MIDPOINTS OF SEGMENTS WHOSE ENDPOINTS BELONG TO TWO STRAIGHT LINES

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Abstract. The paper discusses sets of midpoints of segments whose endpoints belong to two given, different and coplanar or skew lines. The endpoints of these segments in the case of intersecting lines are determined by pencils of lines and concentric circles, whereas in the case of two skew lines by pencils of planes and concentric spheres. The paper proves that these sets are nonsingular or singular conic, for example rectangular hyperbola or a pair of perpendicular straight lines. All the results of study were obtained by synthetic methods.

Keywords: series of points, pencil of lines (planes, circles or spheres), conic, rectangular hyperbola, skew quadric, hyperboloid of one sheet

1 Introduction

The article [3] proves that a set of midpoints of non-zero and constant length segment whose endpoints translate along, respectively, two non-parallel straight lines (intersecting or skew lines) is an ellipse, and when these lines are perpendicular – a circle.

The present paper considers sets of midpoints of variable length segments and their endpoints belonging to the given coplanar or skew lines.

In the case of coplanar lines, the endpoints of segments will be determined by a pencil of lines and a pencil of concentric circles, whereas in the event of skew lines with a bundle of planes plus a pencil of concentric spheres.

2 Coplanar lines

2.1 Pencils of lines

Let the intersecting lines a and b at the point S be given on a plane plus a pencil of lines about the vertex $O \neq S$ (Fig. 1). The lines o_i of pencil{O} intersect with the lines a and b at the points A_i and B_i respectively. If $o_i \neq OS$, then $A_i \neq B_i$.

It will proved that a set of midpoints S_i of segments A_iB_i, if the point O does not belong to the lines a(b), is a hyperbola which belongs to the points O and S whose asymptotes are parallel to the lines a and b, respectively.

With the segments A_iB_i correspond the lines d_i passing through their midpoints and the point $S=a\cdot b$. The lines d_i' which pass through the point S and are parallel to the segments A_iB_i, are directions conjugated with the lines d_i in relation to the lines a and b (Fig. 1).

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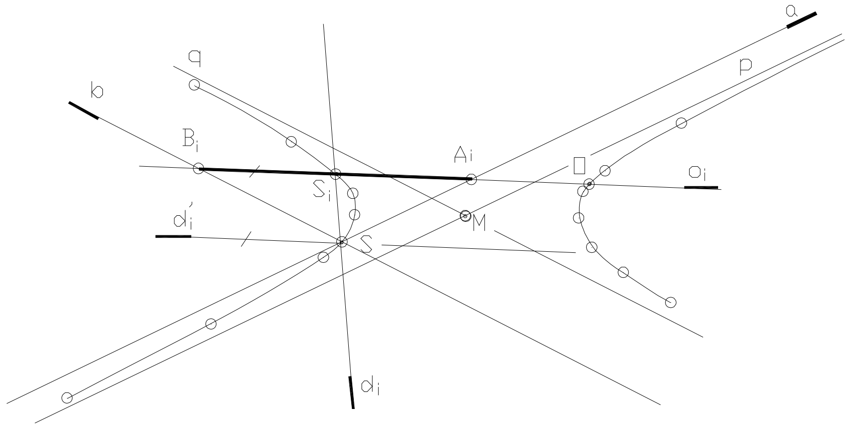

Fig. 1

Let us assume that the lines a and b are asymptotes of one of many hyperbolas which cut out on the lines o_i of pencil {O} chords with common midpoints with the segments A_iB_i.

As a result of this assumption the lines d_i and d_i' are pairs of conjugate diameters of this hyperbola. These diameters form an involution, are pairs of two projective pencils of lines with the common vertex S. The pencil S(d_i,d_i') is projective to the pencil S(d_i',d_i). Because the lines o_i of pencil{O}are parallel to the lines d_i' that the pencils of lines S(d_i) and O(o_i) are projective, thus a set of the points $\{S_i\}=d_i \cap o_i$ is a conic. In addition, we remarked that asymptotes of this conic (one of many) are particular diameters, that is, that the line a(b) is conjugate to one another (each of them is self-conjugate). Hence a conclusion that the line d_a(d_b) of the pencil {S} which coincides with the line a(b), is at once the line d_a'(d_b') conjugate to the line d_a(d_b). The homologous lines o_a and o_b of pencil{O}, are also parallel to d_a'= a and d_b'= b.The two projective pencils of lines{S}and{O} which have two pairs of parallel lines ($o_a \| d_a$'= a and $o_b \| d_b$'= b) generate a hyperbola whose asymptotes are parallel to the lines a and b, respectively.

