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COMPARISON OF APPROXIMATION METHODS OF POSITIVE STABLE CONTINUOUS-TIME LINEAR SYSTEMS BY POSITIVE STABLE DISCRETE-TIME SYSTEMS

The positive asymptotically stable continuous-time linear systems are approximated by corresponding asymptotically stable discrete-time linear systems. Two methods of the approximation are presented and the comparison of the methods is addressed. The considerations are illustrated by three numerical examples and an example of positive electrical circuit.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [5, 8]. The positivity of electrical circuits composed of resistors, coils, condensators and voltage (current) sources has been analyzed in [10].

Stability of positive linear systems has been investigated in [5, 8] and of fractional linear systems in [2-4, 12]. The problem of preservation of positivity by approximation the continuous-time linear systems by corresponding discrete-time linear systems has been addressed in [9]. The approximation of positive stable continuous-time linear systems by positive stable discrete-time linear systems has been considered in [7].

In this paper two methods of approximation of positive stable continuous-time linear systems by positive stable discrete-time linear systems will be presented and a comparison of the methods will be given.

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The paper is organized as follows. In section 2 basic definitions and theorems concerning positive continuous-time and discrete-time linear systems are recalled. Two methods of approximation of positive stable continuous-time linear systems by positive stable discrete-time linear systems and the comparison of the methods are presented in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), M_{nS} - the set of $n\times n$ asymptotically stable Metzler matrices, $\Re_{+S}^{n\times n}$ - the set of $n\times n$ asymptotically stable positive matrices, I_n - the $n\times n$ identity matrix.

2. PRELIMINARIES AND THE PROBLEM FORMULATION

Consider the continuous-time linear system

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0 \tag{2.1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ are the state and input vectors and $A_c \in \Re^{n\times n}$, $B_c \in \Re^{n\times m}$.

Definition 2.1. [5, 8] The system (2.1) is called (internally) positive if $x(t)\in\Re_+^n$, $t\ge 0$ for any initial conditions $x(0) = x_0 \in \Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 2.1. [5, 8] The system (2.1) is positive if and only if

$$A_c \in M_n, \; B_c \in \Re_+^{n\times m}. \tag{2.2}$$

Definition 2.2. [5, 8] The positive system (2.1) is called asymptotically stable if for $u(t) = 0$, $t\ge 0$

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n. \tag{2.3}$$

Theorem 2.2. [5, 8] The positive system (2.1) is asymptotically stable if and only if all coefficients of the polynomial

$$\det[I_n s - A_c] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.4}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,n-1$.

Now let us consider the discrete-time linear system

$$x_{i+1} = A_d x_i + B_d u_i, \quad i\in Z_+, \tag{2.5}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors and $A_d\in\Re^{n\times n}$, $B_d\in\Re^{n\times m}$.

Definition 2.3. [5, 8] The system (2.5) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 2.3. [5, 8] The system (2.5) is positive if and only if

$$A_d \in \Re_+^{n \times n},\ B_d \in \Re_+^{n \times m}. \tag{2.6}$$

Definition 2.4. [5, 8] The positive system (2.5) is called asymptotically stable if for $u_i = 0$, $i \in Z_+$

$$\lim_{i \to \infty} x_i = 0 \ \text{ for all } \ x_0 \in \Re_+^n. \tag{2.7}$$

Theorem 2.4. [5, 8] The positive system (2.5) is asymptotically stable if and only if all coefficients of the polynomial

$$\det[I_n(z+1) - A_d] = z^n + \overline{a}_{n-1}z^{n-1} + ... + \overline{a}_1 z + \overline{a}_0 \tag{2.8}$$

are positive, i.e. $\overline{a}_i > 0$ for $i = 0,1,...,n-1$.

In this paper two methods of approximation of positive stable continuous-time linear systems by positive stable discrete-time linear systems will be presented and a comparison of the methods will be given.

3. METHODS OF APPROXIMATION AND THEIR COMPARISON

3.1. Method 1

In this method the derivative $\dot{x}(t)$ will be approximated by

$$\dot{x}(t) = \frac{x_{i+1} - x_i}{h}, \quad i = 0,1,... \tag{3.1}$$

where $x_i = x(ih)$ and $h > 0$ is the sampling time (step).

