorem we have for any bounded Borel function u on $\mathbb{T}^2$ the equalities

$$\begin{aligned} \int u\, d\mu &= \int \Big(\int u(\lambda^p, w^{-1/p}\lambda^q)\, d\mathbf{m}(\lambda)\Big)\, d\nu(w) \\ &= \int \Big(\int u(\lambda^p, w^{-1/p}\lambda^q)\, d\nu(w)\Big)\, d\mathbf{m}(\lambda). \end{aligned} \tag{2.6}$$

For $\sigma \subset \mathbb{T}$, put

$$\Delta^{pq}_\sigma := \{(\lambda^p, w^{-1/p}\lambda^q) : w \in \sigma, \lambda \in \mathbb{T}\}.$$

As a consequence of (2.1), for $z = \lambda$ we can write the equations

$$z_1 = \lambda^p, \quad z_2 = w^{-1/p}\lambda^q.$$

Solving them we obtain

$$\lambda = z_1^{1/p}, \quad w = z_1^q z_2^{-p}.$$

Define the mappings

$$\Pi_{pq} : \mathbb{T}^2 \ni (z_1, z_2) \mapsto z_1^q z_2^{-p} \in \mathbb{T}, \quad \Lambda_{pq} : \mathbb{T}^2 \ni (z_1, z_2) \mapsto z_1^{1/p} \in \mathbb{T}.$$

By (2.1), we have

$$\varphi_{pq}(\Lambda_{pq}(z_1, z_2), \Pi_{pq}(z_1, z_2)) = (z_1, z_2).$$

Hence, $\Pi^{-1}_{pq}(\sigma) = \Delta^{pq}_\sigma$.

3. A UNITARY OPERATOR ASSOCIATED WITH THE PAIR (U_1, U_2)

Applying our mapping Π_{pq} to the pair (U_1, U_2) we obtain the unitary operator

$$U := (U_2^*)^p U_1^q. \tag{3.1}$$

Although it depends on (p, q), we use here U to simplify the notation.

Proposition 3.1. *The spectral measure F of the above operator U is obtained from E by the formula $F : \mathcal{B}(\mathbb{T}) \ni \sigma \to E(\Pi_{pq}^{-1}(\sigma)) \in \mathcal{L}(L^2(\mu))$.*

Proof. Since $(\Pi_{pq}(U_1, U_2))^a = U^a$ for $a \in \mathbb{Z}$, by the Stone–Weierstrass theorem we have

$$\int f \circ \Pi_{pq}\, dE = f(\Pi_{pq}(U_1, U_2)) = f(U) = \int f\, dF$$

for all bounded Borel functions f on $\mathbb{T}$. Applying this equality to an arbitrary characteristic function χ_σ of a Borel set $\sigma \subset \mathbb{T}$, we get

$$F(\sigma) = \int \chi_\sigma \circ \Pi_{pq}\, dE = \int \chi_{\Pi_{pq}^{-1}(\sigma)}\, dE = E(\Pi_{pq}^{-1}(\sigma)). \qquad \square$$

A pair (V_1, V_2) of isometries is called *compatible* if projections onto the ranges $\mathcal{R}(V_1^m)$, $\mathcal{R}(V_2^n)$ commute for all positive integers m, n (see [5]). Of course any pair of commuting unitary operators is trivially compatible. Pairs of *generalised powers* were defined in [1] where the definition provides a precise but also a little complicated geometrical description. Fortunately, in the case of compatible pairs of isometries, the results of [1] imply the following, more reader – friendly equivalent definition: generalized powers are pairs of compatible unilateral shifts (V_1, V_2) satisfying $V_1^k = UV_2^m$ with some positive integers k, m and a unitary operator U commuting with V_1, V_2.

By [6, Proposition 2.5 (i)], we have for $a \in \mathbb{Z}$

$$\int z_1^{aq} z_2^{-ap}\, d\mu(z_1, z_2) = \int w^a\, d\nu(w) \tag{3.2}$$

and

$$\int z_1^{-k} z_2^{-l}\, d\mu(z_1, z_2) = 0$$

if $(k, l) \neq (aq, -ap)$ for all $a \in \mathbb{Z}$. In particular, the monomials $w_{k,l}(z_1, z_2) := z_1^k z_2^l$ and $z_1^m z_2^n$ will be orthogonal in $L^2(\mu)$, if the difference $(k - m, l - n)$ does not belong to the subgroup $\mathbb{Z} \cdot (q, -p)$ of $\mathbb{Z}^2$. If a subspace of $L^2(\mu)$ contains the monomial $w_{k,l}$ and is invariant under (U_1, U_2), it must contain also $\{w_{k+r,l+s} : r, s \in \mathbb{Z}_+\}$. If it reduces U, it has to contain $\{w_{k+aq,l-ap} : a \in \mathbb{Z}\}$.

