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Semi-Markov decision process as a safety and reliability model of a sea transport operation

Keywords

safety reliability, semi-Markov decision process, transport operation

Abstract

A problem of optimization of a sea transport operation in safety and reliability aspect is discussed in the paper. To describe and solve this problem, a semi-Markov decision processes theory is applied. The semi-Markov decision process as a model of the sea transport operation is constructed. An algorithm which allows to compute the optimal strategy of the operation in safety and reliability aspect is presented.

1. Introduction

Semi-Markov decision processes theory delivers methods giving the opportunity to control an operation processes of the systems. In such kind of problems we want to choose the most rewarding process among some alternatives available for the operation.

2. Semi-Markov decision processes

Semi-Markov decision processes theory was de veloped by Jewell [6], Howard [5], Main & Osaki [9], Gercbacha [2]. Those processes were also discussed by F.Grabski [3].

Semi-Markov (SM) decision process is such SM process with a finite states space, that its trajectory depends on decisions which are made at an initial instant τ_0 and at the states changes moments $\tau_1,...,\tau_n,...$. By $d_i(\tau_n)$ we denote a decision at the moment τ_n, under condition $X(\tau_n)=i$. We assume that a set of decision in each state i, denoting by D_i, is finite. To take decision $k\in D_i$, means to select *k*th row among the alternating rows of the semi-Markov kernels.

$$\{Q_{ij}^{(k)}(t): t\geq 0, k\in D_i, i, j\in S\} \tag{1}$$

where

$$Q_{ij}^{(k)}(t) = p_{ij}^{(k)} F_{ij}^{(k)}(t) \tag{2}$$

It determines the evolution of the system on the interval $[\tau_n, \tau_{n+1})$. More precisely, the decision $d_i(\tau_n)=k\in D_i$ means, that according to the distribution $(p_{ij}^{(k)}: j\in S)$, $S=\{1,2,...,N\}$ a state j is drawn, for which the process jumps at the moment τ_{n+1}, and the length of the interval $[\tau_n, \tau_{n+1})$ is chosen according to distribution given by the *CDF* $F_{ij}^{(k)}(t)$.

A sequence

$$d=\{(d_1(\tau_n),...,d_N(\tau_n)): n=0,1,2,...\} \tag{3}$$

is called a *strategy*. The strategy is said to be Markovian, if for every state $i\in S$, and every moment $\tau_n, n=1,2,...$ of the state change the decision $d_i(\tau_n)\in D_i$ does not depend on process evolution until the moment τ_n. Moreover, if this decision does not depend on n, $d_i(\tau_n)=d_i$, then it is called a stationary decision. In this case we obtain a homogeneous semi-Markov process. Optimization of the semi-Markov decision process consists in choosing the strategy which maximizes the gain of the system.

3. Optimization for a finite states change

We will consider only a problem of semi-Markov process optimization for a finite states change m, so we will investigate the process at time interval $[0, \tau_m)$.

To formulate the optimization problem we have to introduce the reward structure for the process. We assume that the system occupies the state i having chosen a successor state j, it earns a gain (reward) at a rate

$$r_{ij}^{(k)}(x), \quad i, j \in S, k \in D_i \tag{4}$$

at a time x entering state i for a decision $k \in D_i$. The function $r_{ij}^{k}(x)$ is called the "yield rate" of state i at time x when the successor state is j and k is chosen decision [6]. A negative reward at a rate $r_{ij}^{(k)}(x)$ denotes loss or a cost of that one. A value of a function

$$R_{ij}^{(k)}(t) = \int_0^t r_{ij}^{(k)}(x)dx, \quad i, j \in S, k \in D_i \tag{5}$$

denotes the reward the system earns by spending a time t in state i before making a transition to state j for the decision $k \in D_i$. When the transition from the state i to state j for the decision k is actually made , the system earns a bonus a fixed sum. The bonus is denotes by

$$b_{ij}^{(k)}, \quad i, j \in S, k \in D_i \tag{6}$$

A number

$$U_i^{(k)} = \sum_{j\in S} \int_0^\infty (R_{ij}^{(k)}(t) + b_{ij}^{(k)})dQ_{ij}^{(k)}(t) \tag{7}$$

is an expected value of the gain (reward) that is generated by the process in state i at one interval of its realization for the decision $k \in D_i$.