From (2.1) and (3.1) we have

$$x_{i+1} = \overline{A}_d x_i + \overline{B}_d u_i, \quad i = 0,1,... \tag{3.2a}$$

where

$$\overline{A}_d = I_n + hA_c, \quad \overline{B}_d = hB_c \tag{3.2b}$$

and $u_i = u(ih)$, $i = 0,1,…$.

From (3.2b) it follows that if $A_c \in M_n$ then $\overline{A}_d \in \Re_+^{n \times n}$ if and only if

$$h \le \frac{1}{\max\limits_{1 \le i \le n} |a_{i,i}|} \tag{3.3}$$

where $a_{i,i}$ ($i = 1,2,…,n$) is the i-th diagonal entry of A_c.

Therefore, we have the following theorem.

Theorem 3.1. The discrete-time system (3.2a) is (internally) positive if and only if the continuous-time system (2.1) is (internally) positive and the sampling time h satisfy the condition (3.3).

Let the positive continuous-time be asymptotically stable (shortly stable). In this case by Theorem 2.2 the coefficients of the polynomial (2.4) are positive, i.e. $a_k > 0$, $k = 0,1,\dots,n-1$.

Theorem 3.2. The positive discrete-time system (3.2) is stable for any $h > 0$ if and only if the positive continuous-time system (2.1) is stable.

Proof. By Theorem 2.4 the positive discrete-time system (3.2) is stable if and only if all coefficients of the polynomial

$$\det[I_n(z+1)-\overline{A}_d] = \det[I_n(z+1)-(I_n+hA_c)] = \det[I_n z - hA_c] \\ = z^n + \overline{a}_{n-1}z^{n-1} + \dots + \overline{a}_1 z + \overline{a}_0 \tag{3.4}$$

are positive $\overline{a}_k > 0$, $k = 0,1,\dots,n-1$.

It is easy to show that the coefficients a_k and $\overline{a}_k$, $k = 0,1,\dots,n-1$ of the polynomials (2.4) and (3.4) are related by

$$\overline{a}_{n-1} = ha_{n-1},, \overline{a}_{n-2} = h^2 a_{n-2},\dots,\overline{a}_0 = h^n a_0 . \tag{3.5}$$

From (3.5) it follows that $\overline{a}_k > 0$ if and only if $a_k > 0$ for $k = 0,1,\dots,n-1$ and for any $h > 0$. Therefore, by Theorem 2.2 and (2.4) the positive discrete-time system (3.2) is stable if and only if the positive continuous-time system (2.1) is stable. □

Now let assume that the continuous-time system (2.1) is not positive but stable. Let $s_i = -\alpha_i + j\beta_i$ ($i = 1,2,\dots,n$) be the i-th eigenvalue of the matrix A_c. It is well-known [6, 12] that if si is the eigenvalue of A_c then $z_i = 1 + hs_i$ is the eigenvalue of the matrix $\overline{A}_d$ defined by (3.2b). The discrete-time system (3.2) is stable if and only if

$$|z_i| = |1+hs_i| = |1-h\alpha_i + jh\beta_i| < 1 \text{ for } i = 1,2,\dots,n. \tag{3.6}$$

From (3.6) we have

$$(1-h\alpha_i)^2 + (h\beta_i)^2 < 1 . \tag{3.7}$$

Solving (3.7) with respect to h we obtain

$$h < \min_{1\le i\le n} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2} . \tag{3.8}$$

Therefore, the following theorem has been proved.

Theorem 3.3. The discrete-time system (3.2) is stable if and only if the continuous-time system (2.1) is stable and the condition (3.8) is met.