If the point $O \neq S$ belongs to one of the axes of rectangular symmetry of the lines a and b, then the points S and O are vertices of hyperbola, and while in addition the lines a and b are perpendicular, then this hyperbola is equilateral (in this same manner it can be proved that a set of midpoints of segments which are cut out on the lines of pencil by non-degenerated conic is also a non-singular conic).

The trivial cases occur:

- when the point $O \neq S$ lies on the given line a(b), then the lines o_i of pencil{O}intersect the line a(b) at the points A_i(B_i) coinciding with the point O, and the line b(a) in the points B_i(A_i). In this case the midpoints of segments A_iB_i(B_iA_i) about common points A_i=O(B_i=O) belong to the same line $s \| b(a)$;

- while the vertex of pencil{O} is a point at infinity ($O^\infty \neq A^\infty \neq B^\infty$) and different from ideal points of the lines a and b, then the intersecting lines a and b in the real point S, cut out on the parallel lines of pencil $\{O^\infty\}$, parallel segments A_iB_i, whose midpoints S_i belong to the line s belongs to the point $S=a \cap b$;

- if the lines a and b are parallel then the set of midpoints S_i of all segments A_iB_i, whose endpoints belong, respectively, to two given parallel lines, is the central axis of rectangular symmetry of these lines.

2.2 Pencils of concentric circles

The author of the work [1] suggests, inter alia, as an exercise to take on a plane: a conic s together with its center S, one from its chords plus the point $O \neq S$, and next directs a reader to prove that a set of points S_i, as intersection points of the lines o_i passing through the given point O and perpendicular to the chords A_iB_i, intersect diameters of the conic s, belonging to the midpoints M_i of the chords A_iB_i, is a rectangular (equilateral) hyperbola (Fig. 2).

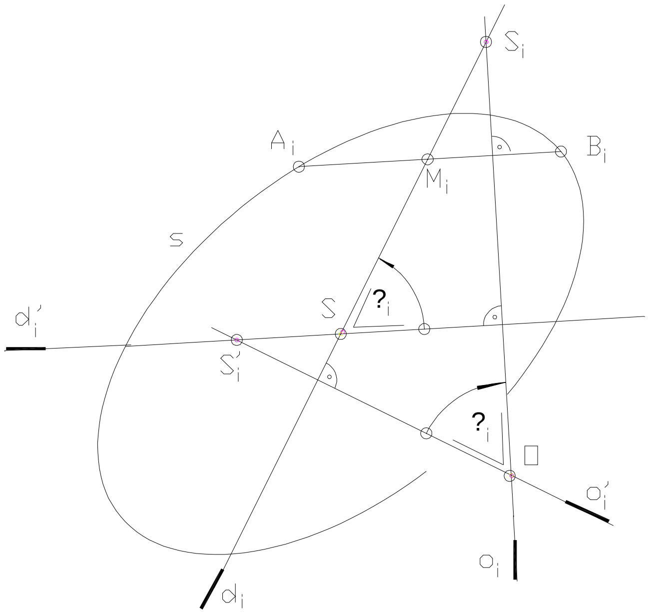

Fig. 2

After supplementing the above assumption with: the line $d_i' \| A_iB_i$ and passing through the point S plus the line o_i' including the point O and perpendicular to the line d_i, it can be seen, that the pencils of lines {O} and {S} are identical (as angles with perpendicular sides),and thus projective, and besides because they are conversely identical (equal), then the common points of the equivalent elements (lines) of those pencils form an equilateral hyperbola- which was to be shown.

Now, this hyperbola can be defined as a set of intersection points of lines passing through the given point $O \neq S$, and perpendicular to its diameters, and with diameters which conjugate with them.

