Opuscula Math. 39, no. 5 (2019), 705–731
https://doi.org/10.7494/OpMath.2019.39.5.705 Opuscula Mathematica

ON THE LEBESGUE AND SOBOLEV SPACES ON A TIME-SCALE

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Communicated by Marek Galewski

Abstract. We consider the generalized Lebesgue and Sobolev spaces on a bounded time-scale. We study the standard properties of these spaces and compare them to the classical known results for the Lebesgue and Sobolev spaces on a bounded interval. These results provide the necessary framework for the study of boundary value problems on bounded time-scales.

Keywords: Lebesgue spaces, Sobolev spaces, modular spaces, time-scales, boundary value problems on time-scales.

Mathematics Subject Classification: 26E70, 46B10.

1. INTRODUCTION

Let $\mathbb{T}$ be a time-scale, i.e. a closed subset of $\mathbb{R}$. We also assume that $\mathbb{T}$ is bounded, since our motivation is the study of boundary value problems on bounded time-scales.

In this paper we consider the so called generalized Lebesgue and Sobolev spaces, which we define in full detail later on, namely

$$L^{p(t)}(\mathbb{T}) = \left\{ u : u \text{ is } \Delta\text{-measurable and } \lim_{\lambda \to 0^+} \int\limits_{\mathbb{T}} |\lambda u(t)|^{p(t)} \Delta t = 0 \right\}$$

and

$$W^{1,p(t)}(\mathbb{T}) = \{u \in L^{p(t)}(\mathbb{T}) : \Delta^w u \text{ exists and } \Delta^w u \in L^{p(t)}(\mathbb{T})\},$$

where $\Delta^w u$ denotes Δ-weak derivative of u and $p \in L^\infty_+(\mathbb{T})$, where

$$L^\infty_+(\mathbb{T}) = \left\{ u \in L^\infty(\mathbb{T}) : \operatorname*{ess\,inf}_{t \in \mathbb{T}} u(t) \geq 1 \right\}.$$

In doing so we will use the properties of the Δ-measure and the Lebesgue Δ-integral introduced in [18].

Our studies are motivated by Fan and Zhao [7]. The authors investigated the concept of spaces $L^{p(t)}(\Omega)$ and $W^{1,p(t)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a measurable subset and $p \in L^\infty_+(\Omega)$ with

$$L^\infty_+(\Omega) = \left\{ u \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{t \in \Omega} u(t) \geq 1 \right\}$$

and which are counterparts of the well known spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$. There has already been some research concerning Sobolev spaces $W^{1,p}(\mathbb{T})$ on time-scales and boundary value problems in the space $W^{1,p}(\mathbb{T})$ with $p > 1$ held constant, see for example [1, 21]. The approach in [1, 21] is different from ours due to possible definitions of measure on $\mathbb{T}$. We adopted the approach from [18], where the measure of an isolated maximum of $\mathbb{T}$, in the case when $\mathbb{T}$ is bounded, is not infinite. Such ideas are more convenient for tackling problems on discrete intervals $\{1, 2, \dots, N\}$ and intervals on the real line in some uniform manner. This is according to the core idea lying behind the introduction of a time-scale. We would like to emphasize here that the Lebesgue and Sobolev spaces $L^{p(t)}(\mathbb{T})$ and $W^{1,p(t)}(\mathbb{T})$ have never been studied before.

This paper consists of five sections. Section 2 reviews the theory of the Lebesgue Δ-integral and the Lebesgue Δ-measure. For a deeper discussion of these concepts, we refer the reader to [1, 3, 11, 18], which serve as our main background reference.

The main results concerning the space $L^{p(t)}(\mathbb{T})$ are given in Section 3. We establish the equivalence between convergence in terms of the modular and in terms of the norm in the space $L^{p(t)}(\mathbb{T})$. Properties and estimations of a modular enable us to prove some new results concerning $L^{p(t)}(\mathbb{T})$, e.g. convergence in Δ-measure for sequence (Theorem 3.7) or reflexivity (Theorem 3.22). It is worth to point out that using Clarkson inequality, we obtain reflexivity only in the case when $p(t) \geq 2$ for Δ-a.e. $t \in \mathbb{T}$. Moreover, we obtain some new inequalities here (see e.g. (3.13), (3.14)). As mentioned before these inequalities would reduce to known ones if we have used the well known settings. See [9] for the discrete case and [5] for the continuous one. We would like to underline that using approach towards measure from [1], we possibly would not obtain all of our results.

In section 4 we study generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$. We analyze the weak convergence in $W^{1,p(t)}(\mathbb{T})$ and the character of linear and continuous functionals defined on $W^{1,p(t)}(\mathbb{T})$. In this section we also consider the Sobolev-like embedding theorem (Theorem 4.9).

Results presented in this paper can be used to discuss boundary value problems on bounded time-scales. An example of such a problem is given in Section 5, where we investigate a Dirichlet problem with $p(t)$-Laplacian (see problem (5.1)). There are many results for problems of this type, see e.g. [8], however, it has been never considered in the context of time-scales. The whole discussion in Section 5 is based on the theory of generalized Lebesgue and Sobolev spaces on time-scales. Within our framework we can apply variational or monotonicity techniques to boundary value problems with $p(t)$-Laplacian on bounded intervals related to the time-scale, see e.g. [6, 14].

Continuous version of problems like (5.1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics [20], electrorheological fluids [17] or image restoration [4]. Variational continuous anisotropic problems have been started by Fan and Zhang in [8] and later considered by many methods and authors, see e.g. [12] for an extensive survey of such boundary value problems. The research concerning the discrete anisotropic problems of variational type have been started in [13] and then continued for example in [10] where some known tools from the critical point theory are modified suitably and then applied in order to get the existence of solutions and also their multiplicity.

Since the research was conducted in discrete and continuous setting separately, it seems interesting to demonstrate that a sort of unification is also possible with the use of a time-scale notion considered with some type of measure that has not been vastly exploited but which appears indispensable. We show by example of a few results that both settings can be tackled in some unified manner. We are of course aware that in the discrete setting one has more options at disposal since all norms are equivalent and in a consequence it does not matter which term in the action functional dominates the other.

2. LEBESGUE Δ-MEASURE AND Δ-INTEGRAL

In this section we recall the notion of the Δ-measure and the Lebesgue Δ-integral as introduced in [18]. Let $\mathbb{T}$ be a bounded time-scale. We denote

$$a = \inf\{s \in \mathbb{T}\}, \quad b = \sup\{s \in \mathbb{T}\}. \tag{2.1}$$

Since $\mathbb{T}$ is bounded, $a, b \in \mathbb{T}$. We consider the time-scale intervals defined as follows

$$[t_1, t_2]_{\mathbb{T}} = [t_1, t_2] \cap \mathbb{T} \quad \text{and} \quad (t_1, t_2)_{\mathbb{T}} = (t_1, t_2) \cap \mathbb{T}$$

for $t_1, t_2 \in \mathbb{T}$. Let $\sigma : \mathbb{T} \to \mathbb{T}$ be a forward jump operator, i.e.,

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} : s > t\} & \text{for } t \in \mathbb{T} \setminus \{b\}, \\ b & \text{for } t = b \end{cases} \tag{2.2}$$

and $\varrho : \mathbb{T} \to \mathbb{T}$ be a backward jump operator, i.e.,

$$\varrho(t) = \begin{cases} \sup\{s \in \mathbb{T} : s < t\} & \text{for } t \in \mathbb{T} \setminus \{a\}, \\ a & \text{for } t = a. \end{cases} \tag{2.3}$$

We introduce the graininess function $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

for $t \in \mathbb{T}$. If $\mu(t) > 0$, we say that $t \in \mathbb{T}$ is right-scattered. If $\mu(t) = 0$, we say that $t \in \mathbb{T}$ is right-dense.

In the sequel we will use the symbol R to denote the set of all right-scattered points of the time-scale $\mathbb{T}$, i.e.,

$$R = \{t \in \mathbb{T} : t < \sigma(t)\}. \tag{2.4}$$

Definition 2.1. Let $f : \mathbb{T} \to \mathbb{R}$. Function $\widehat{f} : [a,b] \to \mathbb{R}$ is a step interpolation of f if

$$\widehat{f}(t) = \begin{cases} f(t) & \text{for } t \in \mathbb{T}, \\ f(s) & \text{for } t \in (s, \sigma(s)), s \in R. \end{cases}$$

The function $\widehat{f}$ extends f to the real interval $[a,b]$ and it allows to establish the equivalence between the Lebesgue Δ-integrable and the Lebesgue integrable functions. With the aid of function $\widehat{f}$, we can calculate the Lebesgue Δ-integral on arbitrary Δ-measurable set as a usual Lebesgue integral on a corresponding Lebesgue measurable set.

We would like to mention that the continuity of a function $f : \mathbb{T} \to \mathbb{R}$ does not imply the continuity of $\widehat{f}$. However, we can formulate the following lemmas.

Lemma 2.2. *Let $f : \mathbb{T} \to \mathbb{R}$. If f is continuous on $\mathbb{T}$, then $\widehat{f}$ is continuous at any point $t \in [a,b]$ such that t is not a left-scattered point of the time-scale $\mathbb{T}$.*

Lemma 2.3. *Let $f : \mathbb{T} \to \mathbb{R}$. If f is continuous on $\mathbb{T}$ and if $f(t) = f(\sigma(t))$ for all $t \in R$, then $\widehat{f}$ is continuous on $[a,b]$.*

Lemma 2.4. *Let $f, g : \mathbb{T} \to \mathbb{R}$. Then*

(a) $\widehat{|f|^g} = |\widehat{f}|^{\widehat{g}}$ *on* $[a,b]$;
(b) $|\widehat{(f-g)}| = |(\widehat{f} - \widehat{g})|$ *on* $[a,b]$.