By $V_i(d_m), i \in S$ we denote the expected value of the gain (reward) that is generated by the process during a time interval $[0, \tau_m)$ under the condition that the initial state is $i \in S$ and a sequence of polices is

$$d_m = ((\delta_1(\tau_n),\dots,\delta_N(\tau_n)) : n = 0,1,\dots,m-1)$$

By $V_j(d_{m-1}), j \in S$ we denote the expected value of the gain (reward) that is generated by the process during a time interval $[\tau_1, \tau_m)$ under the condition that the process has just entered the state $j \in S$ at the moment τ_1 and a sequence of polices

$$d_{m-1} = \{(\delta_1(\tau_n),\dots,\delta_N(\tau_n)) : n = 1,\dots,m-1\},$$

is choosen .

The expected value of the gain during a time interval $[0, \tau_m)$ under the condition that the initial state is $i \in S$ $[0, \tau_m)$ is the expected value of the gain (reward) that is generated by the process during a time interval $[0, \tau_1)$ and the gain (reward) that is generated by the process during a time $[\tau_1, \tau_m)$. Since

$$V_i(d_m) = U_i^{(k)} + \sum_{j\in S} p_{ij}^{(k)} V_j(d_{m-1}), \; i \in S \tag{8}$$

By substitution we obtain

$$V_i(d_m) = \sum_{j\in S} \int_0^\infty (R_{ij}^{(k)}(t) + b_{ij}^{(k)})dQ_{ij}^{(k)}(t) + \\ + \sum_{j\in S} p_{ij}^{(k)} V_j(d_{m-1}), \; i \in S \tag{9}$$

The strategy (the sequence of polices) d_m^* is called optimal in a gain maximum problem on interval $[\tau_0, \tau_m)$ for the semi-Markov decision process which start from a state i, if

$$V_i(d_m^*) = \max_{d_m}[V_i(d_m)]$$

It means that

$$V_i(d_m^*) \geq V_i(d_m)$$

for all strategies d_m.

The optimal strategy we can get by using the dynamic programming technique. Applying Bellman principle of optimality we get an algorithm for obtaining the optimal strategy. This algorithm is defined by the following formulas

$$V_i(d_n^*) = \max_{k\in D_i}[U_i^{(k)} + \sum_{j\in S} p_{ij}^{(k)} V_j(d_{n-1}^*)], \\ i \in S, n = 1,\dots,m \tag{10}$$

$$V_i(d_0^*) = \max_{k\in D_i}[U_i^{(k)}], \; i \in S \tag{11}$$

To obtain the police d_0^* we start from (11). Based on formula

$$V_i(d_1^*) = \max_{k \in D_i}[U_i^{(k)} + \sum_{j \in S} p_{ij}^{(k)} V_j(d_0^*)],\ i \in S$$

we find strategy

$$d_1^* = ((\delta_1^*(\tau_n), \dots, \delta_N^*(\tau_n)) : n = 0,1)$$

in next step Continue this procedure we obtain the optimal strategy

$$d_m^* = ((\delta_1^*(\tau_n), \dots, \delta_N^*(\tau_n)) : n = 0,1,\dots,m-1)$$

4. Decision semi-Markov model of a sea transport operation

The sea transport operation consist of some steps, which are realized in turn. Duration of the each stage is assumed to be positive random variable. Events that cause perturbation of the ship reliability or (and) safety my occur during operation. The perturbations increase the time of operation and the probability of failure as well. The main goal of this paper is to construct semi-Markov decision process describing a simple sea transport operation.

