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A study on the performance measures of a system operated in random environment

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environmental condition, shock process, compound failure rate function, stochastic ordering

Abstract

Most devices (systems) are operated under different environmental conditions. The failure process of a system not only depends on the intrinsic characteristic of the system itself but also on the external environmental conditions under which the system is being operated. In this article we study a stochastic failure model in random environment and investigate the effect of the environmental factors on the failure process of the system. Some stochastic orderings will also be briefly discussed.

1. Introduction

Many of the currently used failure models are developed under the premise that the operating environment is static. In these cases, the basic assumption is that the prevailing environmental conditions either do not change in time or, in case they do, have no effect on the deterioration and failure process of the device. Therefore, in these cases deterministic models not depending on the external environmental conditions are proposed and studied.

However, as a matter of fact, devices often work in varying environments and so their performances will be significantly affected by these varying environmental conditions. For example, jet engines constantly subject to varying atmospheric conditions like pressure, temperature, humidity, and shocks caused by mechanical vibrations during taking-off, cruising, and landing. As another example, some electric devices frequently subject to random shocks caused by fluctuations of unstable electric power. With the intrinsic age of the equipment, in these cases, the changes in external conditions or shocks cause the equipment to deteriorate or age according to certain rules. In this regard, there have been some failure models for a system in random environment. See, for example, Lemoine and Wenocur [1], [2] and Özekici [3], [4].

In this paper, we study a new stochastic failure model, which incorporates the varying external environmental conditions. The external environment is modelled by a Poisson shock process. When the system is used under varying environmental conditions, the intrinsic system performance (e.g., reliability, mean time to failure, etc.) itself has less meaning, and thus should be studied together with the environmental conditions under which the system is being operated. In the next section we shall derive the compound failure rate function of system in random environment and investigate the effect of the environmental factors on the failure process of the system. Some notable remarks are made there. The stochastic orderings between the lifetime distributions of systems operated in different environmental conditions are also briefly addressed in Section 2.

2. Main Results

Let X be the lifetime of an equipment operated in a normal working environment (or laboratory environment which is not exposed to the external environmental conditions) and denote its survival function and failure rate function, which is called the baseline failure rate function, as $\overline{F}_0(t)$ and $r_0(t)$. Let $\{N(t), t \geq 0\}$ be a stochastic counting process describing the occurrence of certain environmental shocks in the time interval $[0,\infty)$. Denote the times of occurrences of these shocks as $0 \equiv T_0 < T_1 < T_2 \cdots$. Let Y be the lifetime of the same equipment operated undergoing this environmental shock process. In this paper it is assumed that

$$P(Y > t \mid N(s), 0 \le s \le t) = \exp\{-\int_0^t [r_0(x) + \sum_{j=1}^{N(t)} \mu I_{[T_j,\infty)}(x)]dx\}, \quad (1)$$

for all $t \ge 0$, where constant $\mu > 0$ and the indicator function $I_{[T_j,\infty)}(x),\ j = 1,2,\cdots,$ is defined by

$$I_{[T_j,\infty)}(x) = \begin{cases} 1, if x \in [T_j,\infty), \\ 0, otherwise. \end{cases}$$

Physically, (1) means that, given the shock process, the failure rate of Y is increased by a fixed amount $\mu > 0$ at the occurrence of each shock. The following result gives the failure rate function of Y, which is called the compound failure rate function of the system.

Theorem 1. Suppose that $\{N(t), t \ge 0\}$ is a Poisson process with intensity λ. Assuming (1), then the failure rate function of Y, denoted as $r(t)$, is given by

$$r(t) = r_0(t) + \lambda(1 - \exp\{-\mu t\}).$$

Proof. Note that $N(t)$ is the number of shocks occurred in the interval $[0,t]$, so

$$\int_0^t \sum_{j=1}^{N(t)} \mu I_{[T_j,\infty)}(x)dx = \mu \sum_{j=1}^{N(t)} \int_0^t I_{[T_j,\infty)}(x)dx$$

$$= \mu \sum_{j=1}^{N(t)} (t - T_j) = \mu t N(t) - \mu \sum_{j=1}^{N(t)} T_j.$$

Thus

$$P(Y > t \mid N(s), 0 \le s \le t) = \bar{F}_0(t) \cdot \exp\{-\mu t N(t) + \mu \sum_{j=1}^{N(t)} T_j\}. \quad (2)$$

From (2) we see that

$$P(Y > t) = \bar{F}_0(t) \cdot E(\exp\{-\mu t N(t) + \mu \sum_{j=1}^{N(t)} T_j\}). \quad (3)$$