Proof. Let us denote $h_1(t) = |f(t)|^{g(t)}$ and $h_2(t) = |f(t) - g(t)|$ for $t \in \mathbb{T}$. Since $\widehat{f}(t) = f(t)$ and $\widehat{g}(t) = g(t)$ for $t \in \mathbb{T}$, we have

$$\widehat{h_1}(t) = h_1(t) = |\widehat{f}(t)|^{\widehat{g(t)}} \quad \text{and} \quad \widehat{h_2}(t) = h_2(t) = |\widehat{f}(t) - \widehat{g}(t)|$$

for $t \in \mathbb{T}$. Let us fix $s \in R$ and take $t \in (s, \sigma(s))$. Then

$$\widehat{h_1}(t) = h_1(s) = |f(s)|^{g(s)} = |\widehat{f}(t)|^{\widehat{g(t)}}$$

and

$$\widehat{h_2}(t) = h_2(s) = |f(s) - g(s)| = |\widehat{f}(t) - \widehat{g}(t)|. \qquad \square$$

In what follows, we recall some background from [18].

We call a function $f : \mathbb{T} \to \mathbb{R}$ Δ-measurable (Δ-integrable) if the extension $\widehat{f} : [a,b] \to \mathbb{R}$ is measurable (integrable) on the interval $[a,b]$ in the Lebesgue sense. We say that $f : \mathbb{T} \to \mathbb{R}$ belongs to $L^1(\mathbb{T})$ if the Δ-integral defined by

$$\int_{\mathbb{T}} f(t)\Delta t = \int_{[a,b]} \widehat{f}(t)dt$$

is finite. $L^1(\mathbb{T})$ is obviously a Banach space with the norm

$$\|f\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} f(t)\Delta t.$$

We say that $A \subset \mathbb{T}$ is Δ-measurable if its characteristic function χ_A is Δ-measurable. We define the notion of Δ-measure $\mu_\Delta(A)$ of $A \subset \mathbb{T}$ by

$$\mu_\Delta(A) = \int\limits_{\mathbb{T}} \chi_A(t)\Delta t = \int\limits_{[a,b]} \widehat{\chi_A}(t)dt.$$

The following property holds $\mu_\Delta(A) = \sum_{t\in R}(\sigma(t) - t) + \mu_L(A)$, where $\mu_L(A)$ denotes the classical Lebesgue measure of $A \subset \mathbb{T}$. Moreover, $\mu_\Delta(A) = \mu_L(A)$ if and only if $A \subset \mathbb{T}$ does not have any right-scattered points.

If $A \subset \mathbb{T}$ is Δ-measurable, we can define the Δ-integral of u over A by

$$\int\limits_A u(t)\Delta t = \int\limits_{\mathbb{T}} u(t)\chi_A(t)\Delta t$$

and then we say that u belongs to $L^1(A)$.

$A \subset \mathbb{T}$ is called Δ-null set if $\mu_\Delta(A) = 0$. We say that some property holds Δ-almost everywhere (Δ-a.e.) on A or for Δ-almost all (Δ-a.a.) $t \in A$ if there is Δ-null set $E \subset A$ such that this property holds on $A \setminus E$.

We would like to note that the only Δ-null subsets of $\mathbb{T}$ are the $\emptyset$ and the unions of single-point sets, which are right-dense points of $\mathbb{T}$. Consequently, we obtain that all subsets of Δ-null sets in $\mathbb{T}$ are Δ-measurable and that Lebesgue Δ-measure μ_Δ is a complete and a no-translation-invariant measure.

Remark 2.5. For each $t_0 \in \mathbb{T} \setminus \{b\}$, the single-point set $\{t_0\}$ is Δ-measurable and $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0)$. For every right-scattered point $t_0 \in \mathbb{T}$ we have $\sigma(t_0) > t_0$. It implies that $\mu_\Delta(\{t_0\}) > 0$ for every $t_0 \in R$. In particular, if $\mathbb{T}$ is a discrete time-scale, then $\mu_\Delta(\{t\}) > 0$ for all $t \in \mathbb{T} \setminus \{b\}$.

Additionally, since we adopted approach to the Δ-measure from [18], we obtain that

$$\mu_\Delta(\{b\}) = \int\limits_{\{b\}} 1\Delta t = \int\limits_{[a,b]} \widehat{\chi}_{\{b\}}(t)dt = \mu_L(\{b\}) = 0,$$

where b is given in (2.1). It gives that all subsets of the time-scale $\mathbb{T}$ containing b are of a finite Δ-measure and it is the main difference from the approach given in [1].

We regard the space $L^1(\mathbb{T})$ as an equivalence class of functions defined Δ-a.e. on $\mathbb{T}$, except possibly on a Δ-null set. In particular, the value of a function $u \in L^1(\mathbb{T})$ need not be well defined at any individual points, which have necessarily Δ-measure zero. In the time-scale setting this means that $u \in L^1(\mathbb{T})$ need not be well defined at every right-dense point, but will be well defined at all right-scattered points.

3. THE SPACE $L^{p(t)}(\mathbb{T})$

In this section we introduce the generalized Lebesgue spaces on $\mathbb{T}$. We denote

$$E = \{u : u \text{ is } \Delta\text{-measurable function on } \mathbb{T}\},$$
$$E_{[a,b]} = \{u : \ u \text{ is measurable function on } [a,b]\},$$

$$\begin{aligned} L^{\infty}(\mathbb{T}) &= \Big\{u \in E\colon \operatorname*{ess\,sup}_{t\in\mathbb{T}} |u(t)| < \infty\Big\},\\ L^{\infty}([a,b]) &= \Big\{u \in E_{[a,b]}\colon \operatorname*{ess\,sup}_{t\in[a,b]} |u(t)| < \infty\Big\},\\ L^{\infty}_{+}(\mathbb{T}) &= \Big\{u \in L^{\infty}(\mathbb{T})\colon \operatorname*{ess\,inf}_{t\in\mathbb{T}} u(t) \geq 1\Big\},\\ L^{\infty}_{+}([a,b]) &= \Big\{u \in L^{\infty}([a,b])\colon \operatorname*{ess\,inf}_{t\in[a,b]} u(t) \geq 1\Big\}. \end{aligned} \tag{3.1}$$

In the sequel we assume that $u \in E$, $p \in L^{\infty}_{+}(\mathbb{T})$ and we define $\phi : \mathbb{T} \times [0,\infty) \to \mathbb{R}$ given by

$$\phi(t,s) = s^{p(t)} \text{ for } t \in \mathbb{T}, s \geq 0. \tag{3.2}$$

We recall that functional $\rho : X \to [0,\infty)$ defined over the vector space X is called a modular if

(M1) $\rho(x) = 0$ if and only if $x = \theta$;
(M2) $\rho(-x) = \rho(x)$ for all $x \in X$;
(M3) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for all $x, y \in X$ and for any $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

The vector space $A_\rho = \{x \in X : \lim_{\alpha\to 0^+} \rho(\alpha x) = 0\}$ is called a modular space. The modular space A_ρ is also called a generalized Orlicz space [15, p. 5].

We see that a function given by the formula

$$\phi(t, |u(t)|) = |u(t)|^{p(t)}$$

for $t \in \mathbb{T}$, where ϕ is defined by (3.2), is a composition of Δ-measurable functions, if $u \in E$, $p \in L^{\infty}_{+}(\mathbb{T})$. It makes it obvious that it is Δ-measurable function of t for every $u \in E$ and that

$$\rho(u) := \int\limits_{\mathbb{T}} \phi(t, |u(t)|)\Delta t \tag{3.3}$$

is a modular.

Since $s \to \phi(t,s)$ is convex on $[0,\infty)$ for Δ-a.e. $t \in \mathbb{T}$, we obtain that ρ is a convex modular over E and that

$$L^{p(t)}(\mathbb{T}) = \Big\{u \in E : \lim_{\lambda\to 0^+} \rho(\lambda u) = 0\Big\}$$

is a modular space. Consequently, $L^{p(t)}(\mathbb{T})$ is also a generalized Orlicz space.

By properties of ϕ, we also get

$$L^{p(t)}(\mathbb{T}) = \{u \in E : \forall\lambda > 0 \rho(\lambda u) < \infty\},$$

which we call the generalized Lebesgue space on a time-scale $\mathbb{T}$.

Observe that for any $p \in L^{\infty}_{+}(\mathbb{T})$ we have

$$\begin{aligned} 1 \leq p^- &:= \operatorname*{ess\,inf}_{t\in\mathbb{T}} p(t) = \operatorname*{ess\,inf}_{t\in[a,b]} \widehat{p}(t)\\ &\leq \operatorname*{ess\,sup}_{t\in[a,b]} \widehat{p}(t) = \operatorname*{ess\,sup}_{t\in\mathbb{T}} p(t) =: p^+ < \infty. \end{aligned} \tag{3.4}$$

Note that for any $u \in L^{p(t)}(\mathbb{T})$ and for any $\lambda > 1$ we get

$$\int\limits_{\mathbb{T}} |u(t)|^{p(t)} \Delta t \leq \lambda^{p^+} \int\limits_{\mathbb{T}} |u(t)|^{p(t)} \Delta t.$$

Thus,

$$\rho(u) \leq \lambda^{p^+} \rho(u).$$

Similarly, for $\lambda \in (0,1)$ we have

$$\lambda^{p^+} \rho(u) \leq \rho(u).$$

For every fixed $u \in L^{p(t)}(\mathbb{T})$, $u \neq \theta$, $\lambda \to \rho(\lambda u)$ is a continuous convex even function and it is increasing on $[0, \infty)$.

Lemma 3.1. *Let $u \in E$ and $p \in L^\infty(\mathbb{T})$. Then*

(a) $u \in L^{p(t)}(\mathbb{T})$ *if and only if* $\widehat{u} \in L^{\widehat{p}(t)}([a,b])$;
(b) $p \in L^\infty_+(\mathbb{T})$ *if and only if* $\widehat{p} \in L^\infty_+([a,b])$.