4.1. Description and assumptions

The sea transport operation consists of 4 stages which following in turn. The stage assume to be: stopover of the ship in the port A , cruise from the port A to port B stopover in the port B, cruise from the port B to port A .

We assume the duration of *i*-th stage for decision $k \in D_i$, is a nonnegative random variable $\xi_i^{(k)}$ with a cumulative probability distribution function

$$F_{\xi_i^{(k)}}(t) = P(\xi_i^{(k)} \le t),\ i = 1,2,3,4,\quad k \in D_i$$

Operation on each step may be perturbed. We assume that no more than one event causing the perturbation of the operation on *i*-th stage for decision $k \in D_i$, may occur. The time to this event is a nonnegative random variable $\eta_i^{(k)}$ with an exponential probability distribution

$$F_{\eta_i^{(k)}}(t) = P(\eta_i^{(k)} \le t) = 1 - e^{-\alpha_i^{(k)} t},\ i = 1,2,3,4,\quad k \in D_i$$

The duration of the perturbed *i*-th stage of the sea transport operation for decision $k \in D_{4+i}$, is a nonnegative random variable $\zeta_i^{(k)}$ with a probability distribution

$$F_{\zeta_i^{(k)}}(t) = P(\zeta_i^{(k)} \le t),\ i = 1,2,3,4,\ k \in D_{4+i}$$

The perturbation degreases the probability of the operation failure. We suppose that time to failure of the perturbed operation on the *i*-th stage for decision $k \in D_i$ is a nonnegative random variable $\nu_i^{(k)}$ that has a exponential distribution with a parameter $\beta_i^{(k)}$

$$P(\nu_i^{(k)} \le t) = 1 - e^{-\beta_i^{(k)} t};\ i = 1,2,3,4,\quad k \in D_{4+i}$$

We assume that all those random variables are mutually independent.

4.2. Model

To construct model we start from a definition of the operation states. Suppose the states of the ship operation are:

1. stopover of the ship, loading and unloading in the port A
2. cruise from the port A to port B
3. stopover, loading and unloading in the port B,
4. cruise from the port B to port A
5. the ship stopover, loading and unloading in the port A with perturbation of its reliability or (and) safety
6. cruise from the port A to port B with perturbation of its reliability or (and) safety
7. stopover, loading and unloading in the port
B with perturbation of its reliability or (and) safety
8. cruise from the port B to port A with perturbation of its reliability or (and) safety
9. failure of the operation

Transition graph for the sea transport operation is shown in *Figure 1.*

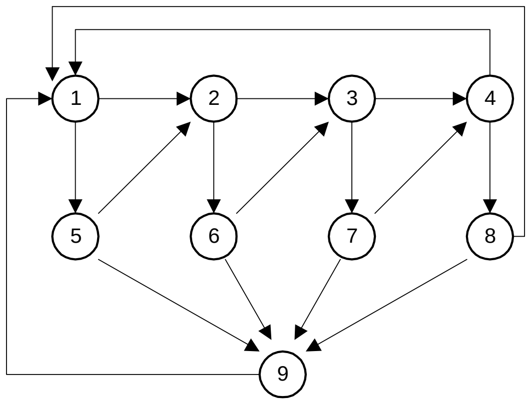

Figure 1. Transition graph for the sea transport operation

To obtain a semi-Markov model we have to define all nonnegative elements of the semi- Markov kernel see [3], [4], [8].

