Opuscula Math. 35, no. 2 (2015), 191–197
http://dx.doi.org/10.7494/OpMath.2015.35.2.191 Opuscula Mathematica

A NEW CHARACTERIZATION OF CONVEX φ-FUNCTIONS WITH A PARAMETER

Bartosz Micherda

Communicated by Henryk Hudzik

Abstract. We show that, under some additional assumptions, all projection operators onto latticially closed subsets of the Orlicz-Musielak space generated by Φ are isotonic if and only if Φ is convex with respect to its second variable. A dual result of this type is also proven for antiprojections. This gives the positive answer to the problem presented in Opuscula Mathematica in 2012.

Keywords: Orlicz-Musielak space, convex function, isotonic operator, projection operator, antiprojection operator.

Mathematics Subject Classification: 41A65, 39B62, 46E30.

1. INTRODUCTION

The isotonicity of projection operators onto closed (or latticially closed) subsets of ordered metric spaces has been investigated since 1986, mainly in Hilbert spaces and Banach spaces (for details and possible applications see [1, 3–5] and the references given there), and in modular spaces [2, 9]. Moreover, antiprojection operators have also been examined from this point of view in [6, 7, 9].

It turns out that the monotonic properties of projections and antiprojections in various function spaces can be determined by two special functional inequalities, named "the properties of four elements", which were defined in [1, 6], and further investigated in [2, 5, 7, 9]. In particular, it has been shown that, under some additional assumption denoted in [9] as $(*)$, both these inequalities are valid in an Orlicz-Musielak space if and only if this space is generated by a convex φ-function [9, Theorem 3.5].

In this note we prove a new, even stronger theorem stating that, under some new assumption on the φ-function Φ, all projections onto latticially closed subsets of the Orlicz-Musielak space L^Φ are isotonic (and all antiprojections onto such sets are antiisotonic) if and only if Φ is convex with respect to its second variable (Theorem 3.4, see also Remark 3.5 and Theorem 3.6). This strengthens the results published in [2,7], and gives the affirmative answer to the question presented as Problem 3.8 in [9].

2. PRELIMINARIES

In this section we review some of the standard facts on Orlicz-Musielak spaces. In particular, we set up notation and terminology used in the following part of the text. For a more detailed summary of the modular spaces theory we refer the reader to [10, 11].

Let us first recall the notion of an Orlicz-Musielak space.

Definition 2.1. Let (Ω, Σ, μ) be a space with nonzero measure. Then $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *φ-function with a parameter* if it satisfies the following properties:

1. for every $t \in \Omega$, $\varphi_t(\cdot) = \Phi(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function such that $\varphi_t(0) = 0$ and $\varphi_t(x) > 0$ for $x > 0$,
2. for every $x \in \mathbb{R}_+$, $\Phi(\cdot, x) : \Omega \to \mathbb{R}_+$ is a Σ-measurable function.

Definition 2.2. Let Φ be a φ-function with a parameter and denote by $M(\Omega;\mathbb{R})$ the set of all real Σ-measurable functions defined on Ω, with equality μ-almost everywhere. For $f \in M(\Omega;\mathbb{R})$ define

$$\rho_\Phi(f) = \int_\Omega \Phi(t, |f(t)|)\, d\mu(t).$$

Then ρ_Φ is the *Orlicz-Musielak modular* generated by Φ. The corresponding modular space

$$L^\Phi = \left\{ f \in M(\Omega;\mathbb{R})\colon \lim_{\alpha\to 0} \rho_\Phi(\alpha f) = 0 \right\}$$

is the *Orlicz-Musielak space*. If the function Φ is independent of t, the modular ρ_Φ is said to be the *Orlicz modular*. Every Orlicz-Musielak space can be ordered in the natural way by the cone of nonnegative functions ($f \geq g$ if and only if $f - g \geq 0$).