Proof. First, we show that relation (a) holds. Let $u \in L^{p(t)}(\mathbb{T})$. Then, obviously $\widehat{u} \in E_{[a,b]}$ and

$$0 = \lim_{\lambda \to 0^+} \rho(\lambda u) = \lim_{\lambda \to 0^+} \int\limits_{\mathbb{T}} |\lambda u(t)|^{p(t)} \Delta t = \lim_{\lambda \to 0^+} \int\limits_{[a,b]} |\lambda \widehat{u}(t)|^{\widehat{p}(t)} dt.$$

Thus, $\widehat{u} \in L^{\widehat{p}(t)}([a,b])$. Assume now that $\widehat{u} \in L^{\widehat{p}(t)}([a,b])$. Hence $u \in E$ and

$$0 \leq \lim_{\lambda \to 0^+} \int\limits_{\mathbb{T}} |\lambda u(t)|^{p(t)} \Delta t = \lim_{\lambda \to 0^+} \int\limits_{[a,b]} |\lambda \widehat{u}(t)|^{\widehat{p}(t)} dt = 0.$$

Consequently, $u \in L^{p(t)}(\mathbb{T})$ and the proof of relation (a) is completed.

Now we shall show that condition (b) holds. Note that if $\mathbb{T}$ does not contain any right-scattered point, then $p = \widehat{p}$ and the thesis is obvious. Assume that $\mathbb{T}$ contains at least one right-scattered point. Let $p \in L^\infty_+(\mathbb{T})$. Then $p \in L^\infty(\mathbb{T})$ and $\operatorname{ess\,inf}_{t \in \mathbb{T}} p(t) \geq 1$. Suppose that $\widehat{p} \notin L^\infty_+([a,b])$. Then $\widehat{p}$ is either not bounded on $[a,b]$ or else $\operatorname{ess\,inf}_{t \in [a,b]} \widehat{p}(t) < 1$. However, if one of these cases holds, there must be some subset $A \subset [a,b] \setminus \mathbb{T}$ of positive measure on which $\widehat{p}$ has at least one of the mentioned properties. Taking the construction of $\widehat{p}$ into account, we obtain that there exists at least one right-scattered point $t_0 \in \mathbb{T}$ such that p is either not bounded at t_0 or else $p(t_0) < 1$. Since $\mu_\Delta(\{t_0\}) > 0$, we reach a contradiction. This proves that the first implication holds.

Now let $\widehat{p} \in L^\infty_+([a,b])$. Then $\widehat{p} \in L^\infty([a,b])$ and $\operatorname{ess\,inf}_{t \in [a,b]} \widehat{p}(t) \geq 1$. It follows that $\widehat{p}|_{\mathbb{T}} \in L^\infty(\mathbb{T})$ and $\operatorname{ess\,inf}_{t \in \mathbb{T}} \widehat{p}(t) \geq 1$. Moreover, $p(t) = \widehat{p}(t)$ for $t \in \mathbb{T}$. Thus, $p \in L^\infty(\mathbb{T})$ and $\operatorname{ess\,inf}_{t \in \mathbb{T}} p(t) \geq 1$. Thus, $p \in L^\infty_+(\mathbb{T})$. $\square$

Remark 3.2. The proof above shows that it is applicable to set $\operatorname{ess\,inf}_{t\in\mathbb{T}} u(t)$ in (3.1). Since $\mu_\Delta(\{t\}) > 0$ for $t \in R$, where R is defined in (2.4), function $p \in L^\infty_+(\mathbb{T})$ to required to achieve values greater or equal 1 in all right-scattered points of the time-scale $\mathbb{T}$.

Lemma 3.3. *The functional* $\|\cdot\|_\rho : L^{p(t)}(\mathbb{T}) \to [0,\infty)$ *defined by*

$$\|u\|_\rho = \|u\|_{L^{p(t)}(\mathbb{T})} = \inf\left\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

for $u \in L^{p(t)}(\mathbb{T})$, *is a norm in* $L^{p(t)}(\mathbb{T})$.

Note that, by (3.3), we obtain

$$\begin{aligned}\|u\|_{L^{p(t)}(\mathbb{T})} &= \inf\left\{\lambda > 0 : \int\limits_{\mathbb{T}} \left|\frac{u(t)}{\lambda}\right|^{p(t)} \Delta t \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int\limits_{[a,b]} \left|\frac{\widehat{u}(t)}{\lambda}\right|^{\widehat{p}(t)} dt \leq 1\right\}.\end{aligned}$$

Thus,

$$\|u\|_{L^{p(t)}(\mathbb{T})} = \|\widehat{u}\|_{L^{\widehat{p}(t)}([a,b])}. \tag{3.5}$$

Since $(L^{\widehat{p}(t)}([a,b]), \|\cdot\|_{L^{\widehat{p}(t)}([a,b])})$ is a Banach space and since (3.5) holds, we can at once obtain that $(L^{p(t)}(\mathbb{T}), \|\cdot\|_\rho)$ is also a Banach space.

Theorem 3.4. *For any* $u \in L^{p(t)}(\mathbb{T}),\ u \neq \theta$,

$$\|u\|_\rho = \alpha > 0$$

if and only if

$$\rho\left(\frac{u}{\alpha}\right) = 1.$$

Proof. Fix $u \in L^{p(t)}(\mathbb{T}),\ u \neq \theta$ and assume that $\|u\|_\rho = \alpha$. Then

$$\alpha = \inf\left\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Let us observe that a function

$$\rho_u(\lambda) = \rho\left(\frac{u}{\lambda}\right) = \int\limits_{\mathbb{T}} \left|\frac{u(t)}{\lambda}\right|^{p(t)} \Delta t$$

is continuous and that it is decreasing on $(0,\infty)$. We see that

$$\alpha = \inf\{\lambda : \lambda \in \rho_u^{-1}([0,1])\}.$$

By the continuity and the monotonicity of ρ_u, we get $1 = \rho_u(\alpha) = \rho\left(\frac{u}{\alpha}\right)$.

Now let us assume that $\rho(\frac{u}{\alpha}) = 1$ for some $\alpha > 0$. Then there exist $\beta \in \mathbb{R}$ such that $\|u\|_\rho = \beta \leq \alpha$ and

$$\rho_u(\beta) = \rho\left(\frac{u}{\beta}\right) \leq 1 = \rho\left(\frac{u}{\alpha}\right) = \rho_u(\alpha).$$

By the monotonicity of ρ_u, we obtain that $\beta \geq \alpha$.

Finally, $\|u\|_\rho = \beta = \alpha$. □

Theorem 3.5. *Let $u \in L^{p(t)}(\mathbb{T}), u \neq \theta$. Then*

(a) $\|u\|_\rho < 1 (= 1, > 1)$ *if and only if* $\rho(u) < 1 (= 1, > 1)$,

(b) *If* $\|u\|_\rho > 1$, *then* $\|u\|_\rho^{p^-} \leq \rho(u) \leq \|u\|_\rho^{p^+}$,

(c) *If* $\|u\|_\rho < 1$, *then* $\|u\|_\rho^{p^+} \leq \rho(u) \leq \|u\|_\rho^{p^-}$.

Proof. We show that condition (a) holds. Assume that $\|u\|_\rho = \alpha_1 < 1$. By Theorem 3.4 and by the monotonicity of the modular, we obtain

$$\rho(u) < \rho\left(\frac{u}{\alpha_1}\right) = 1.$$

By similar reasoning we argue when $\|u\|_\rho > 1$ or $\|u\|_\rho = 1$.

Now we will prove (b). Assume that $\|u\|_\rho = \alpha_2 > 1$. By Theorem 3.4, we get

$$1 = \rho\left(\frac{u}{\alpha_2}\right) = \int\limits_{\mathbb{T}} \left|\frac{u(t)}{\alpha_2}\right|^{p(t)} \Delta t \leq \frac{1}{\alpha_2^{p^-}} \int\limits_{\mathbb{T}} |u(t)|^{p(t)} \Delta t = \frac{1}{\alpha_2^{p^-}} \rho(u).$$

Moreover, we see that

$$1 = \rho\left(\frac{u}{\alpha_2}\right) \geq \frac{1}{\alpha_2^{p^+}} \int\limits_{\mathbb{T}} |u(t)|^{p(t)} \Delta t = \frac{1}{\alpha_2^{p^+}} \rho(u).$$

Therefore

$$\|u\|_\rho^{p^-} = \alpha_2^{p^-} \leq \rho(u) \leq \alpha_2^{p^+} = \|u\|_\rho^{p^+}.$$

The proof that (c) holds may be performed in a similar way as (b). □

Now we provide a theorem, which relates convergence in norm and convergence obtained via a modular. As expected, this two notions coincide.

Theorem 3.6. *Let $u \in L^{p(t)}(\mathbb{T})$ and $u_k \in L^{p(t)}(\mathbb{T})$ for $k \in \mathbb{N}$. Then*

$$\lim_{k \to \infty} \|u_k - u\|_\rho = 0$$

if and only if

$$\lim_{k \to \infty} \rho(\lambda(u_k - u)) = 0$$

for every $\lambda \geq 0$.

Proof. Let us assume that $\lim_{k\to\infty}\|u_k - u\|_\rho = 0$. Then, of course, we get $\lim_{k\to\infty}\|\lambda(u_k - u)\|_\rho = 0$ for all $\lambda \geq 0$. Now let $\lambda \geq 0$ and let $\varepsilon \in (0,1)$. Then there exists $k_0 \in \mathbb{N}$ such that $\|\lambda\,(u_k - u)\,\|_\rho < \varepsilon$ for all $k > k_0$.

By Theorem 3.5, for every $k > k_0$ we get

$$\rho\,(\lambda\,(u_k - u)) \leq \|\lambda(u_k - u)\|_\rho^{p^-} \leq \|\lambda(u_k - u)\|_\rho < \varepsilon.$$

Thus, $\lim_{k\to\infty}\rho\,(\lambda\,(u_k - u)) = 0$. Since λ and ε were arbitrarily fixed, we obtain that the convergence of a sequence in $L^{p(t)}(\mathbb{T})$ implies the convergence in terms of modular.