Example 3.1. Consider the positive continuous-time system (2.1) with the matrices

$$A_c = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.9}$$

For $h = 0.4$ the corresponding discrete-time system (3.2) is also positive since

$$\overline{A}_d = I_2 + hA_c = \begin{bmatrix} 0.6 & 0.4 \\ 0 & 0.2 \end{bmatrix} \in \Re_+^{2\times 2}, \quad \overline{B}_d = hB_c = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \in \Re_+^2, \tag{3.10}$$

but for $h = 1$ it is not positive since

$$\overline{A}_d = I_2 + hA_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \in M_2, \quad \overline{B}_d = hB_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \Re_+^2. \tag{3.11}$$

Note that for $h = 0.4$ the condition (3.3) is satisfied $h \le \dfrac{1}{\max\limits_{1\le i\le 2} |a_{i,i}|} = \dfrac{1}{2}$ but it is not satisfied for $h = 1$.

The positive continuous-time system (2.1) with (3.9) is stable since by Theorem 2.2 the polynomial

$$\det[I_2 s - A_c] = \begin{vmatrix} s+1 & -1 \\ 0 & s+2 \end{vmatrix} = s^2 + 3s + 2 \tag{3.12}$$

has all positive coefficients.

The corresponding positive discrete-time system for $h = 0.4$ is also stable since by Theorem 2.4 all coefficients of the polynomial

$$\det[I_2 z - hA_c] = \begin{vmatrix} z+0.4 & -0.4 \\ 0 & z+0.8 \end{vmatrix} = z^2 + 1.2z + 0.32 \tag{3.13}$$

are positive.

Figures 3.1 presents the step response of the continuous-time and discrete-time systems and discrete-time (green) systems with matrices (3.10) for $h = 0.4$ and discrete-time (green) systems with matrices (3.11) for $h = 1$.

Example 3.2. Consider the continuous-time system (2.1) with the matrices

$$A_c = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.14}$$

The system is stable but not positive.

By Theorem 3.3 the corresponding discrete-time system is also stable for the sampling time h satisfying the condition (3.8). In this case $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = \beta_2 = 0$ and from (3.8) we have

$$h < \min_{1\le i\le 2} \frac{2\alpha_i}{\alpha_i^2 + \beta_i^2} = \frac{2}{3}. \tag{3.15}$$

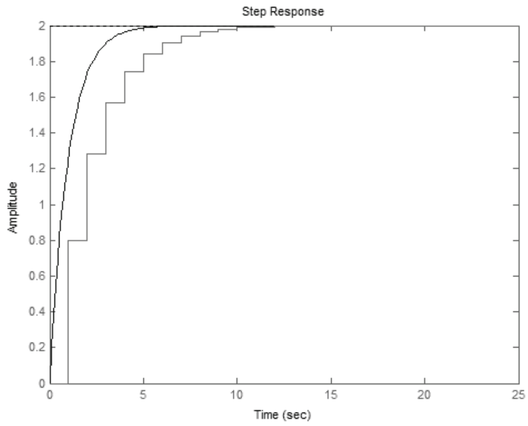

Fig. 3.1a. Step response of continuous-time (blue) with matrices (3.9)

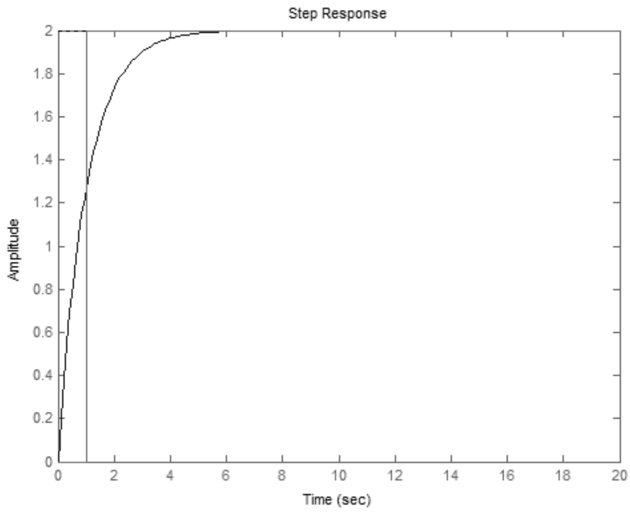

Fig. 3.1b. Step response of continuous-time (blue) with matrices (3.9)

For h = 0.5 we obtain

$$\overline{A}_d = I_2 + hA_c = \begin{bmatrix} 0 & -1 \\ 0 & -0.5 \end{bmatrix} \in \Re^{2\times 2}. \tag{3.16}$$

Therefore, the discrete-time system with (3.16) is stable.