In a set of pairs with conjugate diameters of a conic, there is only one pair of perpendicular diameters called the axes of conic.

The line of pencil{O} perpendicular to one of the axes intersects the other (second) in the point at infinity, thus, axes of conic s are directions of asymptotes of this hyperbola.

Let us now take on a plane two distinct lines a and b which belong to the same point S plus a pencil of concentric circles with the center $O \neq S$. The circles o_i of this pencil intersect the lines a and b, generally in the two real and different points ${}^{1,2}A_i$ and ${}^{1,2}B_i$.

We proved that the midpoints ${}^{j}S_i$ (j=1,2,3,4) of segments ${}^{j}A_i\, {}^{j}B_i$ (i=1,2,…,n; j=1,2) obtained in this way belong to an equilateral hyperbola (Fig. 3).

The midpoints of these segments correspond to the lines ${}^{j}d_i = S\, {}^{j}S_i$, and with them conjugate directions – the lines ${}^{j}d_i'$ passing through the point $S=a \cdot b$ and parallel to the segments ${}^{j}A_i\, {}^{j}B_i$. The segments ${}^{1}A_i\, {}^{1}B_i$ were selected for the graphical illustration. The lines o_i of pencil {O}, perpendicular to those segments, and ipso facto, to the diameters ${}^{1}d_i'$ of singular conic, intersect diameters conjugate with them ${}^{1}d_i$ of pencil {S} precisely in the points ${}^{1}S_i$ (the point

O is the center of circles, and the segments ^1A_i ^1B_i its chords), and the lines o_i' perpendicular to the diameters ^1d_i, intersect diameters conjugate with them $^1d_i'$ in the points ^1S_i. The bundles of lines {O} and {S}, as in the case of nonsingular conic are projective and conversely identical with pencils, hence, generate an equilateral hyperbola whose directions of asymptotes are the axes of rectangular symmetry of the given lines a and b (the line of pencil {O} perpendicular to one of them is parallel to the other).