Assume now that for any $\lambda \geq 0$ we have

$$\lim_{k\to\infty}\rho(\lambda\,(u_k - u)) = 0.$$

Then there exists $k_0 \in \mathbb{N}$ such that

$$\rho\left(\frac{u_k - u}{\frac{1}{\lambda}}\right) = \rho(\lambda\,(u_k - u)) < 1$$

for $k > k_0$. Therefore $\|u_k - u\|_\rho \leq \frac{1}{\lambda}$ for $k > k_0$ and every $\lambda > 0$. This proves that

$$\lim_{k\to\infty}\|u_k - u\|_\rho = 0. \qquad \square$$

From the well known Riesz Theorem convergence in Δ-measure of some sequence implies that the this sequence is convergent, up to a subsequence, Δ-almost everywhere. We can prove however the following result.

Theorem 3.7. *Let $u \in L^{p(t)}(\mathbb{T})$ and $u_k \in L^{p(t)}(\mathbb{T})$ for $k \in \mathbb{N}$. If*

$$\lim_{k\to\infty}\|u_k - u\|_\rho = 0,$$

then

$$u_k \to u \text{ in } \Delta\text{-measure}$$

and $(u_k)_{k\in\mathbb{N}}$ contains subsequence $(u_{k_i})_{i\in\mathbb{N}}$, which is Δ-a.e. convergent to u on $\mathbb{T}$.

Proof. Let us fix $\alpha > 0$ and let us define a measure

$$\vartheta(A) = \int_A \phi(t,\alpha)\Delta t, \tag{3.6}$$

where $A \subset \mathbb{T}$ and ϕ is defined by (3.2). We see that the Δ-measure is absolutely continuous with respect to the measure given in (3.6). If $\vartheta(A) = 0$ for some $A \subset \mathbb{T}$, then we conclude that the set A cannot contain any right-scattered point and

$$\int_A \phi(t,\alpha)\; dt = 0.$$

Since ϕ is a non-negative function, we obtain that

$$\mu_L(A) = \mu_\Delta(A) = 0.$$

Let $\varepsilon \in (0,1)$. Now if $\lim_{k\to\infty} \|u_k - u\|_\rho = 0$, then by Theorem 3.6, there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have

$$\rho(u_k - u) < \varepsilon^{2p^+}.$$

We denote

$$A_k = \{t \in \mathbb{T} : |u_k(t) - u(t)| \geq \varepsilon\}$$

for $k > k_0$. Then

$$\int\limits_{A_k} \phi(t,\varepsilon)\Delta t = \int\limits_{A_k} \varepsilon^{p(t)}\Delta t \leq \int\limits_{A_k} |u_k(t) - u(t)|^{p(t)}\Delta t \leq \rho(u_k - u) < \varepsilon^{2p^+}$$

and

$$\mu_\Delta(A_k) = \frac{1}{\varepsilon^{p^+}} \int\limits_{A_k} \varepsilon^{p^+}\Delta t \leq \frac{1}{\varepsilon^{p^+}} \int\limits_{A_k} \varepsilon^{p(t)}\Delta t \leq \varepsilon^{p^+}$$

for $k > k_0$. Hence we conclude that

$$\lim_{k\to\infty} \mu_\Delta(A_k) = 0.$$

The sequence $(u_k)_{k\in\mathbb{N}}$ is convergent to u in Δ-measure on $\mathbb{T}$. Moreover, since $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(t)}(\mathbb{T})$, we can show that $(u_k)_{k\in\mathbb{N}}$ is Cauchy sequence in Δ-measure. By the Riesz theorem it follows that $(u_k)_{k\in\mathbb{N}}$ contains a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ that is convergent Δ-a.e. on the time-scale $\mathbb{T}$. □

Lemma 3.8. *Let $v, w \in L^{p(t)}(\mathbb{T})$. Then v, w satisfy the following inequalities:*

(a) $|v(t) + w(t)|^{p(t)} \leq 2^{p^+-1}\left(|v(t)|^{p(t)} + |w(t)|^{p(t)}\right)$,
(b) $|v(t) - w(t)|^{p(t)} \leq 2^{p^+-1}\left(|v(t)|^{p(t)} + |w(t)|^{p(t)}\right)$

for Δ-a.e. $t \in \mathbb{T}$.

Proof. By the Jensen inequality, we get

$$\begin{aligned}|v(t) + w(t)|^{p(t)} &= 2^{p(t)}\left|\frac{v(t)}{2} + \frac{w(t)}{2}\right|^{p(t)} \leq 2^{p^+}\left(\left|\frac{v(t)}{2}\right|^{p(t)} + \left|\frac{w(t)}{2}\right|^{p(t)}\right)\\ &\leq 2^{p^+}\left(\frac{|v(t)|^{p(t)}}{2} + \frac{|w(t)|^{p(t)}}{2}\right) = 2^{p^+-1}\left(|v(t)|^{p(t)} + |w(t)|^{p(t)}\right)\end{aligned}$$

for Δ-a.e. $t \in \mathbb{T}$. Condition (b) can be obtained in a similar way. □

Lemma 3.9. *Let $(u_k)_{k\in\mathbb{N}} \subset L^{p(t)}(\mathbb{T})$ be a sequence convergent to a certain function $u \in L^{p(t)}(\mathbb{T})$. Then there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}} \subset L^{p(t)}(\mathbb{T})$ such that*

$$\lim_{l\to\infty} u_{k_l}(t) = u(t)$$

for Δ-a.e. $t \in \mathbb{T}$ and there exists a function $g \in L^{p(t)}(\mathbb{T})$ such that $|u_{k_l}(t)| \leq g(t)$ for $l \in \mathbb{N}$ and Δ-a.e. $t \in \mathbb{T}$.

Proof. The first part of this lemma follows by Theorem 3.7. We consider the following monotone sequence of real valued, Δ-measurable functions $(g_m)_{m\in\mathbb{N}}$ defined by

$$g_m(t) = |u_{k_1}(t)| + \sum_{l=1}^{m} |u_{k_l}(t) - u_{k_{l+1}}(t)|$$

for $m \in \mathbb{N}$ and $t \in \mathbb{T}$. Then by Lemma 3.8 (a),

$$\begin{aligned} \int\limits_{\mathbb{T}} |g_m(t)|^{p(t)} \Delta t &= \int\limits_{\mathbb{T}} \left| |u_{k_1}(t)| + \sum_{l=1}^{m} |u_{k_l}(t) - u_{k_{l+1}}(t)| \right|^{p(t)} \Delta t \\ &\leq 2^{p^+} \left(\int\limits_{\mathbb{T}} |u_{k_1}(t)|^{p(t)} \Delta t + M \right) \end{aligned} \tag{3.7}$$

for some $M \geq 0$. Moreover, $(g_m)_{m\in\mathbb{N}}$ is convergent to a certain function g in Δ-measure. By (3.7) and the Lebesgue Dominated Convergence Theorem, we obtain that $g \in L^{p(t)}(\mathbb{T})$. Apart from this, by the monotonicity of $(g_m)_{m\in\mathbb{N}}$, we obtain that for every $l \in \mathbb{N}$ we can find $m \in \mathbb{N}$ such that

$$\begin{aligned} |u_{k_l}(t)| &= |u_{k_1}(t) + u_{k_2}(t) - u_{k_1}(t) + \ldots + u_{k_l}(t) - u_{k_{l-1}}(t)| \\ &\leq |u_{k_1}(t)| + |u_{k_2}(t) - u_{k_1}(t)| + \ldots + |u_{k_l}(t) - u_{k_{l-1}}(t)| \leq g_m(t) \leq g(t) \end{aligned}$$

for Δ-a.e. $t \in \mathbb{T}$. □

Theorem 3.10. *If $p_1, p_2 \in L^{\infty}_{+}(\mathbb{T})$ and $p_1(t) \leq p_2(t)$ for Δ-a.e. $t \in \mathbb{T}$, then the embedding $L^{p_2(t)}(\mathbb{T}) \hookrightarrow L^{p_1(t)}(\mathbb{T})$ is continuous.*

Proof. By [7, p. 430], we know that if $\widehat{p_1}(t) \leq \widehat{p_2}(t)$ for almost all $t \in [a,b]$, where $\widehat{p_1}, \widehat{p_2} \in L^{\infty}_{+}([a,b])$, then the embedding $L^{\widehat{p_2}(t)}([a,b]) \hookrightarrow L^{\widehat{p_1}(t)}([a,b])$ is continuous. Now if $p_1, p_2 \in L^{\infty}_{+}(\mathbb{T})$ and $p_1(t) \leq p_2(t)$ for Δ-a.e. $t \in \mathbb{T}$, then $\widehat{p_1}, \widehat{p_2} \in L^{\infty}_{+}([a,b])$ and $\widehat{p_1}(t) \leq \widehat{p_2}(t)$ for almost all $t \in [a,b]$. By Lemma 3.1 and by (3.5), we obtain that the embedding $L^{p_2(t)}(\mathbb{T}) \hookrightarrow L^{p_1(t)}(\mathbb{T})$ is continuous. □

Now let $u \in L^{p(t)}(\mathbb{T})$ and let us define

$$u_n(t) = \begin{cases} u(t) & \text{if } |u(t)| \leq n, \\ 0 & \text{if } |u(t)| > n \end{cases}$$

for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \rho(u_n - u) = 0$$

and since Theorem 3.6 holds, we obtain the following result.

Theorem 3.11. *The set of all bounded measurable functions defined on $\mathbb{T}$ is dense in $(L^{p(t)}(\mathbb{T}), \|\cdot\|_{\rho})$.*

We now discuss the uniform convexity of $L^{p(t)}(\mathbb{T})$ which property is crucial for the further applications of variation methods in this setting.

Theorem 3.12. *Let $p \in L^{\infty}_{+}(\mathbb{T})$ and $p(t) \geq 2$ for Δ-a.e. $t \in \mathbb{T}$, then $L^{p(t)}(\mathbb{T})$ is uniformly convex* (*and thus reflexive*).

Proof. Let us recall that a Banach space X is uniformly convex [2, p. 76], if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $\|x\|_X \leq 1$, $\|y\|_X \leq 1$ and $\|x - y\|_X > \varepsilon$, we have $\left\|\frac{x+y}{2}\right\|_X < 1 - \delta$.