The semi-Markov kernel corresponding to the graph shown in *Figure 1* take a form

$$Q^{(k)}(t)=\begin{bmatrix} Q_{12}^{(k)} & 0 & 0 & 0 & Q_{15}^{(k)} & 0 & 0 & 0 & 0 \\ 0 & Q_{23}^{(k)} & 0 & 0 & 0 & Q_{26}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & Q_{34}^{(k)} & 0 & 0 & 0 & Q_{37}^{(k)} & 0 & 0 \\ Q_{41}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & Q_{48}^{(k)} & 0 \\ 0 & Q_{52}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & Q_{59}^{(k)} \\ 0 & 0 & Q_{63}^{(k)} & 0 & 0 & 0 & 0 & 0 & Q_{69}^{(k)} \\ 0 & 0 & 0 & Q_{74}^{(k)} & 0 & 0 & 0 & 0 & Q_{79}^{(k)} \\ Q_{81}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{89}^{(k)} \\ Q_{91}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Based on assumptions we have to define functions

$$Q_{ij}^{(k)}=Q_{ij}^{(k)}(t)=p_{ij}^{(k)}F_{ij}^{(k)}(t), \quad t\ge 0$$

$$Q_{12}^{(k)}(t)=P(\xi_1^{(k)}\le t,\xi_1^{(k)}<\eta_1^{(k)}\}= \\ =\int_0^t e^{-\alpha_1^{(k)}x}dF_{\xi_1^{(k)}}(x), \quad k\in D_1$$

$$Q_{15}^{(k)}(t)=P(\eta_1^{(k)}\le t,\eta_1^{(k)}<\xi_1^{(k)}\}= \\ =\alpha_1^{(k)}\int_0^t[1-F_{\xi_1^{(k)}}(x)]e^{-\alpha_1^{(k)}x}dx, \quad k\in D_1$$

$$Q_{23}^{(k)}(t)=P(\xi_2^{(k)}\le t,\xi_2^{(k)}<\eta_2^{(k)}\}= \\ =\int_0^t e^{-\alpha_2^{(k)}x}dF_{\xi_2^{(k)}}(x), \quad k\in D_2$$

$$Q_{34}^{(k)}(t)=P(\xi_3^{(k)}\le t,\xi_3^{(k)}<\eta_3^{(k)}\}= \\ =\int_0^t e^{-\alpha_3^{(k)}x}dF_{\xi_3^{(k)}}(x), \quad k\in D_3$$

$$Q_{37}^{(k)}(t)=P(\eta_3^{(k)}\le t,\eta_3^{(k)}<\xi_3^{(k)}\}= \\ =\alpha_3^{(k)}\int_0^t[1-F_{\xi_3^{(k)}}(x)]e^{-\alpha_3^{(k)}x}dx, \quad k\in D_3$$

$$Q_{41}^{(k)}(t)=P(\xi_4^{(k)}\le t,\xi_4^{(k)}<\eta_4^{(k)}\}= \\ =\int_0^t e^{-\alpha_4^{(k)}x}dF_{\xi_4^{(k)}}(x), \quad k\in D_4$$

$$Q_{48}^{(k)}(t)=P(\eta_4^{(k)}\le t,\ \eta_4^{(k)}<\xi_4^{(k)}\}= \\ =\alpha_4^{(k)}\int_0^t[1-F_{\xi_4^{(k)}}(x)]e^{-\alpha_4^{(k)}x}dx, \quad k\in D_4$$

$$Q_{52}^{(k)}(t)=P(\varsigma_1^{(k)}\le t,\varsigma_1^{(k)}<\nu_1^{(k)}\}= \\ =\int_0^t e^{-\beta_1^{(k)}x}dF_{\varsigma_1^{(k)}}(x), \quad k\in D_5$$

$$Q_{59}^{(k)}(t)=P(\nu_1^{(k)}\le t,\nu_1^{(k)}<\varsigma_1^{(k)}\}= \\ =\beta_1^{(k)}\int_0^t[1-F_{\varsigma_1^{(k)}}(x)]e^{-\beta_1^{(k)}x}dx, \quad k\in D_5$$