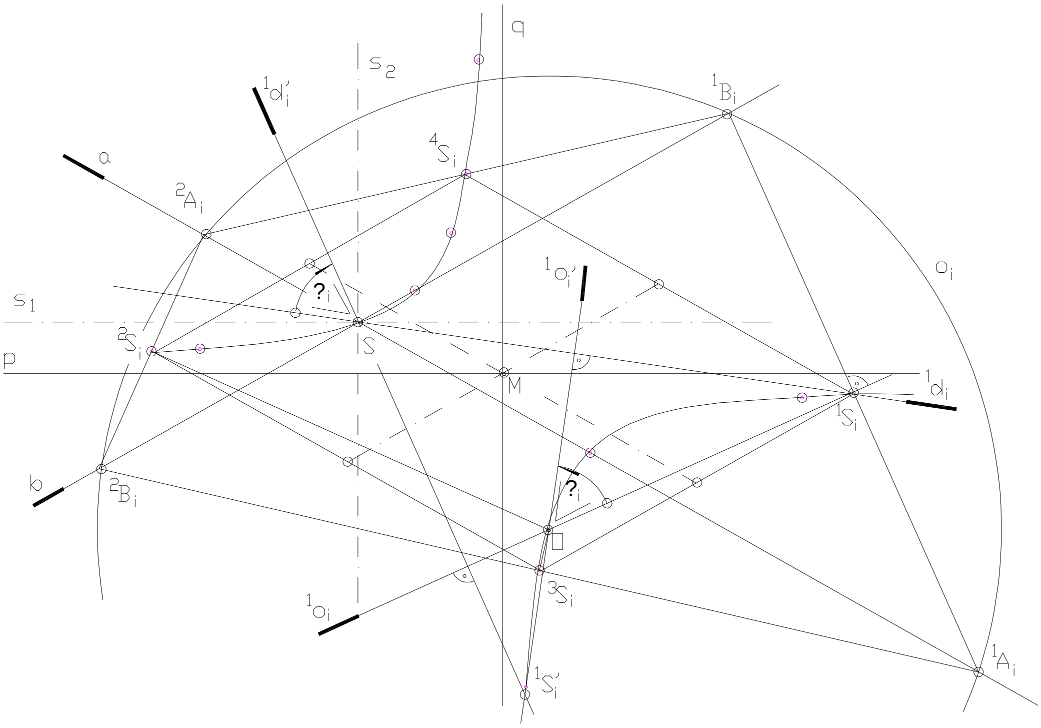

Fig. 3

In general, we can also say that the set of midpoints ^jS_i,(j=1,2,3,4) of the segments ^jA_i ^jB_i(j=1,2), whose endpoints are intersection points of the lines a and b, with circles of pencil of concentric circles with center in the point O≠S is a rectangular hyperbola (Fig. 3). Its asymptotes p and q are parallel, respectively, to the axes of rectangular symmetry of the lines a and b($p\|s_1$, $q\|s_2$). It is easy to see that the midpoints ^jS_i (j=1,2,3,4) of the segments ^jA_i ^jB_i (j =1,2) are vertices of parallelograms inscribed in this hyperbola, thus the lines joining the midpoints of its opposite sides form an involutional pencil of pairs conjugate diameters about the vertex M - the center of hyperbola (in this same manner it can be shown that a set of midpoints of segments whose endpoints are intersection points of non-degenerated conic with circles of pencil of concentric circles is a rectangular hyperbola).

If the point O = S, then this hyperbola is reduced to the two perpendicular lines $^{1,2}s$ and $^{3,4}s$, coinciding, respectively, with the axes of rectangular symmetry of the lines a and b (Fig. 4).

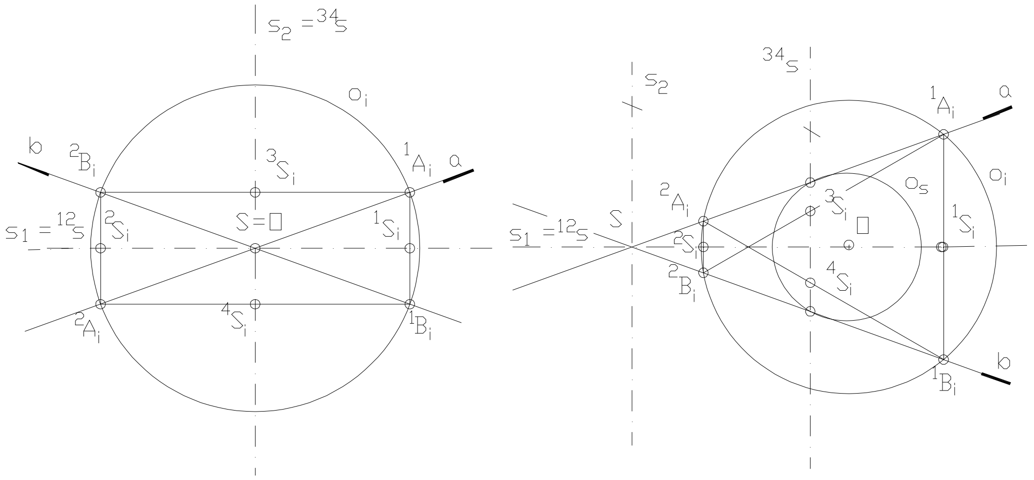

Fig. 4

And while the point O≠S lies on one of those axes, then this hyperbola degenerates also to two perpendicular lines, of which only one coincides with the symmetry of one axis, whereas the other is parallel to the other axis ($^{1,2}s/\ ^{3,4}s = s_1(s_2)$, and $^{3,4}s(^{1,2}s)\|s_2(s_1)$). The lines $^{1,2}s$ and $^{3,4}s$ are parallel, respectively, to the axes s_1 and s_2 pass through the contact points of circles with the centers $O_{1,2}$, and at the same time tangent to the lines a and b (Fig. 4).

3 Two skew lines

3.1 Pencils of planes

Next let us consider sets of midpoints of segments whose endpoints are intersection points of the lines a and b with planes of the bundle {l}.

Let us assume that one of distance planes of the considered skew lines, for example the plane α including the line a is a plane of drawing and at the same time the first (horizontal) projection plane of the mongean system of reference (Fig. 5).