Let $\varepsilon \in (0,1)$ and let $u, v \in L^{p(t)}(\mathbb{T})$ be such that $\|u\|_\rho \leq 1$, $\|v\|_\rho \leq 1$ and $\|u - v\|_\rho > \varepsilon$. Then by Theorem 3.5, we get that $\rho(u) \leq 1$, $\rho(v) \leq 1$. Using the Clarkson inequality [2], we obtain that

$$\left|\frac{u(t)+v(t)}{2}\right|^{p(t)} + \left|\frac{u(t)-v(t)}{2}\right|^{p(t)} \leq \frac{1}{2}\left(|u(t)|^{p(t)} + |v(t)|^{p(t)}\right) \tag{3.8}$$

holds for Δ-a.e. $t \in \mathbb{T}$. Integrating both sides of (3.8) and applying Theorem 3.5, we have

$$\rho\left(\frac{u+v}{2}\right) + \rho\left(\frac{u-v}{2}\right) \leq \frac{1}{2}\left(\rho(u) + \rho(v)\right) \leq 1.$$

Hence

$$\rho\left(\frac{u+v}{2}\right) \leq 1 - \rho\left(\frac{u-v}{2}\right).$$

Since $\varepsilon \in (0,1)$, we have

$$\frac{1}{\varepsilon^{p^+}}\rho(u-v) = \frac{1}{\varepsilon^{p^+}}\int_{\mathbb{T}} |u(t)-v(t)|^{p(t)}\Delta t \geq \rho\left(\frac{u-v}{\varepsilon}\right) \geq 1. \tag{3.9}$$

By (3.9), we obtain that

$$\rho\left(\frac{u+v}{2}\right) \leq 1 - \rho\left(\frac{u-v}{2}\right) \leq 1 - \frac{\varepsilon^{p^+}}{2^{p^+}} < 1.$$

Applying Theorem 3.5, we get that there exists $\delta > 0$ such that

$$\left\|\frac{u+v}{2}\right\|_\rho < 1 - \delta.$$

Applying similar considerations for $\varepsilon > 1$, we obtain that

$$\rho\left(\frac{u+v}{2}\right) \leq 1 - \rho\left(\frac{u-v}{2}\right) \leq 1 - \frac{\varepsilon^{p^-}}{2^{p^+}} < 1.$$

By Theorem 3.5, we have the assertion. Since every uniformly convex Banach space is reflexive, we obtain that $L^{p(t)}(\mathbb{T})$ is reflexive and the proof is finished. $\square$

Lemma 3.13 ([15, p. 48])**.** *Let $p, q : \mathbb{T} \to \mathbb{R}$ be conjugative functions on $\mathbb{T}$, i.e.,*

$$\frac{1}{p(t)} + \frac{1}{q(t)} = 1$$

for Δ-a.e $t \in \mathbb{T}$. Then

$$\alpha\beta \leq \frac{\alpha^{p(t)}}{p(t)} + \frac{\beta^{q(t)}}{q(t)} \leq \frac{\alpha^{p(t)}}{p^-} + \frac{\beta^{q(t)}}{q^-}$$

for any $\alpha, \beta > 0$ and for Δ-a.e. $t \in \mathbb{T}$, where p^- is defined in (3.4) *and $q^- = \operatorname{ess\,inf}_{t\in\mathbb{T}} q(t)$.*

Applying Lemma 3.13, we have the following.

Corollary 3.14. *Let $p, q : \mathbb{T} \to \mathbb{R}$ be conjugative functions on $\mathbb{T}$. Then*

$$\int\limits_{\mathbb{T}} |u(t)v(t)|\Delta t \leq \frac{1}{p^-} \int\limits_{\mathbb{T}} |u(t)|^{p(t)}\Delta t + \frac{1}{q^-} \int\limits_{\mathbb{T}} |v(t)|^{q(t)}\Delta t$$

for $u \in L^{p(t)}(\mathbb{T})$, $v \in L^{q(t)}(\mathbb{T})$.

Now applying theory of classes N and Φ described in [15], we discuss the conjugate space $\left(L^{p(t)}(\mathbb{T})\right)^*$. Let us recall [15, p. 108] that $\varphi : \mathbb{T} \times [0,\infty) \to \mathbb{R}$ belongs to class Φ, if it satisfies the following conditions:

(Φ1) $\varphi(t, y) = 0$ if and only if $y = 0$ for every $t \in \mathbb{T}$,
(Φ2) $\varphi(t, y)$ is a non-decreasing, continuous function of y for every $t \in \mathbb{T}$,
(Φ3) $\varphi(t, y)$ is a Δ-measurable function of t for all $y \geq 0$.

Next, we say that a function $\varphi \in \Phi$ belongs to class N [15, p. 82], if for every $t \in \mathbb{T}$, φ is a convex function of y and

(C1) $\lim_{y\to 0^+} \frac{\varphi(t,y)}{y} = 0$,
(C2) $\lim_{y\to\infty} \frac{\varphi(t,y)}{y} = \infty$.

Lemma 3.15 ([15, p. 82])**.** *If a function $\varphi : \mathbb{T} \times [0,\infty) \to \mathbb{R}$ belongs to class Φ and it is a convex function of the variable $y \in [0,\infty)$ for every $t \in \mathbb{T}$, then φ is of the form*

$$\varphi(t, y) = \int\limits_0^{|y|} g(t, s)ds, \tag{3.10}$$

where $g(t, y)$ is right-handed derivative of $\varphi(t, y)$ for a fixed $t \in \mathbb{T}$.

Definition 3.16 ([15, p. 82])**.** Let φ belongs to class N and be of the form defined in (3.10) with

$$g^*(t, y) = \sup\{s : g(t, s) \leq y\}.$$

Then the function

$$\varphi^*(t,y) = \int_0^{|y|} g^*(t,s)ds$$

is called a complementary of φ in the sense of Young.

Theorem 3.17 ([16, p. 104])**.** *Let φ^* be a complementary function in the sense of Young to φ. Then φ^* and φ are convex and they satisfy Young inequality*

$$\alpha\beta \le \varphi(t,\alpha) + \varphi^*(t,\beta) \tag{3.11}$$

for all α, $\beta > 0$, $t \in \mathbb{T}$.

We will assume $p, q : \mathbb{T} \to \mathbb{R}$ to be conjugative on $\mathbb{T}$ in the sequel. Set

$$\phi_p(t,s) = \frac{1}{p(t)} s^{p(t)}$$

for $t \in \mathbb{T}, s > 0$. Then $\phi_p \in \Phi$. Moreover,

$$\lim_{s\to 0^+} \frac{\phi_p(t,s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{\phi_p(t,s)}{s} = \infty.$$

Consequently, $\phi_p \in N$. Writing

$$\rho_p(u) = \int_{\mathbb{T}} \phi_p(t, |u(t)|)\Delta t$$

and

$$\|u\|_p = \inf\left\{\lambda > 0 : \rho_p\left(\frac{u}{\lambda}\right) \le 1\right\},$$

we obtain an equivalent norm to $\|\cdot\|_\rho$ on $L^{p(t)}(\mathbb{T})$.

Now notice that for $t \in \mathbb{T}$ and $s > 0$ we have

$$\phi_p(t,s) = \frac{1}{p(t)} s^{p(t)} = \int_0^{|s|} r^{p(t)-1} dr$$

and

$$\sup\{s : s^{p(t)-1} \le r\} = \sup\{s : s \le r^{\frac{1}{p(t)-1}}\} = r^{\frac{1}{p(t)-1}}.$$

Thus,

$$\phi_p^*(t,s) = \int_0^{|s|} r^{\frac{1}{p(t)-1}} dr = \frac{p(t)-1}{p(t)} s^{\frac{p(t)}{p(t)-1}} = \frac{1}{q(t)} s^{q(t)}$$

for any $t \in \mathbb{T}$ and any $s > 0$.

Writing

$$\rho_p^*(v) = \int\limits_{\mathbb{T}} \frac{1}{q(t)} |v(t)|^{q(t)} \Delta t = \int\limits_{\mathbb{T}} \phi_p^*(t, |v(t)|) \Delta t,$$

we obtain

$$L^{q(t)}(\mathbb{T}) = \left\{ v \in E : \int\limits_{\mathbb{T}} \frac{1}{q(t)} |v(t)|^{q(t)} \Delta t < \infty \right\} = \left\{ v \in E : \lim_{\lambda \to 0^+} \rho_p^* (\lambda v) = 0 \right\}.$$

Consequently, using Corollary 13.14 and Theorem 13.17 in [15] we can formulate the following theorems.

Theorem 3.18. *For every $v \in L^{q(t)}(\mathbb{T})$ functional $f : L^{p(t)}(\mathbb{T}) \to \mathbb{R}$ defined by*

$$f(u) = \int\limits_{\mathbb{T}} u(t) v(t) \Delta t \tag{3.12}$$

is a continuous and linear on $L^{p(t)}(\mathbb{T})$.

Theorem 3.19. *For every continuous and linear functional f on $L^{p(t)}(\mathbb{T})$, there is a unique element $v \in L^{q(t)}(\mathbb{T})$ such that f is defined by* (3.12).

Consequently, the following theorem holds

Theorem 3.20. $(L^{p(t)}(\mathbb{T}))^* = L^{q(t)}(\mathbb{T})$.

Remark 3.21. By Theorem 3.12, we know that if $p \in L_+^\infty(\mathbb{T})$ and $p(t) \geq 2$ for Δ-a.e. $t \in \mathbb{T}$, then $L^{p(t)}(\mathbb{T})$ is reflexive. Notice, that $(L^{p(t)}(\mathbb{T}))^* = L^{q(t)}(\mathbb{T})$, where $p, q : \mathbb{T} \to \mathbb{R}$ are conjugative on $\mathbb{T}$ and $q(t) \in (1, 2]$ for Δ-a.e. $t \in \mathbb{T}$. Since a Banach space is reflexive if and only if its dual space is reflexive, we obtain that $L^{p(t)}(\mathbb{T})$ is reflexive for function $p \in L_+^\infty(\mathbb{T})$ satisfying $p(t) \in (1, 2]$ for Δ-a.e. $t \in \mathbb{T}$.