Example 3.3. Consider the electrical circuit shown in Figure 3.2 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.

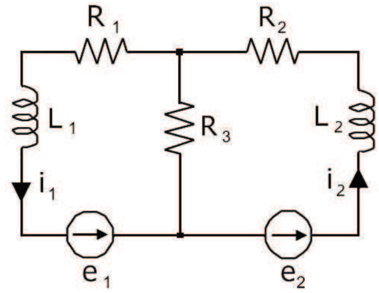

Fig. 3.2. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{di_1}{dt}, \\ e_2 &= R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{di_2}{dt} \end{aligned} \tag{3.17}$$

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.18a}$$

where

$$A_c = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B_c = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{3.18b}$$

The electrical circuit is positive since the matrix A_c is Metzler matrix and the matrix B_c has nonnegative entries. It is also stable since the polynomial

$$\begin{aligned} \det[I_2 s - A_c] &= \begin{vmatrix} s + \frac{R_1 + R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & s + \frac{R_2 + R_3}{L_2} \end{vmatrix} \\ &= s^2 + \left(\frac{R_1 + R_3}{L_1} + \frac{R_2 + R_3}{L_2} \right) s + \frac{R_1(R_2 + R_3) + R_2 R_3}{L_1 L_2} \end{aligned} \tag{3.19}$$

has all positive coefficients.

The corresponding discrete-time electrical circuit (3.2) for the sampling time $h > 0$ has the matrices

$$\overline{A}_d = I_2 + hA_c = \begin{bmatrix} 1-\frac{h(R_1+R_3)}{L_1} & \frac{hR_3}{L_1} \\ \frac{hR_3}{L_2} & 1-\frac{h(R_2+R_3)}{L_2} \end{bmatrix}, \ \overline{B}_d = hB_c = \begin{bmatrix} \frac{h}{L_1} & 0 \\ 0 & \frac{h}{L_2} \end{bmatrix} \quad (3.20)$$

and it is stable for any $h > 0$ since the characteristic polynomial

$$\det[I_2 z - hA_c] = \begin{vmatrix} z+\frac{h(R_1+R_3)}{L_1} & -\frac{hR_3}{L_1} \\ -\frac{hR_3}{L_2} & z+\frac{h(R_2+R_3)}{L_2} \end{vmatrix}$$
$$= z^2 + h\left(\frac{R_1+R_3}{L_1} + \frac{R_2+R_3}{L_2}\right)z + \frac{h^2(R_1(R_2+R_3)+R_2R_3)}{L_1L_2} \quad (3.21)$$

has all positive coefficients.

Extending the result of Example 3.3 and using Theorem 3.2 we obtain the following important conclusion.

Conclusion 3.1. The approximation of positive stable continuous-time electrical circuits by the use of Method 1 yields for any $h > 0$ corresponding positive stable discrete-time electrical circuits.

3.2. Method 2

It is well-known [7] that if the sampling is applied to the continuous-time system (2.1) then the corresponding discrete-time system (2.5) has the matrices

$$A_d = e^{A_c h}, \ B_d = \int_0^h e^{A_c t} B_c dt \quad (3.22)$$

where $h > 0$ is the sampling time.

In this paper the following approximation of the matrix A_d defined by (2.23) will be applied

$$\overline{A}_d = [A_c + I_n\alpha][I_n\alpha - A_c]^{-1} \quad (3.23)$$

where the coefficients $\alpha = \alpha(h) = \frac{\overline{\alpha}}{h} > 0$ is chosen so that $[A_c + I_n\alpha] \in \Re_+^{n\times n}$. It is well-known [1] that if $A_c \in M_{nS}$ then $\det[I_n\alpha - A_c] \neq 0$ for any $\alpha > 0$.