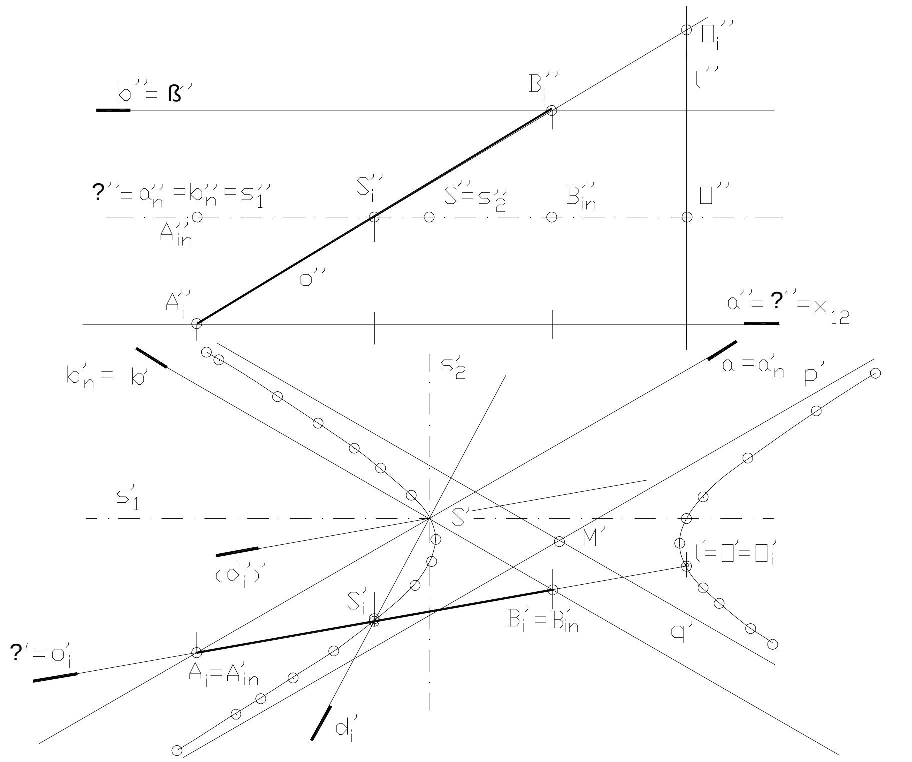

Fig.5

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In addition, let us assume that the axis l of the pencil of planes is perpendicular to the given skew lines a and b as well as oblique in relation to them. The planes ω_i of the bundle{l} intersect the lines a and b**,** respectively, at the points A_i and B_i. The lines $o_i=A_iB_i$ intersect the line l in the points O_i. The midpoints of all the segments, whose endpoints belong, respectively, to the distance planes always lie on the equidistant plane σ. Projecting the lines a, b, l plus o_i together with points O_i, A_i, B_i and S_i onto the plane σ in direction which is parallel to the axis l, we obtain on this projection plane a configuration shown in Fig. 1 and described in section 2.1. The orthogonal projection of the line l and its points is the point $O=l\cdot\sigma$. The projections a^n and b^n of the lines a and b intersect in the point $S=S^n$, and the projections o_i^n of the lines o_i are elements of the bundle of lines{O}. It must be noted, that, the segments A_iB_i and their projections have the common midpoints $S_i=S_i^n$. As shown in the section 2.1, the pencils of lines {S} and {O} belonging to the same plane σ are projective, hence their product is a conic being hyperbola, because these pencils of lines have two pairs of parallel lines.

It is clear that the lines $o_i=A_iB_i$ intersect the three lines (a,b and l), oblique in pairs, and hence generate lines of a skew quadric, whose intersection with plane σ, is a hyperbola, because this plane is parallel to the lines a and b (Fig. 5).

Now let us assume that the line l – axis of a bundle of planes {l} has a general position in relation to the skew lines a and b (Fig. 6). The planes ω_i $(m_i\|n_i)$ of bundle {l} intersect the lines a and b in the points A_i and B_i, respectively. If the series $a(A_i)$ and $b(B_i)$ are perspective to the same pencil of planes $l(\omega_i)$, then these ranks of points are projective (the ending elements in this perspective chain are projective as stated above).