By Remark 3.21 and Theorem 3.20, we obtain the following theorem.

Theorem 3.22. *If $p \in L_+^\infty(\mathbb{T})$ and $p(t) > 1$ for Δ-a.e. $t \in \mathbb{T}$, then $L^{p(t)}(\mathbb{T})$ is reflexive.*

Theorem 3.23. *Let $v \in L^{q(t)}(\mathbb{T})$ be such that $p, q \in L_+^\infty(\mathbb{T})$ and p, q are conjugative on $\mathbb{T}$. Let us define*

$$\|v\|' = \sup \left\{ \int\limits_{\mathbb{T}} |u(t) v(t)| \Delta t : u \in L^{p(t)}(\mathbb{T}) \text{ and } \|u\|_{L^{p(t)}(\mathbb{T})} \leq 1 \right\}.$$

Then

$$\|v\|_{L^{q(t)}(\mathbb{T})} \leq \|v\|' \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|v\|_{L^{q(t)}(\mathbb{T})}.$$

Proof. For fixed $v \in L^{q(t)}(\mathbb{T})$, $u \in L^{p(t)}(\mathbb{T})$ we set $\alpha = \|v\|_{L^{q(t)}(\mathbb{T})}$, $\beta = \|u\|_{L^{p(t)}(\mathbb{T})} \leq 1$. Then by (3.11), we obtain

$$\int\limits_{\mathbb{T}} \frac{|u(t)v(t)|}{\alpha\beta}\Delta t \leq \int\limits_{\mathbb{T}} \frac{1}{p(t)}\left|\frac{u(t)}{\beta}\right|^{p(t)}\Delta t + \int\limits_{\mathbb{T}} \frac{1}{q(t)}\left|\frac{v(t)}{\alpha}\right|^{q(t)}\Delta t$$
$$\leq \frac{1}{p^-}\int\limits_{\mathbb{T}}\left|\frac{u(t)}{\beta}\right|^{p(t)}\Delta t + \frac{1}{q^-}\int\limits_{\mathbb{T}}\left|\frac{v(t)}{\alpha}\right|^{q(t)}\Delta t = \frac{1}{p^-} + \frac{1}{q^-}.$$

Consequently, we have

$$\int\limits_{\mathbb{T}} |u(t)v(t)|\Delta t \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\alpha\beta \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\alpha. \tag{3.13}$$

Thus, we get

$$\|v\|' \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|v\|_{L^{q(t)}(\mathbb{T})}.$$

On the other hand, for $v \in L^{q(t)}(\mathbb{T})$ such that $\|v\|_{L^{q(t)}(\mathbb{T})} = \alpha$ we define

$$u(t) = \left|\frac{v(t)}{\alpha}\right|^{q(t)-1} \operatorname{sgn} v(t).$$

Then $u \in L^{p(t)}(\mathbb{T})$ and

$$\int\limits_{\mathbb{T}} |u(t)|^{p(t)}\Delta t = \int\limits_{\mathbb{T}}\left|\frac{v(t)}{\alpha}\right|^{q(t)}\Delta t = 1 = \|u\|_{L^{p(t)}(\mathbb{T})}.$$

Therefore

$$\int\limits_{\mathbb{T}} |u(t)v(t)|\Delta t = \int\limits_{\mathbb{T}}\left|\frac{v(t)}{\alpha}\right|^{q(t)-1} v(t)\Delta t = \int\limits_{\mathbb{T}} \alpha\left|\frac{v(t)}{\alpha}\right|^{q(t)}\Delta t = \alpha = \|v\|_{L^{q(t)}(\mathbb{T})}.$$

Consequently, we obtain

$$\|v\|' \geq \|v\|_{L^{q(t)}(\mathbb{T})}.$$

□

Remark 3.24. Inequality (3.13) leads to the Hölder inequality

$$\int\limits_{\mathbb{T}} |u(t)v(t)|\Delta t \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_\rho \|v\|_{L^{q(t)}(\mathbb{T})} \tag{3.14}$$

for $u \in L^{p(t)}(\mathbb{T}), v \in L^{q(t)}(\mathbb{T})$. Apart from this, by Theorem 3.23, we obtain the equivalence of norms $\|v\|_{L^{q(t)}(\mathbb{T})}, \|v\|'$ in $L^{q(t)}(\mathbb{T})$.

4. THE SPACE $W^{1,p(t)}(\mathbb{T})$

In this section we shall give some basic properties of the generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$, where $p \in L_+^\infty(\mathbb{T})$. We denote

$$\varphi^\sigma(t) = \varphi(\sigma(t))$$

for $t \in \mathbb{T}$, where $\sigma : \mathbb{T} \to \mathbb{T}$ is defined in (2.2) and $\varphi : \mathbb{T} \to \mathbb{R}$.

Let us denote $\mathbb{T}^\kappa = \mathbb{T} \setminus (\varrho(\sup \mathbb{T}), \sup \mathbb{T}]$, where $\varrho : \mathbb{T} \to \mathbb{T}$ is defined in (2.3). In this way, we remove from the time-scale $\mathbb{T}$ a Δ-null single-point set, which consists of a left-scattered maximum of $\mathbb{T}$. Alternatively, it can be written as

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} & \text{if } b \text{ is not an isolated point,} \\ \mathbb{T} \setminus \{b\} & \text{if } b \text{ is an isolated point.} \end{cases}$$

We recall that function $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}^\kappa$ if there exists a finite number $f^\Delta(t)$ with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We denote by $C^1_{rd}(\mathbb{T})$ the set of functions defined on $\mathbb{T}$, which are Δ-differentiable on $\mathbb{T}^\kappa$ and their Δ-derivatives are rd-continuous on $\mathbb{T}^\kappa$.

Definition 4.1. Let $u \in E$. We say that a function $g : \mathbb{T}^\kappa \to \mathbb{R}$ is Δ-weak derivative of u if

$$\int_{\mathbb{T}} (u \cdot \varphi^\Delta)(s) \Delta s = - \int_{\mathbb{T}} (g \cdot \varphi^\sigma)(s) \Delta s$$

for every $\varphi \in C^1_{0,\mathrm{rd}}(\mathbb{T})$ with

$$C^1_{0,\mathrm{rd}}(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : f \in C^1_{\mathrm{rd}}(\mathbb{T}) \text{ and } f(a) = f(b) = 0\}.$$

We denote

$$g = \Delta^w u.$$

Having the definition of Δ-weak derivative we introduce generalized Sobolev space on a time-scale $\mathbb{T}$.

Definition 4.2. Let $p \in L_+^\infty(\mathbb{T})$. We say that u belongs to generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$ if and only if $u \in L^{p(t)}(\mathbb{T})$ and $\Delta^w u$ exists and belongs to $L^{p(t)}(\mathbb{T})$.

By Lemma 3.1 and Theorem 3.10, we obtain the following result.

Theorem 4.3. *Let $p_1, p_2 \in L_+^\infty(\mathbb{T})$. If $p_1(t) \leq p_2(t)$ for Δ-a.e $t \in \mathbb{T}$, then the embedding $W^{1,p_2(t)}(\mathbb{T}) \hookrightarrow W^{1,p_1(t)}(\mathbb{T})$ is continuous.*

An immediate consequence of Theorem 4.3 is that the following continuous embeddings hold

$$W^{1,p^+}(\mathbb{T}) \hookrightarrow W^{1,p(t)}(\mathbb{T}) \hookrightarrow W^{1,p^-}(\mathbb{T}).$$

Since any element of $W^{1,p^-}(\mathbb{T})$ is absolutely continuous, we see that same holds for any $u \in W^{1,p(t)}(\mathbb{T})$.

For generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$ we introduce the norm

$$\|u\|_{1,p(t)} = \|u\|_{W^{1,p(t)}(\mathbb{T})} = \|u\|_{L^{p(t)}(\mathbb{T})} + \|\Delta^w u\|_{L^{p(t)}(\mathbb{T})}. \tag{4.1}$$

Using basic properties of generalized Lebesgue spaces $L^{p(t)}(\mathbb{T})$, we can prove the following theorems.

Theorem 4.4. $(W^{1,p(t)}(\mathbb{T}), \|\cdot\|_{1,p(t)})$ *is a Banach space.*

Theorem 4.5. *The generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$ is reflexive.*

Proof. Observe that the product space $L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T})$ considered with the norm

$$\|(u,v)\|_{p(t),p(t)} = \|u\|_{L^{p(t)}(\mathbb{T})} + \|v\|_{L^{p(t)}(\mathbb{T})}$$

is reflexive. The operator $T : W^{1,p(t)}(\mathbb{T}) \to L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T})$ defined by

$$T(u) = [u, \Delta^w u] \tag{4.2}$$

is an isometry. Since $W^{1,p(t)}(\mathbb{T})$ is a Banach space, $T(W^{1,p(t)}(\mathbb{T}))$ is closed subspace of $L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T})$. Consequently, $T(W^{1,p(t)}(\mathbb{T}))$ and $W^{1,p(t)}(\mathbb{T})$ are reflexive. $\square$

By (4.1), we may conclude the following result holds.

Lemma 4.6. *Let $u_n \in W^{1,p(t)}(\mathbb{T})$ for $n \in \mathbb{N}$. Then $u_n \to u$ in $W^{1,p(t)}(\mathbb{T})$ if and only if $u_n \to u$ in $L^{p(t)}(\mathbb{T})$ and $\Delta^w u_n \to \Delta^w u$ in $L^{p(t)}(\mathbb{T})$.*

Since we are going to consider weak-convergence in $W^{1,p(t)}(\mathbb{T})$, we define the character of linear and continuous functional defined on $W^{1,p(t)}(\mathbb{T})$.