Considering conditional expectation, we have

$$E(\exp\{-\mu t N(t) + \mu \sum_{j=1}^{N(t)} T_j\} \mid N(t) = n) = \exp\{-n\mu t\} \times E(\exp\{\mu \sum_{j=1}^{n} T_j\} \mid N(t) = n). \quad (4)$$

It is known that the joint distribution of $(T_1, T_2, \cdots, T_n)$ given $N(t) = n$ is the same as the joint distribution of $(U_{(1)}, U_{(2)}, \cdots, U_{(n)})$, where $U_{(1)} \le U_{(2)} \le \cdots \le U_{(n)}$ are the order statistics of i.i.d. random variables $U_1, U_2, \cdots, U_n$, which are uniformly distributed on the interval $[0,t]$. Hence in the equation (4),

$$E(\exp\{\mu \sum_{j=1}^{n} T_j\} \mid N(t) = n) = E(\exp\{\mu \sum_{j=1}^{n} U_{(j)}\}) = E(\exp\{\mu \sum_{j=1}^{n} U_j\}) = [E(\exp\{\mu U_1\})]^n, \quad (5)$$

where the second equality follows from $\sum_{j=1}^{n} U_{(j)} = \sum_{j=1}^{n} U_j$, and the third equality follows from the independence of $U_1, U_2, \cdots, U_n$ and that U_i 's are uniformly distributed on $[0,t]$.

According to (5) we obtain

$$\begin{aligned} & E(\exp\{-\mu t N(t) + \mu \sum_{j=1}^{N(t)} T_j\}) \\ &= E[E(\exp\{-\mu t N(t) + \mu \sum_{j=1}^{N(t)} T_j\} \mid N(t))] \\ &= \sum_{n=0}^{\infty} \exp\{-\mu t n\} \times E(\exp\{\mu \sum_{j=1}^{n} T_j\} \mid N(t) = n) \\ &\times P(N(t) = n) \\ &= \sum_{n=0}^{\infty} \exp\{-\mu t n\} \cdot [E(\exp\{\mu U_1\})]^n \\ &\times (\lambda t)^n / n! \cdot \exp\{-\lambda t\} \\ &= \exp\{-\lambda t\} \times \sum_{n=0}^{\infty} [\exp\{-\mu t\} \cdot E(\exp\{\mu U_1\}) \cdot \lambda t]^n / n! \\ &= \exp\{-\lambda t\} \times \exp\{\exp\{-\mu t\} \cdot E(\exp\{\mu U_1\}) \cdot \lambda t\}. \end{aligned} \quad (6)$$

It is easy to see that

$$E(\exp\{\mu U_1\}) = \int_0^t \exp\{\mu x\} \cdot 1/t \, dx$$
$$= 1/\mu t (\exp\{\mu t\} - 1). \tag{7}$$

Combining (3), (6), and (7), we obtain

$$P(Y > t) = \exp\{-\lambda t\}$$
$$\times \exp\{\exp\{-\mu t\} \cdot 1/\mu t (\exp\{\mu t\} - 1) \cdot \lambda t\}$$
$$\times \bar{F}_0(t) \tag{8}$$
$$= \exp\{-\lambda t\} \times \exp\{\lambda/\mu (1 - \exp\{-\mu t\})\}$$
$$\times \bar{F}_0(t).$$

From (8), we have

$$\ln P(Y > t) = -\lambda t$$
$$+ \lambda/\mu (1 - \exp\{-\mu t\}) - \int_0^t r_0(x) dx.$$

The compound failure rate function $r(t)$ is thus given by

$$r(t) = -d/dt \ln P(Y > t)$$
$$= -d/dt \{-\int_0^t r_0(x) dx - \lambda t + \lambda/\mu (1 - \exp\{-\mu t\})\}$$
$$= r_0(t) + \lambda + \lambda/\mu (-\mu \exp\{-\mu t\})$$
$$= r_0(t) + \lambda(1 - \exp\{-\mu t\}).$$

↑

We will use $S(r_0(t), \lambda, \mu)$ to denote the system with the baseline failure rate function $r_0(t)$ operated under the environmental condition (λ, μ). In order to emphasize the dependence on (λ, μ) we further denote the lifetime, compound failure rate function, and reliability function of the system $S(r_0(t), \lambda, \mu)$ as $Y_{(\lambda,\mu)}$, $r(t; \lambda, \mu)$, and $R(t; \lambda, \mu)$ when it is needed.