If $\det A_c \neq 0$ then from (3.22) we have

$$B_d = A_c^{-1}[e^{A_c h} - I_n]B_c. \quad (3.24)$$

Theorem 3.4. If the positive continuous-time system (2.1) is asymptotically stable then the corresponding discrete-time positive system (2.5) is also asymptotically stable.

It is well-known [6, 7, 11] that if s_k, $k = 1,2,\ldots,n$ are eigenvalues of the matrix $A_c \in M_n$ then the eigenvalues z_k, $k = 1,2,\ldots,n$ of the matrix $\overline{A}_d \in \Re_+^{n\times n}$ defined by (3.23) are given by

$$z_k = \frac{s_k + \alpha}{\alpha - s_k} \text{ for } k = 1,2,\ldots,n. \tag{3.25}$$

If the positive continuous-time system (2.1) is asymptotically stable then the real parts $-\alpha_k$ of its eigenvalues $s_k = -\alpha_k \pm j\beta_k$, $k = 1,2,\ldots,n$ are negative. In this case using (3.25) we obtain

$$|z_k| = \left|\frac{\alpha - \alpha_k \pm j\beta_k}{\alpha + \alpha_k \mp j\beta_k}\right| = \frac{|\alpha - \alpha_k \pm j\beta_k|}{|\alpha + \alpha_k \mp j\beta_k|} < 1 \tag{3.26}$$

and the discrete-time system (2.5) is also asymptotically stable.

Theorem 3.5. If the continuous-time system (2.1) is positive and stable then the discrete-time system (2.5) with the matrix (3.23) is also positive for any sampling time $h > 0$.

Proof. If the continuous-time system (2.1) is positive and stable then $A_c \in M_{nS}$ and there exists such $\alpha > 0$ that $[A_c + I_n\alpha] \in \Re_+^{n\times n}$. If $A_c \in M_{nS}$ then $\det[I_n\alpha - A_c] \neq 0$ for any $\alpha > 0$ and $[I_n\alpha - A_c]^{-1} \in \Re_+^{n\times n}$. In this case $\overline{A}_d = [A_c + I_n\alpha][I_n\alpha - A_c]^{-1} \in \Re_+^{n\times n}$ and by Theorem 2.3 the discrete-time system (2.5) is positive. □

Example 3.3. Consider the positive stable continuous-time system (2.1) with the matrices

$$A_c = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix},\ B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.27}$$

Using (3.23) and (3.24) we obtain for $h = 1$ and $\alpha = 4$ the matrices $\overline{A}_d$ and B_d of the corresponding discrete-time system (2.5) of the forms

$$\overline{A}_d = [A_c + I_2\alpha][I_2\alpha - A_c]^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 0 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 0.3333 & 0.1905 \\ 0 & 0.1429 \end{bmatrix} \tag{3.28a}$$

and

$$B_d = A_c^{-1}[e^{A_c h} - I_2]B_c = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}^{-1}\begin{bmatrix} -0.8647 & 0.0855 \\ 0 & -0.9502 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1156 \\ 0.3167 \end{bmatrix}. \tag{3.28b}$$

The eigenvalues z_1, z_2 of the matrix A_d can be computed by the use of (3.25)

$$z_1 = \frac{s_1 + \alpha}{\alpha - s_1} = \frac{-2+4}{4+2} = \frac{1}{3} = 0.3333,\ z_2 = \frac{s_2 + \alpha}{\alpha - s_2} = \frac{-3+4}{4+3} = \frac{1}{7} = 0.1429. \tag{3.29}$$

From (3.28) and (3.29) it follows that the discrete-time system is positive and stable.
Figure 3.2 presents the step response of the continuous-time and discrete-time systems and discrete-time (green) systems with matrices (3.28) for $h = 1$ and $\alpha = 4$.