Theorem 4.7. *Let $q \in L^\infty_+(\mathbb{T})$ be a conjugative to $p \in L^\infty_+(\mathbb{T})$ on $\mathbb{T}$. Functional $F : W^{1,p(t)}(\mathbb{T}) \to \mathbb{R}$ is linear and continuous if and only if there exist functions $g_1, g_2 \in L^{q(t)}(\mathbb{T})$ such that F is defined by the following formula*

$$F(u) = \int_{\mathbb{T}} u(t) g_1(t) \Delta t + \int_{\mathbb{T}} \Delta^w u(t) g_2(t) \Delta t$$

for $u \in W^{1,p(t)}(\mathbb{T})$.

Proof. We assume that functional $F : W^{1,p(t)}(\mathbb{T}) \to \mathbb{R}$ is linear and continuous. Then

$$\widetilde{F} = F \circ T^{-1} : L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T}) \to \mathbb{R},$$

where $T : W^{1,p(t)}(\mathbb{T}) \to L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T})$ is given in (4.2), is linear and continuous. Consequently, there exist functions $g_1, g_2 \in L^{q(t)}(\mathbb{T})$ such that $\widetilde{F}$ is given by the following formula

$$\widetilde{F}(f_1, f_2) = \int\limits_{\mathbb{T}} f_1(t)g_1(t)\Delta t + \int\limits_{\mathbb{T}} f_2(t)g_2(t)\Delta t \tag{4.3}$$

for $f_1, f_2 \in L^{p(t)}(\mathbb{T})$. Notice that

$$F(u) = (\widetilde{F} \circ T)(u) = \int\limits_{\mathbb{T}} u(t)g_1(t)\Delta t + \int\limits_{\mathbb{T}} \Delta^w u(t)g_2(t)\Delta t$$

for $u \in W^{1,p(t)}(\mathbb{T})$.

Now we take the following functional

$$F(u) = \int\limits_{\mathbb{T}} u(t)g_1(t)\Delta t + \int\limits_{\mathbb{T}} \Delta^w u(t)g_2(t)\Delta t$$

for $u \in W^{1,p(t)}(\mathbb{T})$. By Theorem 3.18, the functional $\widetilde{F} : L^{p(t)}(\mathbb{T}) \times L^{p(t)}(\mathbb{T}) \to \mathbb{R}$ defined by (4.3) is linear and continuous. Since $F = \widetilde{F} \circ T$, we have that the functional F is continuous and linear as a composition. □

By Theorem 4.7, we obtain the following result.

Theorem 4.8. *Let $u_n \in W^{1,p(t)}(\mathbb{T})$ for $n \in \mathbb{N}$. Then $u_n \rightharpoonup u$ in $W^{1,p(t)}(\mathbb{T})$ if and only if $u_n \rightharpoonup u$ in $L^{p(t)}(\mathbb{T})$ and $\Delta^w u_n \rightharpoonup \Delta^w u$ in $L^{p(t)}(\mathbb{T})$.*

Now we focus our attention on compact and continuous embeddings of generalized Sobolev space $W^{1,p(t)}(\mathbb{T})$.

Theorem 4.9. *The embedding $W^{1,p(t)}(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ is compact.*

Proof. Let u belong to the unit ball in $W^{1,p(t)}(\mathbb{T})$. Then

$$\|\Delta^w u\|_\rho \leq 1. \tag{4.4}$$

Let $t_1, t_2 \in \mathbb{T}$, $\varepsilon > 0$ and let $q \in L^\infty_+(\mathbb{T})$ be such that p, q are conjugative on $\mathbb{T}$. We recall that $q^- = \operatorname{ess\,inf}_{t\in\mathbb{T}} q(t)$ and $q^+ = \operatorname{ess\,sup}_{t\in\mathbb{T}} q(t)$. Applying (4.4), the Fundamental Theorem of Calculus and Hölder inequality (3.14) we estimate

$$\begin{aligned} |u(t_1) - u(t_2)| &= \left| \int\limits_{t_1}^{t_2} \Delta^w u(t)\Delta t \right| \leq \int\limits_{t_1}^{t_2} |\Delta^w u|\Delta t \\ &\leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|\mathbf{1}\|_{L^{q(t)}([t_1,t_2]_{\mathbb{T}})} = \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|\widehat{\mathbf{1}}\|_{L^{\widehat{q}(t)}([t_1,t_2])} \end{aligned}$$

with extension $\widehat{q}:[t_1,t_2]\to\mathbb{R}$ of function q, number p^- defined in (3.4) and $\mathbf{1}(t)=1$ for $t\in\mathbb{T}$. Observe that if $\alpha=\|\widehat{\mathbf{1}}\|_{L^{\widehat{q}(t)}([t_1,t_2])}\le 1$, then

$$1=\rho\left(\frac{\mathbf{1}}{\alpha}\right)=\int\limits_{t_1}^{t_2}\left(\frac{1}{\alpha}\right)^{\widehat{q}(t)}dt\le\int\limits_{t_1}^{t_2}\left(\frac{1}{\alpha^{q^+}}\right)dt=\frac{1}{\alpha^{q^+}}|t_2-t_1|.$$

Hence

$$|u(t_1)-u(t_2)|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\frac{1}{\alpha^{q^+}}|t_2-t_1|.$$

Analogously, if $\alpha=\|\widehat{\mathbf{1}}\|_{L^{\widehat{q}(t)}([t_1,t_2])}>1$, then

$$1=\rho\left(\frac{\mathbf{1}}{\alpha}\right)=\int\limits_{t_1}^{t_2}\left(\frac{1}{\alpha}\right)^{\widehat{q}(t)}dt\le\int\limits_{t_1}^{t_2}\frac{1}{\alpha^{q^-}}dt=\frac{1}{\alpha^{q^-}}|t_2-t_1|.$$

Thus,

$$|u(t_1)-u(t_2)|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\alpha\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\alpha^{q^-}\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|t_2-t_1|.$$

Taking

$$\delta=\min\left\{\frac{\varepsilon\alpha^{q^+}}{\left(\frac{1}{p^-}+\frac{1}{q^-}\right)},\frac{\varepsilon}{\left(\frac{1}{p^-}+\frac{1}{q^-}\right)}\right\}$$

we obtain that if $|t_1-t_2|<\delta$, then $|u(t_1)-u(t_2)|\le\varepsilon$. Thus, by the Ascoli–Arzelà Theorem we have that unit ball in $W^{1,p(t)}(\mathbb{T})$ has compact closure in $C(\mathbb{T})$. $\square$

Moreover, since embeddings

$$C(\mathbb{T})\hookrightarrow L^{p^-}(\mathbb{T})$$

and

$$C(\mathbb{T})\hookrightarrow L^{p^+}(\mathbb{T})$$

are continuous we have that embeddings

$$W^{1,p(t)}(\mathbb{T})\hookrightarrow L^{p^-}(\mathbb{T})\tag{4.5}$$

and

$$W^{1,p(t)}(\mathbb{T})\hookrightarrow L^{p^+}(\mathbb{T})$$

are compact.

5. THE DIRICHLET PROBLEM DRIVEN BY $p(t)$-LAPLACIAN

A function $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is called Δ-Caratheodory function if it satisfies the following conditions

(C1) for all $y\in\mathbb{R}$ the function $t\to f(t,y)$ is Δ-measurable on $\mathbb{T}$,
(C2) for Δ-a.e. $t\in\mathbb{T}$ the function $y\to f(t,y)$ is continuous on $\mathbb{R}$.

A function $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is called L^1-Caratheodory function over $\mathbb{T}\times\mathbb{R}$ if it is Δ-Caratheodory function and for every $d>0$ there exists $f_d\in L^1(\mathbb{T})$ such that for Δ-a.e. $t\in\mathbb{T}$ and for all $u\in[-d,d]$ we have $|f(t,u)|\le f_d(t)$.

We recall that $C_{rd}(\mathbb{T})$ denotes the set of functions which are rd-continuous on $\mathbb{T}$, i.e. which are continuous at right-dense points of the time-scale $\mathbb{T}$ and posses finite left-sided limits at left-dense points of $\mathbb{T}$.

Let $p\in L^{\infty}_{+}(\mathbb{T})\cap C_{rd}(\mathbb{T})$ be such that $p(t)\ge 2$ for $t\in\mathbb{T}$ and let $W_0^{1,p(t)}(\mathbb{T})$ denote the closure of $C^{\infty}_{0,rd}(\mathbb{T})$ in $W^{1,p(t)}(\mathbb{T})$, where

$$C^{\infty}_{0,\mathrm{rd}}(\mathbb{T})=\{u\in C^{\infty}_{\mathrm{rd}}(\mathbb{T}):u(a)=u(b)=0\}.$$

In this section we discuss the following problem

$$\begin{cases}-\Delta_{p(t)}u(t)=-\frac{\Delta}{\Delta t}\left(|\Delta^w u(t)|^{p(t)-2}\Delta^w u(t)\right)=f(t,u^{\sigma}(t)) \quad \text{for } t\in\mathbb{T}^{\kappa},\\ u(a)=u(b)=0,\end{cases}\tag{5.1}$$

where $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is L^1-Caratheodory function over $\mathbb{T}\times\mathbb{R}$, a and b are defined in (2.1), $\sigma:\mathbb{T}\to\mathbb{T}$ is a forward jump operator given in (2.2) and $u\in W_0^{1,p(t)}(\mathbb{T})$.

Definition 5.1. Function $u\in W_0^{1,p(t)}(\mathbb{T})$ is a weak solution to the problem given in (5.1) if

$$\int_{\mathbb{T}}|\Delta^w u(t)|^{p(t)-2}\Delta^w u(t)\Delta^w v(t)\Delta t=\int_{\mathbb{T}}f(t,u^{\sigma}(t))v^{\sigma}(t)\Delta t\tag{5.2}$$

for every $v\in W_0^{1,p(t)}(\mathbb{T})$.

Note that the definition of the weak solution to the Dirichlet problem on time-scale is different from the classical one. The definition of Δ-derivative implies that we multiply both sides of (5.1) by the superposition $v\circ\sigma$ in order to integrate it by parts. Apart from this the Δ-integral on the left side of (5.2) can not be simply considered as the classical Lebesgue integral. It is caused by lack of equivalence between the weak derivative of the extended function $\widehat{u}$ and the extension of Δ-weak derivative of function u defined over time-scale.