Remark 1. The compound failure rate function $r(t; \lambda, \mu)$ has the following properties:
(a) $r(t; \lambda, \mu)$ strictly increases in $\lambda > 0$,
(b) $r(t; \lambda, \mu)$ strictly increases in $\mu > 0$.
The physical meanings of these are clear.

Remark 2. If $\lim_{t \to \infty} r_0(t) \equiv r_0(\infty) \le \infty$ exists, then

$$r(\infty; \lambda, \mu) \equiv \lim_{t \to \infty} r(t; \lambda, \mu) \equiv r_0(\infty) + \lambda.$$

It is surprising that in the limiting case 'μ' has no contribution to the compound failure rate $r(t; \lambda, \mu)$ of the system $S(r_0(t), \lambda, \mu)$. In the contrast, 'λ' contributes a lot. Furthermore, it is also notable that if $r_0(\infty) < \infty$ then

$$r(\infty; \lambda, \mu) \equiv \lim_{t \to \infty} r(t; \lambda, \mu) < \infty,$$

even though $N(t) \to \infty$ as $t \to \infty$ with probability 1.

Remark 3. Lifetime distributions are often classified into different classes such as IFR (Increasing Failure Rate), CFR (Constant Failure Rate), and DFR (Decreasing Failure Rate) distributions. The closure properties of these classes are very useful and important in reliability theory and its applications. From the results obtained in the above we notice that the class to which the lifetime distribution of a device belongs is not necessarily preserved when it is operated under varying environmental condition. For example, suppose that $r_0(t) \equiv r_0, \forall t \ge 0$ (i.e., the lifetime distribution is CFR). In this case, the class of the lifetime distribution of Y is IFR according to Theorem 1. Practically, this implies that even though the lifetime distribution of X is exponential (i.e., the system never deteriorates), preventive maintenance actions should be considered when it is operated in random environment.

In the next result, we compare the performances of a system undergoing different shock processes. For this purpose, we need the concept of the hazard rate ordering (see, e.g., Shaked and Shanthikumar [5]).

Definition 1. Let Z_1 and Z_2 be two nonnegative continuous random variables with hazard rate functions $q_1(\cdot)$ and $q_2(\cdot)$ such that

$$q_1(t) \ge q_2(t), \forall t \ge 0.$$

Then Z_1 is said to be smaller than Z_2 in the hazard rate order denoted as $Z_1 \le_{hr} Z_2$.

Theorem 2.

(I) Consider systems $S(r_0(t), \lambda_1, \mu)$ and $S(r_0(t), \lambda_2, \mu)$ with $\lambda_1 > \lambda_2$. Then the following holds:

(i) $Y_{(\lambda_1,\mu)} \leq_{hr} Y_{(\lambda_2,\mu)}$.

(ii) $R(t;\lambda_1,\mu) \leq R(t;\lambda_2,\mu), \forall t \geq 0.$

(iii) $E[Y_{(\lambda_1,\mu)}] \leq E[Y_{(\lambda_2,\mu)}]$.

(II) Consider systems $S(r_0(t), \lambda, \mu_1)$ and $S(r_0(t), \lambda, \mu_2)$ with $\mu_1 > \mu_2$. Then the following holds:

(i) $Y_{(\lambda,\mu_1)} \leq_{hr} Y_{(\lambda,\mu_2)}$.

(ii) $R(t;\lambda,\mu_1) \leq R(t;\lambda,\mu_2), \forall t \geq 0.$

(iii) $E[Y_{(\lambda,\mu_1)}] \leq E[Y_{(\lambda,\mu_2)}]$.

Proof. It is clear that $Y_{(\lambda_1,\mu)} \leq_{hr} Y_{(\lambda_2,\mu)}$ and $Y_{(\lambda,\mu_1)} \leq_{hr} Y_{(\lambda,\mu_2)}$ by *Theorem 1*. The other results are thus readily obtained.

3. Conclusion

When a system is used under varying environmental conditions, its performances should be studied together with the environmental factors. In the previous section, we studied the performances of the system, which is operated in random environment governed by a Poisson shock process. From the compound failure rate function, it can be seen that the influence of the occurrence rate of shock process is more significant than that of the amount of failure rate increment caused by each shock.

In the future study, more general environmental factors, such as pressure, temperature, humidity, and so on, could be modelled via various stochastic processes to describe the operational characteristics of the real world. Also note that in much of the previous research work on statistical inferences, only the parameters in the baseline failure rate function received attention. Since the environmental factors are essential elements of the system performances, more attention should be paid to parameters describing environmental conditions in the future research.

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