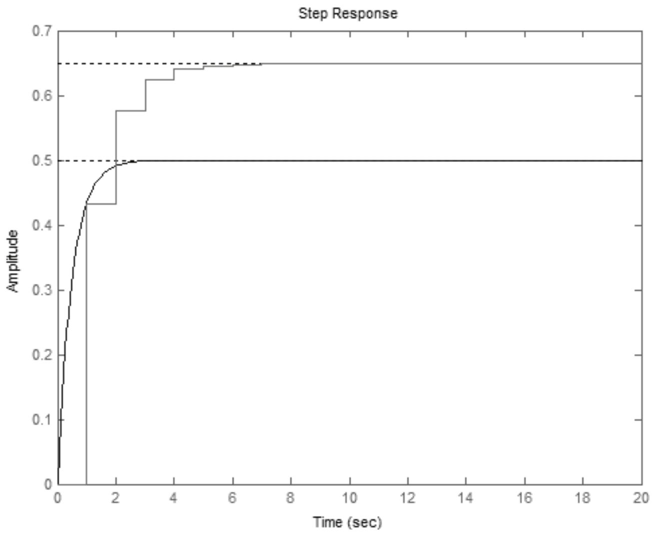

Fig. 3.2. Step response of continuous-time (blue) with matrices (3.27)

3.3. Comparison of the methods

From (3.24) we have

$$z_k = \frac{\alpha + s_k}{\alpha - s_k} = \frac{1 + \frac{s_k}{\alpha}}{1 - \frac{s_k}{\alpha}} = \left(1 + \frac{s_k}{\alpha}\right)\left[1 + \frac{s_k}{\alpha} + \left(\frac{s_k}{\alpha}\right)^2 + \ldots\right] = 1 + 2\left[\frac{s_k}{\alpha} + \left(\frac{s_k}{\alpha}\right)^2 + \ldots\right] \quad (3.30)$$

for $k = 1,2,\ldots,n$. In this case from (3.30) and (3.23) we obtain [6, 12]

$$\overline{A}_d = [A_c + I_n\alpha][I_n\alpha - A_c]^{-1} = I_n + 2\left[\frac{1}{\alpha}A_c + \frac{1}{\alpha^2}A_c^2 + \ldots\right]. \quad (3.31)$$

Comparison of (3.2b) and (3.31) yields $\overline{A}_d$ representing for $h = \frac{2}{\alpha}$ only the linear part of $\overline{A}_d$ defined by (3.23).

Using (3.24) and $\overline{B}_d$ defined by (3.2b) and taking into account that

$$e^{A_c h} = I_n + \frac{A_c h}{1!} + \frac{(A_c h)^2}{2!} + \ldots \tag{3.32}$$

we obtain

$$B_d = A_c^{-1}[e^{A_c h} - I_2]B_c = B_c h + \frac{A_c B_c h^2}{2!} + \ldots. \tag{3.33}$$

From (3.33) it follows that $\overline{B}_d = B_c h$ represents only the linear part of the matrix B_d.

Therefore, we have the following important conclusions.

Conclusion 3.2. The method 2 gives better approximation of the positive stable continuous-time linear systems than the method 1.

Conclusion 3.3. By Theorem 3.1 the method 1 provides a positive approximation (3.2a) of the positive continuous-time system if and only if the sampling time *h* satisfies the condition (3.3) and by Theorem 3.4 the method 2 for any sampling time *h*.

Conclusion 3.4. By Theorem 3.2 and 3.4 for both methods the positive discrete-time approximation (3.2) is stable for any $h > 0$ if and only if the positive continuous-time system (2.1) is stable.

4. CONCLUDING REMARKS

The problem of approximation of positive asymptotically stable continuous-time linear system by positive asymptotically stable discrete-time linear system has been addressed. Two method of the approximation have been presented. The comparison of the methods has shown that the method 2 provides the better approximation of the positive stable continuous-time linear systems by positive stable discrete-time linear system than the method 1 (Conclusion 3.2). For both methods the positive discrete-time approximation is stable for any sampling time h if and only if the positive continuous-time system is stable (Conclusion 3.4). The considerations have been illustrated by three numerical examples and one example of positive electrical circuit.

The considerations can be applied to the positive electrical circuits (Conclusion 3.1). An open problem is an extension of these considerations to the fractional positive linear systems [12].

5. ACKNOWLEDGMENT

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