$$Q_{63}^{(k)}(t)=P(\varsigma_2^{(k)}\le t,\varsigma_2^{(k)}<\nu_2^{(k)}\}= \\ =\int_0^t e^{-\beta_2^{(k)}x}dF_{\varsigma_2^{(k)}}(x), \quad k\in D_6$$

$$Q_{69}^{(k)}(t)=P(\nu_2^{(k)}\le t,\nu_2^{(k)}<\varsigma_2^{(k)}\}= \\ =\beta_2^{(k)}\int_0^t[1-F_{\varsigma_2^{(k)}}(x)]e^{-\beta_2^{(k)}x}dx, \quad k\in D_6$$

$$Q_{74}^{(k)}(t)=P(\varsigma_3^{(k)}\le t,\varsigma_3^{(k)}<\nu_3^{(k)}\}= \\ =\int_0^t e^{-\beta_3^{(k)}x}dF_{\varsigma_3^{(k)}}(x), \quad k\in D_7$$

$$Q_{79}^{(k)}(t)=P(\nu_3^{(k)}\le t,\nu_3^{(k)}<\varsigma_3^{(k)}\}= \\ =\beta_3^{(k)}\int_0^t[1-F_{\varsigma_3^{(k)}}(x)]e^{-\beta_3^{(k)}x}dx, \quad k\in D_7$$

$$Q_{81}^{(k)}(t)=P(\varsigma_4^{(k)}\le t,\varsigma_4^{(k)}<\nu_4^{(k)}\}= \\ =\int_0^t e^{-\beta_4^{(k)}x}dF_{\varsigma_4^{(k)}}(x), \quad k\in D_8$$

$$Q_{89}^{(k)}(t)=P(\nu_4^{(k)}\le t,\ \nu_4^{(k)}<\varsigma_4^{(k)}\}= \\ =\beta_4^{(k)}\int_0^t[1-F_{\beta_4^{(k)}}(x)]e^{-\beta_4^{(k)}x}dx, \quad k\in D_8$$

Having the semi-Markov kernel we can find transition matrix of the embedded Markov chain. Elements of this matrix are given by the formula

$$p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t), \quad t \ge 0$$

Since we have

$$P^{(k)} = \begin{bmatrix} p_{12}^{(k)} & 0 & 0 & 0 & p_{15}^{(k)} & 0 & 0 & 0 & 0 \\ 0 & p_{23}^{(k)} & 0 & 0 & 0 & p_{26}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & p_{34}^{(k)} & 0 & 0 & 0 & p_{37}^{(k)} & 0 & 0 \\ p_{41}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & p_{48}^{k4} & 0 \\ 0 & p_{52}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & p_{59}^{(k)} \\ 0 & 0 & p_{63}^{(k)} & 0 & 0 & 0 & 0 & 0 & p_{69}^{(k)} \\ 0 & 0 & 0 & p_{74}^{(k)} & 0 & 0 & 0 & 0 & p_{79}^{(k)} \\ p_{81}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{89}^{(k)} \\ p_{91}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$p_{12}^{(k)} = \int_0^\infty e^{-\alpha_1^{(k)} x} dF_{\xi_1^{(k)}}(x), \quad k \in D_1$$

$$p_{15}^{(k)} = \alpha_1^{(k)} \int_0^\infty [1 - F_{\xi_1^{(k)}}(x)] e^{-\alpha_1^{(k)} x} dx, \quad k \in D_1$$

$$p_{23}^{(k)} = \int_0^\infty e^{-\alpha_2^{(k)} x} dF_{\xi_2^{(k)}}(x), \quad k \in D_2$$

$$p_{26}^{(k)} = \alpha_2^{(k)} \int_0^\infty [1 - F_{\xi_2^{(k)}}(x)] e^{-\alpha_2^{(k)} x} dx, \quad k \in D_2$$

$$p_{34}^{(k)} = \int_0^\infty e^{-\alpha_3^{(k)} x} dF_{\xi_3^{(k)}}(x), \quad k \in D_3$$