Let us consider for $n \in \mathbb{Z}$ the following subspaces H_n, $\mathcal{H}$ of $L^2(\mu)$:

$$H_n = \overline{\operatorname{span}}\{z_1^{aq+r} z_2^{-ap+s}, a \in \mathbb{Z}, nq \le r < (n+1)q, 0 \le s < p\},$$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} H_n.$$

Observe that

$$L^2(\mu) = \bigoplus_{n=-\infty}^{\infty} H_n \tag{3.3}$$

and

$$H_n = \bigoplus_{s=0}^{p-1} \bigoplus_{r=nq}^{(n+1)q-1} H^{rs}, \tag{3.4}$$

where $H^{rs} := \overline{\operatorname{span}}\{z_1^{aq+r} z_2^{-ap+s}, a \in \mathbb{Z}\}$.

Corollary 3.2. *The subspaces H_k $(k \in \mathbb{Z})$ and $\mathcal{H}$ are reducing for U. Moreover $\mathcal{H}$ is invariant for (U_1, U_2) and the restrictions $(U_1|_{\mathcal{H}}, U_2|_{\mathcal{H}})$ form a compatible pair of generalized powers.*

Proof. In view of the preceding remarks, it remains to show that (U_1, U_2) form a compatible pair of isometries which means the commutativity of the projections: $P_{1,m}$ onto the range of U_1^m with $P_{2,n}$ -onto $\mathcal{R}(U_2^n)$. Let

$$A = \{(aq + r, -ap + s) \in \mathbb{Z}^2 : a \in \mathbb{Z}, r, s \in \mathbb{Z}_+\},$$

so that $\mathcal{H} = \overline{\operatorname{span}}\{z_1^k z_2^l : (k,l) \in A\}$. Also let us define two sets

$$A_{1,m} := \{(k+m, l) : (k,l) \in A\} \quad \text{and} \quad A_{2,n} := \{(k, l+n) : (k,l) \in A\}.$$

The range of U_1^m is spanned by $\{z_1^k z_2^l : (k,l) \in A_{1,m}\}$, while $\mathcal{R}(U_2^n)$ is spanned by $\{z_1^k z_2^l : (k,l) \in A_{2,n}\}$. These two sets $A_{2,n}, A_{1,m}$ are saturated with respect to the equivalence relation modulo subgroup $\{(aq, -ap) : a \in \mathbb{Z}\}$ of $\mathbb{Z}^2$, so is their set-theoretical differences and intersection - which we denote

$$B_{m,n} := A_{2,n} \cap A_{1,m}.$$

The monomials $z_1^k z_2^l$ with $(k,l) \in A_{1,m} \setminus A_{2,n}$ are orthogonal to those with $(k,l) \in B_{m,n}$. Analogous orthogonality takes place for the monomials corresponding to $(k,l) \in A_{2,n} \setminus A_{1,m}$ and $(k', l') \in B_{m,n}$ (and for $(k', l') \in A_{1,m} \setminus A_{2,n}$). The subspaces of $\mathcal{H}$ spanned by such monomials are also orthogonal and from this one can easily deduce that both $P_{1,m}P_{2,n}$ and $P_{2,n}P_{1,m}$ are equal to the projection onto $\overline{\operatorname{span}}\{z_1^k z_2^l : (k,l) \in B_{m,n}\}$. Hence these projections commute. □

4. UNITARY OPERATORS WITH CYCLIC VECTORS

The following property of unitary operators seems known, but uneasy to find in the literature, so we include its proof for the sake of completeness.

Proposition 4.1. *If a unitary operator on a Hilbert space has a cyclic vector, then all its invariant subspaces are reducing.*

Proof. Let $U \in \mathcal{L}(H)$ be an arbitrary unitary operator with a cyclic vector $x_0 \in H$ and let $H_0 \subset H$ be reducing for U. Hence P_{H_0} commutes with U and with the polynomials $\varphi(U)$. Now any vector $y \in H_0$ can be approximated by $\varphi(U)x_0$ for some polynomial φ. Then $P_{H_0}\varphi(U)x_0 = \varphi(U)P_{H_0}x_0$ approximates $P_{H_0}y = y$.

Hence for any reducing subspace H_0 for U the vector $P_{H_0}x$ is cyclic for $U|_{H_0}$. In particular, if there is a U wandering vector $y \in H$, then $H_0 = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}U^n y$

reduces U, so $U|_{H_0}$ has a cyclic vector. However, $U|_{H_0}$ is a bilateral shift of multiplicity one, so it is equivalent to $M_z \in \mathcal{L}(L^2(\mathbb{T}))$. If there is a cyclic vector $f \in L^2(\mathbb{T})$, then $M_f H^2(\mathbb{T}) = L^2(\mathbb{T})$ which is impossible.