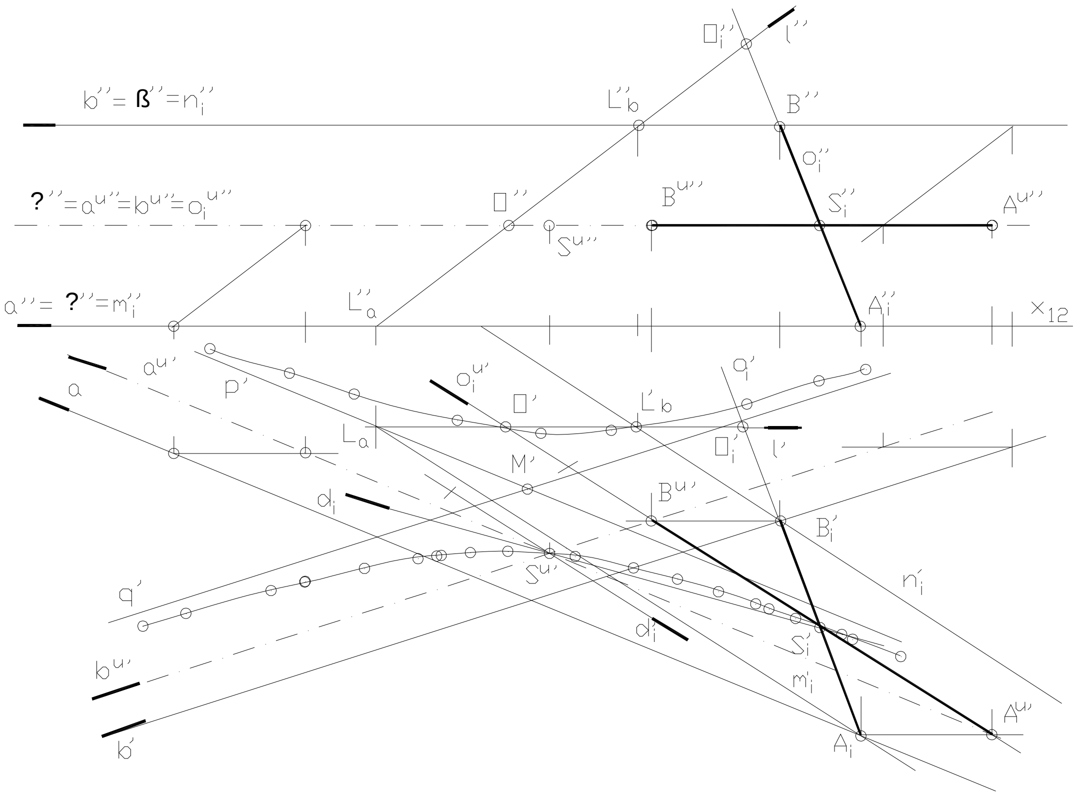

Fig.6

A skew quadric can be obtained as a sum of all the lines joining the equivalent points of the two projective series of points with skew bases. If in these series of points, the ideal points are a pair of homologous points, then the skew quadric is of parabolic type, and if they do not correspond - then it is of hyperbolic type [2].

In the case considered above, the plane ω of pencil {l} parallel to the line a(b) intersects the line a(b) in the point at infinity, and the line b(a) in the real point. The conclusion is that a skew quadric which is the sum of the lines $o_i=A_iB_i$ is a hyperboloid of one sheet (points ∞ of those projective series do not correspond with one another). The planes σ at the same time parallel to their two lines a and b (generating lines of skew quadric) intersect this hyperboloid along the hyperbola which is a set of midpoints S_i of segments A_iB_i.