Remark 2.3. By the Lebesgue dominant convergence theorem, it is easy to show that

$$L^\Phi = \{ f \in M(\Omega;\mathbb{R})\colon \rho_\Phi(\alpha f) < \infty \text{ for some } \alpha > 0 \}.$$

The following definition of the projection and the antiprojection onto a subset of an Orlicz-Musielak space agrees with the one widely used in metric space theory.

Definition 2.4. Let D be any subset of L^Φ and choose $f \in L^\Phi$. Then

$$P_D(f) = \left\{ y \in D\colon \rho_\Phi(f - y) = \inf_{d\in D} \rho_\Phi(f - d) \right\}$$

and

$$P_D^a(f) = \left\{ z \in D\colon \rho_\Phi(f - z) = \sup_{d\in D} \rho_\Phi(f - d) \right\}.$$

The sets $P_D(f)$ and $P_D^a(f)$ (both of which can be empty) are called the *projection of f onto D* and the *antiprojection of f onto D*, respectively.

Definition 2.5. We say that the projection operator P_D is *isotonic* if, for any $x, y \in L^\Phi$ such that $x \leq y$ and both $P_D(x)$, $P_D(y)$ are non-empty, there exist $w \in P_D(x)$ and $v \in P_D(y)$ satisfying $w \leq v$. In particular, if $x \leq y$ and $P_D(x)$, $P_D(y)$ are both singletons, then $P_D(x) \leq P_D(y)$. The antiprojection operator P_D^a is *antiisotonic* if, for any $x, y \in L^\Phi$ such that $x \leq y$ and both $P_D^a(x)$, $P_D^a(y)$ are non-empty, there exist $w \in P_D^a(x)$ and $v \in P_D^a(y)$ with $w \geq v$. In the case when $x \leq y$ and $P_D^a(x)$, $P_D^a(y)$ are both singletons, we get $P_D^a(x) \geq P_D^a(y)$.

Definition 2.6. A set $D \subset L^\Phi$ is called *latticially closed* if $\min(x, y) \in D$ and $\max(x, y) \in D$ for all $x, y \in D$.

Various examples of such sets are given in [7, Ex. 2.4] and [8, Ex. 1.4].

3. MAIN RESULTS

From the results proved in [2, 7], the following theorem can been derived.

Theorem 3.1. *If Φ is convex with respect to its second variable, then:*

(1) *for each latticially closed set $D \subset L^\Phi$, the projection operator P_D is isotonic,*
(2) *for each latticially closed set $D \subset L^\Phi$, the antiprojection operator P_D^a is antiisotonic.*

Moreover, it is known that the analogue of the above theorem does not have to be true if Φ is not a convex function of x, see [9, Ex. 3.7] for the counterexample.

Now we are going to show that the assumption of convexity in Theorem 3.1 is, in fact, the essential one. We will need the following additional (but not too restrictive) assumption.

Assumption $(**)$**.** For any $y_1, y_2, y_3, y_4 > 0$, $t \in \Omega$ and $\varepsilon > 0$, there exist two disjoint sets $\Omega_1, \Omega_2 \in \Sigma$ such that:

(a) $\mu(\Omega_1) = \mu(\Omega_2) \in (0, \infty)$,
(b) $|\Phi(w, y_i) - \Phi(t, y_i)| < \varepsilon$ for $w \in \Omega_1 \cup \Omega_2$ and $1 \leq i \leq 4$.

Remark 3.2. The condition $(**)$ is satisfied in various important cases, the examples of which are listed below.

1. (the Orlicz case) If $\Phi(t, x) = \Phi(x)$, then for all y_1, y_2, y_3, y_4, t and ε, we can choose the same pair of disjoint sets $\Omega_1, \Omega_2 \in \Sigma$ with equal, positive and finite measure.
2. (the sequence case with some additional assumptions) Let $\Omega = \mathbb{N}$ with the counting measure and assume that, given $t \in \mathbb{N}$ and $M > 0$, we may find $t_1 \neq t$ such that

$$|\Phi(t_1, y) - \Phi(t, y)| < \varepsilon \text{ for all } y \in [0, M].$$

Then we put $\Omega_1 = \{t\}$, $\Omega_2 = \{t_1\}$, where t_1 is chosen for $M = \max_i y_i$.