$$p_{37}^{(k)} = \alpha_3^{(k)} \int_0^\infty [1 - F_{\xi_3^{(k)}}(x)] e^{-\alpha_3^{(k)} x} dx, \quad k \in D_3$$

$$p_{41}^{(k)} = \int_0^\infty e^{-\alpha_4^{(k)} x} dF_{\xi_4^{(k)}}(x), \quad k \in D_4$$

$$p_{48}^{(k)} = \alpha_4^{(k)} \int_0^\infty [1 - F_{\xi_4^{(k)}}(x)] e^{-\alpha_4^{(k)} x} dx, \quad k \in D_4$$

$$p_{52}^{(k)} = \int_0^\infty e^{-\beta_1^{(k)} x} dF_{\varsigma_1^{(k)}}(x), \quad k \in D_5$$

$$p_{59}^{(k)} = \beta_1^{(k)} \int_0^\infty [1 - F_{\varsigma_1^{(k)}}(x)] e^{-\beta_1^{(k)} x} dx, \quad k \in D_5$$

$$p_{63}^{(k)} = \int_0^\infty e^{-\beta_2^{(k)} x} dF_{\varsigma_2^{(k)}}(x), \quad k \in D_6$$

$$p_{69}^{(k)} = \beta_2^{(k)} \int_0^\infty [1 - F_{\varsigma_2^{(k)}}(x)] e^{-\beta_2^{(k)} x} dx, \quad k \in D_6$$

$$p_{74}^{(k)} = \int_0^\infty e^{-\beta_3^{(k)} x} dF_{\varsigma_3^{(k)}}(x), \quad k \in D_7$$

$$p_{79}^{(k)} = \beta_3^{(k)} \int_0^\infty [1 - F_{\varsigma_3^{(k)}}(x)] e^{-\beta_3^{(k)} x} dx, \quad k \in D_7$$

$$p_{81}^{(k)} = \int_0^\infty e^{-\beta_4^{(k)} x} dF_{\varsigma_4^{(k)}}(x), \quad k \in D_8$$

$$p_{89}^{(k)} = \beta_4^{(k)} \int_0^\infty [1 - F_{\beta_4^{(k)}}(x)] e^{-\beta_4^{(k)} x} dx, \quad k \in D_8$$

To construct the semi-Markov decision model of the sea transport operations we need also to assume sets of decisions D_i, $i = 1,2,...,9,$ which generate parameters. For simplicity we suppose that each set D_i, $i = 1,2,...,9,$ consists of two components only: $D_i = \{1, 2\} \quad i = 1,2,...,9.$
We assume

$$r_{ij}^{(k)}(x) = r_{ij}^{(k)} \in R, \quad k \in D_i,$$
$$i, j \in S = \{1,2,..., 9\}$$

From (5) and (7) we obtain

$$R_{ij}^{(k)}(t) = r_{ij}^{(k)} t, \quad i, j \in S, k \in D_i$$

and

$$U_i^{(k)} = \sum_{j\in S} p_{ij}^{(k)} (r_{ij}^{(k)} m_{ij}^{(k)} + b_{ij}^{(k)})$$

where $m_{ij}^{(k)} = E(T_{ij}^{(k)})$ denotes the expectation of the holding time of the state i if the successor state will be j.

4.3. Algorithm of choosing optimal strategy for the sea transport operations

1. For all $i \in S = \{1,2,...,9\}$ and $k \in D_i = \{1_i, 2_i\}$ compute

$$U_i^{(k)} = \sum_{j\in S} p_{ij}^{(k)} (r_{ij}^{(k)} m_{ij}^{(k)} + b_{ij}^{(k)})$$

2. For all $i \in S = \{1,2,...,9\}$ find d_0^* such that

$$V_i(d_0^*) = \max_{k\in D_i} [U_i^{(k)}]$$

3. For *l=1,2,…,m-1* and for all $i \in S$ find d_l^* such that

$$V_i(d_l^*) = \max_{k \in D_i}[U_i^{(k)} + \sum_{j \in S} p_{ij}^{(k)} V_j(d_{l-1}^*)]$$