This spatial problem, as in the case discussed above, can be changed to a planar, by projecting the lines a, b,l and o_i together with their points on the plane **σ,** in the direction which is parallel to the line l. The oblique projections a^u and b^u of the lines a and b belong to the same point S^u, the point $O = l\cdot\sigma$ is an oblique projection of the line l and its points, The lines o_i^u are projections o the lines o_i, and the segments A_iB_i and their oblique projections have the common midpoints ($S_i^u=S_i$). The pencils of lines $\{S^u\}$and {O}, as that shown in the section 2.1 are projective and have two pairs of homologous parallel lines, thus generate a hyperbola.

If the axis l of pencil of planes intersects the line a(b), or is parallel to line $s_1(s_2)$ or is a line at infinity, then the midpoints S_i of segments A_iB_i determined by these pencils of planes, lie on the lines (analogy to the trivial cases described in section 2.1).

3.2 Pencils of concentric spheres

It has been shown that the spatial problem, when determining a set of midpoints of segments, whose endpoints are intersection points of two skew lines with spheres of pencil of concentric spheres, may be expressed as coplanar lines and a pencil of concentric circles.

Let us assume that the given two skew lines a and b belong, respectively, to the distance planes α and β, and the point O, the center of concentric spheres does not coincide with the midpoint S of distance segment (the shortest segment of skew lines), as well as does not lie on either axes of rectangular symmetry of the given two skew lines.

The planes σ passing through the midpoint of distance segment of lines a and b and perpendicular to it is the plane of rectangular symmetry of distance planes, include the axes of rectangular symmetry of those lines, and also the midpoints of all the segments, whose endpoints belong to the planes α and β, respectively.

The spheres Ω_i of pencil $O(\Omega_i,R_i)$ intersect the lines a and b, respectively, in the points ${}^jA_i$ and ${}^jB_i(j=1,2)$. These points belong to the circles ${}^ao_i$ and ${}^bo_i$, being intersections of the distance plane α and β with spheres Ω_i. The orthogonal projections of lines a and b on the plane σ belong to the same point S, the projections of centers ${}^aO_i$ and ${}^bO_i$ of circles ${}^ao_i$ and ${}^bo_i$ unite with the projection of center O of spheres Ω_i, and the orthogonal projections of those circles form the pencil of concentric circles. The segments ${}^jA_i\ {}^jB_i$ and their projections have the common midpoint ${}^jSi = {}^jS_i^n$. When the point O, the center of concentric spheres Ω_i lies on the plane σ, then the circles ${}^ao_i$ and ${}^bo_i$ are identical figures and their orthogonal projections coincide. As a result o this projection obtained on the plane σ ,the two intersecting lines a^n and b^n at the point S as well as the pencil of concentric circles with the centre O – hence, the case described in section 2.2 and shown in Fig. 3,4.