3. (the continuous atomless case) Suppose that $\Omega \subset \mathbb{R}^n$ is an open set with positive Lebesgue measure and that, for each $x \in \mathbb{R}_+$, the function $\Phi(\cdot, x)$ is continuous. Then there exist positive numbers ε_i with

$$|\Phi(w, y_i) - \Phi(t, y_i)| < \varepsilon \text{ for } w \in B(t, \varepsilon_i) \text{ and } 1 \leq i \leq 4,$$

where B denotes the closed ball in $\mathbb{R}^n$. In this case, we choose any disjoint sets $\Omega_1, \Omega_2 \subset B(t, \min_i \varepsilon_i)$ which satisfy (a).

Moreover, without loss of generality we can assume that $\mu(\Omega_1) = \mu(\Omega_2) = 1$ (replacing the measure μ with $\mu/\mu(\Omega_1)$, if necessary).

Remark 3.3. It is also worth mentioning that our new assumption $(**)$ implies properties (1) and (3) required in $(*)$ (see [9, p. 172]), with $A^{y_i}_{t,\varepsilon} = \Omega_1$ (or Ω_2). Additionally, in all of the cases presented in the previous remark the remaining condition (2) is also satisfied.

Our new result, which improves Theorem 3.1, reads as follows.

Theorem 3.4. *Let $\Phi\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a φ-function with a parameter such that $(**)$ is satisfied. Then the following conditions are equivalent:*

(i) *for each $t \in \Omega$, φ_t is a convex function,*
(ii) *for each $t \in \Omega$, φ_t satisfies the Lim inequality: if $a, b \geq 0$ and $c \geq a$, then*

$$\varphi_t(a) + \varphi_t(b + c) \geq \varphi_t(a + b) + \varphi_t(c),$$

(iii) *for any latticially closed set $D \subset L^\Phi$, the projection operator P_D is isotonic,*
(iv) *for any latticially closed set $D \subset L^\Phi$, the antiprojection operator P^a_D is antiisotonic.*

Proof. Let us first point out that the equivalence (i)⇔(ii) has already been observed and used in [9, Theorem 3.5], while (i)⇒(iii) and (i)⇒(iv) hold by Theorem 3.1. Consequently, we only need to prove that both (iii) and (iv) imply (ii).

Suppose, on the contrary, that φ_t does not satisfy the Lim inequality for some $t \in \Omega$. This gives

$$\varphi_t(a) + \varphi_t(b + c) < \varphi_t(a + b) + \varphi_t(c) - 4\varepsilon \tag{3.1}$$

with appropriately chosen $a, b, \varepsilon > 0$ and $c > a$. By $(**)$ and the following remark, for such t, ε and $y_1 = a$, $y_2 = b + c$, $y_3 = a + b$, $y_4 = c$ we can find the corresponding sets Ω_1, Ω_2 of measure 1.

Set $A = \Omega_1 \cup \Omega_2$, $d = c/a > 1$ and define four simple functions on Ω:

$$x = (a + b + c) \cdot I_{\Omega_2}, \quad y = \frac{c(d-1)}{d} \cdot I_{\Omega_1} + (b + 2c) \cdot I_{\Omega_2},$$
$$z = c \cdot I_A, \quad w = (b + c) \cdot I_A,$$

where I_D denotes the characteristic function of D. Then it is easy to observe that all these mappings are elements of the Orlicz-Musielak space L^Φ and that $x \leq y$, $z(t) < w(t)$ for $t \in A$.