5. Concluding remarks

To formulate the real optimization problem of the sea transport operation we have to know decisions in each state and the corresponding parameters. For example, for the state 2 the set of decision D2 could consist of two options 1 and 2 where,

1 - normal cruise
2 - fast cruise

Transition probabilities from the state 2 are:

$p_{23}^{(1)}, \quad p_{26}^{(1)} = 1 - p_{23}^{(1)}$ for decision 1

and

$p_{23}^{(2)}, \quad p_{26}^{(2)} = 1 - p_{23}^{(2)}$ for decision 2

It seems to be true inequality

$p_{23}^{(1)} > p_{23}^{(2)}$ which imply inequality $p_{26}^{(1)} < p_{26}^{(2)}$.

Those parameters or characteristics which define (generate) them, we can assess using one of statistics or expert methods.
The "yield rates" denoting the cost rates for state 2 are negative numbers:

$r_{23}^{(1)}, \quad r_{26}^{(1)}$ for decision 1

and

$r_{23}^{(2)}, \quad r_{26}^{(2)}$ for decision 2

The expectations of the holding times are positive numbers:

$m_{23}^{(1)}, \quad m_{26}^{(1)}$ for decision 1

and
$m_{23}^{(2)}, \quad m_{26}^{(2)}$ for decision 2

The bonus denoting the gain in the state 2 which are positive numbers:

$b_{23}^{(1)}, \quad b_{26}^{(1)}$ for decision 1

and

$b_{23}^{(2)}, \quad b_{26}^{(2)}$ for decision 2

should be known for a decision maker.

The expected value of the gain (reward) which is generated by the process in state *2* at one interval of its realization for the decision 1 and 2 are

$$U_2^{(1)} = p_{23}^{(1)}(r_{23}^{(1)} m_{23}^{(1)} + b_{23}^{(1)}) + p_{26}^{(1)}(r_{26}^{(1)} m_{26}^{(1)} + b_{26}^{(1)})$$

$$U_2^{(k)} = p_{23}^{(2)}(r_{23}^{(2)} m_{23}^{(2)} + b_{23}^{(2)}) + p_{26}^{(2)}(r_{26}^{(2)} m_{26}^{(2)} + b_{26}^{(2)})$$

References

[1] Aven, T. (2008). *Risk Analysis*. Wiley.
[2] Gercbach, J.B. (1982). *Modeli profilaktiki*. Sovetskoe Radio, Moskva.
[3] Grabski, F. (1982). Teoria semi-markowskich procesów eksploatacji obiektów technicznych. *Zeszyty Naukowe AMW*, nr 75 A.
[4] Grabski, F. (2002). Semi-markowskie modele niezawodności i eksploatacji. Polska Akademia Nauk, IBS. Warszawa 2002. Seria: *Badania Systemowe* , tom 30.
[5] Howard, R.A.. (1964). Research in semi - Markovian decision structures. *J. Oper. Res. Soc*. 6, nr 4, p.163-199.
[6] Howard, R.A. (1971). *Dynamic Probabilistic Systems*. Volume II Semi-Markov and Decision Processes. John Wiley, New York, London, Sydney, Toronto.
[7] Jewell, W.S. (1963). Markov-renewal programming. *Operation Research* 11, 938-971.
[8] Korolu, V.S. & Turbin, A.F. (1976). *Polumarkovskije processy i ich priłożenija*. Naukova Dumka, Kijev.
[9] Mine, H. & Osaki, S. (1970). *Markovian decision processes*. AEPCI, Nev York.
[10] Silvestrov, D.S. (1980). *Polumarkovskije processy s diskretnym możestwom sostojanij*. Sovetskoe Radio, Moskva.