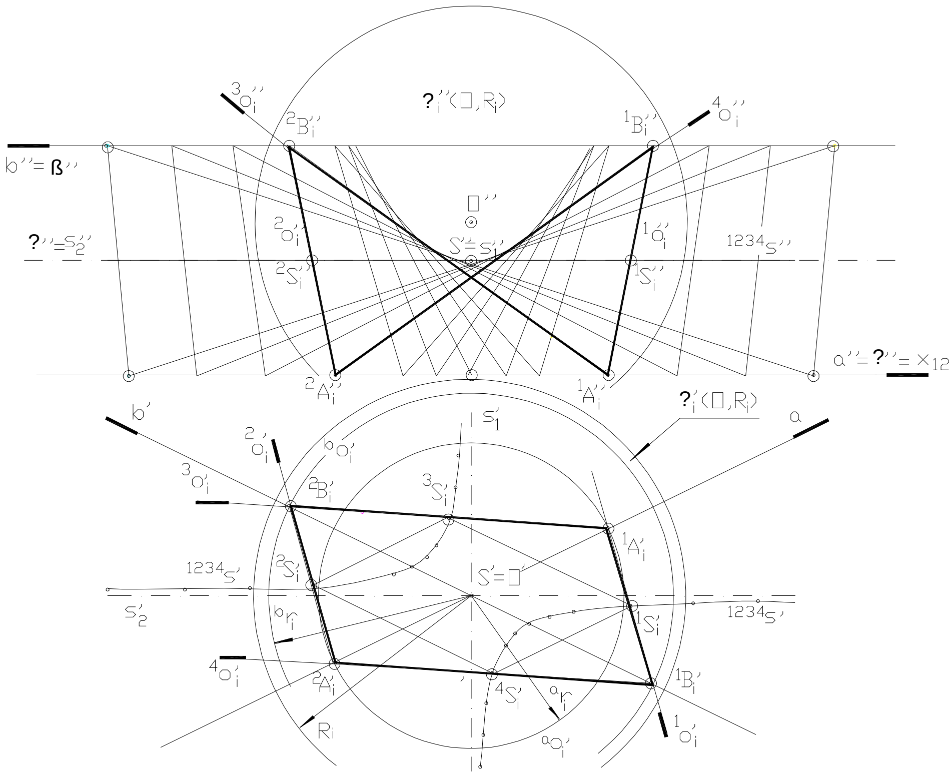

Fig. 7

Treating this problem as spatial, it is necessary to point out, that the spheres Ω_i of pencil{O}cut out on the skew lines a and b of involutional ranks of points a($^{1,2}A_i$) and b($^{1,2}B_i$). It is a particular case of an involution called similarity (in other words, a symmetry in relation to the center).These ranks o points about of oblique fundamental lines are projective. The lines ^jo_i(j=1,2,3,4) joining homologous points of those projective ranks, are generating lines of a skew quadric which is intersected by the plane σ parallel to its two lines (a and b) along the rectangular hyperbola (Fig. 7).

The domain of this paper can be extended and these studies can be continued. The endpoints of segments can be determined on a plane, for example, with a of pencil of coaxial tangential circles or belonging to two different and constant points, and spatially by a pencil of coaxial tangential spheres or belonging to the same circle.

Making use of one of the pencils of circles, with real power axis to determine endpoints of segments, one can prove that their midpoints belong, inter alia, to a parabola.

Finally, the results of those studies demonstrate that curves of the second order can also be considered as midpoints of segments, whose endpoints belong to two lines both coplanar and skew lines, respectively.

References :

[1] Coxeter H.S.M.: *The real projective plane*. Cambridge at the University Press,1955.

[2] Otto F., Otto E.: *Podręcznik geometrii wykreślnej* (*The Handbook of Descriptive Geometry*). PWN, Warszawa, 1976.

[3] Ochoński S.: *The Equilateral Triangles of a Given Side whose Vertices Belong to Three Non Coplanar Straight Lines*. The Journal BIULETYN of Polish Society for Geometry and Engineering Graphics, Volume 19 (2009), 15-26.

ŚRODKI ODCINKÓW O KOŃCACH NALEŻĄCYCH DO DWÓCH PROSTYCH

Artykuł przedstawia wyniki badań zbiorów środków odcinków, o końcach należących do dwóch prostych komplanarnych, jak i skośnych.
Punkty ograniczające te odcinki, na płaszczyźnie, wyznaczane są za pomocą: a) pęku prostych, b) pęku koncentrycznych okręgów, zaś w przestrzeni c) pęku płaszczyzn i d) pęku współśrodkowych sfer.
Wykazano, że w przypadkach a) i c), środki tak wyznaczonych odcinków należą do prostej bądź hiperboli, która może być hiperbolą prostokątną, a w b) i d) są zawsze punktami hiperboli równobocznej lub pary prostopadłych prostych jako zdegenerowanej stożkowej.
Ponadto zwrócono uwagę, iż zakres tej pracy może być znacznie rozszerzony, a badania kontynuowane. Punkty ograniczające rozważane odcinki mogą być bowiem wyznaczane również na płaszczyźnie za pomocą: pęku współosiowych i stycznych okręgów bądź przechodzących przez dwa stałe punkty, a w przestrzeni - pęku współosiowych, stycznych sfer lub zawierających ten sam okrąg.
Można wykazać, że w przypadku jednego z pęków okręgów o właściwej osi potęgowej, środki tak wyznaczonych odcinków należą między innymi do paraboli.
Reasumując stwierdzono, iż stożkowe mogą być również rozpatrywane, jako środki odcinków o końcach należących odpowiednio do dwóch prostych zarówno komplanarnych, jak i skośnych

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