Moreover, by (**) and (3.1),

$$\begin{aligned}\rho_\Phi(x-z) &= \int_{\Omega_1} \Phi(w,c)\,dw + \int_{\Omega_2} \Phi(w,a+b)\,dw \\ &\geq \int_{\Omega_1} [\Phi(t,c)-\varepsilon]\,dw + \int_{\Omega_2} [\Phi(t,a+b)-\varepsilon]\,dw = \varphi_t(c)+\varphi_t(a+b)-2\varepsilon \\ &> \varphi_t(b+c)+\varphi_t(a)+2\varepsilon = \int_{\Omega_1} [\Phi(t,b+c)+\varepsilon]\,dw + \int_{\Omega_2} [\Phi(t,a)+\varepsilon]\,dw \\ &\geq \int_{\Omega_1} \Phi(w,b+c)\,dw + \int_{\Omega_2} \Phi(w,a)\,dw = \rho_\Phi(x-w)\end{aligned}$$

and

$$\begin{aligned}\rho_\Phi(y-z) &= \int_{\Omega_1} \Phi(w,a)\,dw + \int_{\Omega_2} \Phi(w,b+c)\,dw \\ &\leq \int_{\Omega_1} [\Phi(t,a)+\varepsilon]\,dw + \int_{\Omega_2} [\Phi(t,b+c)+\varepsilon]\,dw = \varphi_t(a)+\varphi_t(b+c)+2\varepsilon \\ &< \varphi_t(a+b)+\varphi_t(c)-2\varepsilon = \int_{\Omega_1} [\Phi(t,a+b)-\varepsilon]\,dw + \int_{\Omega_2} [\Phi(t,c)-\varepsilon]\,dw \\ &\leq \int_{\Omega_1} \Phi(w,a+b)\,dw + \int_{\Omega_2} \Phi(w,c)\,dw = \rho_\Phi(y-w).\end{aligned}$$

We have shown that, for the latticially closed set $D=\{z,w\}$,

$$P_D(x)=\{w\}, \quad P_D(y)=\{z\}, \quad \text{and so} \quad P_D(x)(t)>P_D(y)(t) \text{ for } t\in A,$$

and

$$P_D^a(x)=\{z\}, \quad P_D^a(y)=\{w\}, \quad \text{which gives} \quad P_D^a(x)<P_D^a(y) \text{ on } A.$$

Therefore, the projection P_D is not isotonic, while the antiprojection P_D^a is not antiisotonic. In consequence, neither (iii) nor (iv) can be valid. This completes the proof of our theorem. □

Remark 3.5. We have actually proved a little more, namely: If there exist $t\in\Omega$ with φ_t being non-convex, and $\Omega_1,\Omega_2\in\Sigma$ satisfying (a) and such that, for $w\in\Omega_1\cup\Omega_2$, φ_w is "similar to φ_t" in the sense of (b), then (iii) and (iv) cannot hold.

Similar considerations lead us to the following additional result for the sequence case.

Theorem 3.6. *Let $\Phi\colon \mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+$ be a φ-function with a parameter and assume that there exist natural numbers $t_1\neq t_2$ such that the system of inequalities*

$$\begin{cases}\varphi_{t_2}(a+b)-\varphi_{t_2}(a)>\varphi_{t_1}(b+c)-\varphi_{t_1}(c)\\ \varphi_{t_1}(a+b)-\varphi_{t_1}(a)>\varphi_{t_2}(b+c)-\varphi_{t_2}(c)\end{cases}\tag{3.2}$$

has a solution (a,b,c) with $a,b,c>0$, $c>a$. Then the conditions (iii) *and* (iv) *from Theorem 3.4 are not satisfied.*

Proof. Set $\Omega_1=\{t_1\}$, $\Omega_2=\{t_2\}$, $A=\Omega_1\cup\Omega_2$ and define simple functions $x,y,z,w\in L^\Phi$ as in the proof of Theorem 3.4. Then, by (3.2),

$$\rho_\Phi(x-z)=\varphi_{t_1}(c)+\varphi_{t_2}(a+b)>\varphi_{t_1}(b+c)+\varphi_{t_2}(a)=\rho_\Phi(x-w)$$

and

$$\rho_\Phi(y-z)=\varphi_{t_1}(a)+\varphi_{t_2}(b+c)<\varphi_{t_1}(a+b)+\varphi_{t_2}(c)=\rho_\Phi(y-w).$$