Suppose that U has a nontrivial invariant subspace H' which is nonreducing. Then $U|_{H'}$ has a nonzero unilateral shift part and consequently has a nonzero wandering vector which leads to the contradiction. □

5. SZEGÖ SINGULAR MEASURES

For any non-negative regular Borel measure η on $\mathbb{T}^d$, $d = 1, 2$ we denote by $H^2(\eta)$ the closure in $L^2(\eta)$ of the algebra of all analytic polynomials (i.e. spanned by z^k, resp. by $z_1^k z_2^l$, $k, l \in \mathbb{Z}_+$). We say that η is *Szegö singular* if $H^2(\eta) = L^2(\eta)$. In the case of the unit circle $\mathbb{T}$ we say that η is a *Szegö measure*, if for any $\omega \in \mathcal{B}(\mathbb{T})$ the inclusion $\chi_\omega L^2(\eta) \subset H^2(\eta)$ implies $\eta(\omega) = 0$. For preliminaries on Szegö measures see [2–4]. By U we denote the unitary operator on $L^2(J_{pq}(\nu))$ defined in (3.1).

Theorem 5.1. *If ν is Szegö singular measure, then every invariant for U subspace is reducing.*

Proof. Take an arbitrary $\varepsilon > 0$ and $k \in \mathbb{N} \cup \{0, -1\}$. Since ν is Szegö singular, we can find a polynomial $h_k(w)$ divisible by w^{k+1} (i.e. with vanishing first $k + 1$ coefficients) and such that

$$\int |h_k(w) - w^k|^2 \, d\nu(w) < \varepsilon. \tag{5.1}$$

Using the equality

$$|h_k(w) - w^k|^2 = (h_k(w) - w^k)(\overline{h_k(w)} - w^{-k})$$

and applying (3.2) to each monomial of its right hand side, by (5.1) we have

$$\int |h_k(z_1^{-q} z_2^p) - z_1^{-kq} z_2^{kp}|^2 \, d\mu(z_1, z_2) < \varepsilon.$$

Since ε was arbitrary, we conclude that $z_1^{-kq} z_2^{kp} \in H_0$. By induction also $z_1^{aq} z_2^{-ap} \in H_0$ for $a \in \mathbb{N}$. For $r, s \in \mathbb{Z}$ we have

$$\int |(h_k(z_1^{-q} z_2^p) - z_1^{-kq} z_2^{kp}) z_1^r z_2^s|^2 \, d\mu(z_1, z_2) \le \int |h_k(z_1^{-q} z_2^p) - z_1^{-kq} z_2^{kp}|^2 \, d\mu(z_1, z_2) < \varepsilon.$$

Since $z_1^r z_2^s h_k(z_1^{-q} z_2^p) \in H_n$, we conclude for any $k \in \mathbb{N} \cup \{0, -1\}$ and $n \ge 0$ that

$$z_1^{r-kq} z_2^{kp-s} \in \bigvee_{m=1}^{\infty} U^m (z_1^{r-kq} z_2^{kp-s}).$$

This implies that $U|_{H_n^{rs}}$ has a cyclic vector. By Proposition 4.1, we conclude that every invariant for U subspace $H' \subset H_n$ is reducing. By (3.3), (3.4) and Corollary 3.2, we obtain the desired conclusion. □

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Zbigniew Burdak
rmburdak@cyf-kr.edu.pl
https://orcid.org/0000-0002-6097-5761

Department of Applied Mathematics
University of Agriculture
ul. Balicka 253c, 30-198 Kraków, Poland

Marek Kosiek
Marek.Kosiek@im.uj.edu.pl
https://orcid.org/0000-0002-1455-6275

Jagiellonian University
Faculty of Mathematics and Computer Science
ul. Prof. St. Łojasiewicza 6, 30-348 Kraków, Poland

Patryk Pagacz
patryk.pagacz@gmail.com
https://orcid.org/0000-0001-9866-1613

Jagiellonian University
Faculty of Mathematics and Computer Science
ul. Prof. St. Łojasiewicza 6, 30-348 Kraków, Poland

Krzysztof Rudol (corresponding author)
grrudol@cyfronet.pl
https://orcid.org/0000-0001-7030-653X

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

Marek Słociński
Marek.Slocinski@im.uj.edu.pl
https://orcid.org/0000-0002-8499-5126

Jagiellonian University
Faculty of Mathematics and Computer Science
ul. Prof. St. Łojasiewicza 6, 30-348 Kraków, Poland

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