Putting $D=\{z,w\}$ and reasoning as above, we show that the operators P_D and P_D^a do not possess the required properties, which proves the theorem. □

Remark 3.7. If the system (3.2) has the desired solution, then the function $\varphi_{t_1}+\varphi_{t_2}$ cannot be convex, and so at least one of its components is also a non-convex function.

Our final example shows that the assumptions of Theorem 3.6 can be satisfied even in the case when $(**)$ is not valid (and, in consequence, Theorem 3.4 does not work).

Example 3.8.

1. Suppose that there exist two distinct numbers $t_1,t_2\in\mathbb{N}$ such that $\varphi_{t_1}=\varphi_{t_2}$ is a non-convex function. Then it does not satisfy the Lim inequality, and consequently (3.2) has a solution required in the previous theorem. Let us observe that in this case Remark 3.5 could also be applied.
2. Let $\varphi_{t_1}(x)=\sqrt{x}$ and $\varphi_{t_2}(x)=x$. Then (3.2) leads to the system of inequalities

$$\begin{cases}b>\sqrt{b+c}-\sqrt{c},\\ \sqrt{a+b}-\sqrt{a}>b,\end{cases}$$

   which is solved by any positive numbers a,b,c such that

$$\sqrt{a+b}+\sqrt{a}<1<\sqrt{b+c}+\sqrt{c}.$$

3. Define $\varphi_{t_1}(x)=\ln(x+1)$ and $\varphi_{t_2}(x)=x/2$. Then (3.2) can be rewritten as

$$\ln\frac{b+c+1}{c+1}<b/2<\ln\frac{a+b+1}{a+1},$$

   and this problem also has infinitely many solutions, e.g. $a<d$, $b=2$, $c>d$ for $d=\dfrac{3-e}{e-1}$.

REFERENCES

[1] G. Isac, *On the order monotonicity of the metric projection operator*, [in:] Approximation Theory, Wavelets and Applications, NATO ASI Series, Kluwer Acad. Publ., Dordrecht (1995), 365–379.

[2] G. Isac, G. Lewicki, *On the property of four elements in modular spaces*, Acta Math. Hungar. **83** (1999) 4, 293–301.

[3] G. Isac, A.B. Németh, *Every generating isotone projection cone is latticial and correct*, J. Math. Anal. Appl. **147** (1990) 1, 53–62.

[4] G. Isac, A.B. Németh, *Isotone projection cones in Euclidean spaces*, Ann. Sci. Math. Québec **16** (1992) 1, 35–52.

[5] G. Isac, L.E. Persson, *On an inequality related to the isotonicity of the projection operator*, J. Approx. Theory **86** (1996) 2, 129–143.

[6] G. Isac, L.E. Persson, *Inequalities related to isotonicity of projection and antiprojection operators*, Math. Inequal. Appl. **1** (1998) 1, 85–97.

[7] B. Micherda, *The properties of four elements in Orlicz-Musielak spaces*, Math. Inequal. Appl. **4** (2001) 4, 599–608.

[8] B. Micherda, *On the latticity of projection and antiprojection sets in Orlicz-Musielak spaces*, Acta Math. Hungar. **119** (2008) 1–2, 165–180.

[9] B. Micherda, *A characterization of convex φ-functions*, Opuscula Math. **32** (2012) 1, 171–178.

[10] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer-Verlag, Berlin, 1983.

[11] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker, Inc., New York, 1991.

Bartosz Micherda
bmicherda@ath.bielsko.pl

University of Bielsko-Biała
Department of Mathematics
ul. Willowa 2, 43–309 Bielsko-Biała, Poland

Received: April 13, 2014.
Accepted: June 18, 2014.