Example 5.2. Let $\mathbb{T}=[1,3]\cup[4,5]$ and $u(t)=t$ for $t\in\mathbb{T}$. Then

$$\widehat{\Delta^w u}(t)=1$$

for $t \in [1,5]$ and

$$D\widehat{u}(t) = \begin{cases} 1 & \text{for } t \in [1,3) \cup (4,5], \\ 0 & \text{for } t \in (3,4), \end{cases}$$

where $D\widehat{u}$ denotes the weak derivative of the extended function $\widehat{u} : [1,5] \to \mathbb{R}$.

Definition 5.3 ([19, p. 500])**.** Let X be real Banach space and $L : X \to X^*$ be an operator. Then

(a) L is called uniformly monotone if

$$(Lu - Lv)\,(u - v) \geq f_1(\|u - v\|_X)\|u - v\|_X \text{ for all } u, v \in X,$$

where $f_1 : [0,\infty) \to [0,\infty)$ is continuous, strictly monotone increasing with $f_1(0) = 0$ and $\lim_{t\to\infty} f_1(t) = \infty$,

(b) L is called coercive if

$$\lim_{\|u\|_X \to \infty} \frac{(Lu)\,(u)}{\|u\|_X} = \infty,$$

(c) L is called hemicontinuous if the real function $t \to L(u + tw, v)$ is continuous on $[0,1]$ for all $u, w, v \in X$.

Theorem 5.4 ([19, p. 554])**.** *If the operator $L : X \to X^*$ is monotone, coercive and hemicontinuous on the reflexive Banach space X, then L is surjective.*

Theorem 5.5 ([19, p. 557])**.** *Let $L : X \to X^*$ be a uniformly monotone, coercive and hemicontinuous operator on the real, reflexive Banach space X. Then the inverse operator $L^{-1} : X^* \to X$ exists and it is continuous.*

Lemma 5.6. *There exists function $f_1 : [0,\infty) \to [0,\infty)$ which is continuous, strongly increasing, $f_1(0) = 0$ and $\lim_{t\to} f_1(t) = \infty$ such that*

$$\rho(u) = f_1(\|u\|_\rho)\|u\|_\rho \quad \textit{for all } u \in L^{p(t)}\,(\mathbb{T})\,.$$

Proof. Let $u \in L^{p(t)}\,(\mathbb{T})$. Then Theorem 3.5 enables us to conclude that $\rho(u) \leq \|u\|_\rho^{p^+-1}\|u\|_\rho$ if $\|u\|_\rho \geq 1$ and $\rho(u) \leq \|u\|_\rho^{p^--1}\|u\|_\rho$ if $\|u\|_\rho < 1$. We define function

$$f_1(t) = \begin{cases} t^{p^--1}, & \text{if } t < 1, \\ t^{p^+-1}, & \text{if } t \geq 1, \end{cases}$$

which is the desired function. □

Lemma 5.7. *The norm $\|u\|_\Delta = \|\Delta^w u\|_\rho$ is an equivalent to the norm $\|u\|_{1,p(t)}$ in $W_0^{1,p(t)}(\mathbb{T})$.*

Proof. It is sufficient to show that there exists $c \in \mathbb{R}$ such that $\|u\|_\rho \leq c\|\Delta^w u\|_\rho$. Indeed, since $p^+ < \infty$ we may find functions $p_i \in C_{rd}(\mathbb{T})$ for $i = 0, 1, 2, \ldots, k$ such that $p(t) =: p_0(t) \geq p_1(t) \geq p_2(t) \geq \ldots \geq p_k(t) := 1$ for $t \in \mathbb{T}$. Since embeddings

$W^{1,p_{i+1}(t)}(\mathbb{T}) \hookrightarrow L^{p_i(t)}(\mathbb{T})$ are continuous for $i = 0, 1, \dots, k-1$ we have that there exist $C_0, C_1, \dots, C_{k-1}, \widetilde{C}_0, \dots, \widetilde{C}_{k-1} > 0$ such that

$$\begin{aligned}
\|u\|_{L^{p(t)}(\mathbb{T})} &\le C_0 \left(\|u\|_{L^{p_1(t)}(\mathbb{T})} + \|\Delta^w u\|_{L^{p_1(t)}(\mathbb{T})}\right) \\
&\le C_0\|u\|_{L^{p_1(t)}(\mathbb{T})} + \widetilde{C_0}\|\Delta^w u\|_{L^{p(t)}(\mathbb{T})}, \\
\|u\|_{L^{p_1(t)}(\mathbb{T})} &\le C_1 \left(\|u\|_{L^{p_2(t)}(\mathbb{T})} + \|\Delta^w u\|_{L^{p_2(t)}(\mathbb{T})}\right) \\
&\le C_1\|u\|_{L^{p_2(t)}(\mathbb{T})} + \widetilde{C_1}\|\Delta^w u\|_{L^{p(t)}(\mathbb{T})}, \\
&\dots \\
\|u\|_{L^{p_{k-1}(t)}(\mathbb{T})} &\le C_{k-1} \left(\|u\|_{L^{p_k(t)}(\mathbb{T})} + \|\Delta^w u\|_{L^{p_k(t)}(\mathbb{T})}\right) \\
&\le C_{k-1}\|u\|_{L^{p_k(t)}(\mathbb{T})} + \widetilde{C}_{k-1}\|\Delta^w u\|_{L^{p_k(t)}(\mathbb{T})}.
\end{aligned}$$

Moreover, one has

$$u(t) = u(a) + \int_a^t \Delta^w u(s)\Delta s \le \left|\int_a^t \Delta^w u(s)\Delta s\right| \le \|\Delta^w u\|_{L^1(\mathbb{T})}$$

for $t \in \mathbb{T}$. Thus, $\sup_{t\in\mathbb{T}} |u(t)| < \|\Delta^w u\|_{L^1(\mathbb{T})}$. Consequently,

$$\|u\|_{L^{p_k(t)}(\mathbb{T})} = \|u\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} u(t)\Delta t \le \mu_\Delta(\mathbb{T})\|\Delta^w u\|_{L^1(\mathbb{T})} \le C_k\|\Delta^w u\|_{L^{p(t)}(\mathbb{T})}$$

with $C_k > 0$. Hence there exists $c \in \mathbb{R}$ such that

$$\|u\|_{L^{p(t)}(\mathbb{T})} \le c\|\Delta^w u\|_{L^{p(t)}(\mathbb{T})}. \tag{5.3}$$

□

We denote

$$X := W_0^{1,p(t)}(\mathbb{T})$$

and consider operator $L : X \to X^*$ defined by

$$(Lu)(v) = \int_{\mathbb{T}} |\Delta^w u(t)|^{p(t)-2}\Delta^w u(t)\Delta^w v(t)\Delta t \tag{5.4}$$

for $v \in X$. For operator L we can formulate the following theorem.

Theorem 5.8. *L is an uniformly monotone, coercive and hemicontinuous operator.*

Proof. Let $u, v, w \in X$. By [8], we know that for every $\alpha, \beta \in \mathbb{R}$ and $p > 2$ we have

$$\left(|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta\right)(\alpha - \beta) \ge \left(\frac{1}{2}\right)^p |\alpha - \beta|^p. \tag{5.5}$$

If we apply (5.5) to $\Delta^w u, \Delta^w v$ and the function p, we obtain

$$(Lu - Lv)(u - v) \geq \frac{1}{2^{p^+}} \int_{\mathbb{T}} |\Delta^w u(t) - \Delta^w v(t)|^{p(t)} \Delta t = \frac{1}{2^{p^+}} \rho(\Delta^w u - \Delta^w v). \quad (5.6)$$

Combining (5.6) with Lemmas 5.6 and 5.7, we obtain uniformly monotonicity. Theorem 3.5 and (5.3) lead to

$$\frac{(Lu)(u)}{\|u\|_{1,p(t)}} = \frac{\rho(\Delta^w u)}{\|u\|_\rho + \|\Delta^w u\|_\rho} \geq \frac{\rho(\Delta^w u)}{c_1 \|\Delta^w u\|_\rho} \geq c_2 \|\Delta^w u\|_\rho^{p^- - 1}$$

with $c_1, c_2 \in \mathbb{R}$. Thus L is coercive. Now we prove hemicontinuity. Let $t_n \in [0,1]$ for $n \in \mathbb{N}$ be such that $t_n \to t \in [0,1]$ as $n \to \infty$. By Theorem 3.7, $u + t_n w \to u + tw$ in Δ-measure. Moreover,

$$\begin{aligned}
&L(u + t_n w)(v) \\
&= \int_{\mathbb{T}} |\Delta^w u(t) + t_n \Delta^w w(t)|^{p(t)-2} (\Delta^w u(t) + t_n \Delta^w w(t)) \Delta^w v(t) \Delta t \\
&\leq \int_{\mathbb{T}} |\Delta^w u(t) + t_n \Delta^w w(t)|^{p(t)-1} \Delta^w v(t) \Delta t \\
&\leq \int_{\mathbb{T}} |\Delta^w u(t) + \Delta^w w(t)|^{p(t)-1} \Delta^w v(t) \Delta t
\end{aligned}$$

for $n \in \mathbb{N}$. Applying Young inequality (3.11) and Corollary 3.14, we have

$$\begin{aligned}
&\int_{\mathbb{T}} |\Delta^w u(t) + \Delta^w w(t)|^{p(t)-1} \Delta^w v(t) \Delta t \\
&\leq \frac{1}{q^-} \int_{\mathbb{T}} \left(|\Delta^w u(t) + \Delta^w w(t)|^{p(t)-1} \right)^{q(t)} \Delta t + \frac{1}{p^-} \int_{\mathbb{T}} |\Delta^w v(t)|^{p(t)} \Delta t \\
&= \frac{1}{q^-} \int_{\mathbb{T}} |\Delta^w u(t) + \Delta^w w(t)|^{p(t)} \Delta t + \frac{1}{p^-} \int_{\mathbb{T}} |\Delta^w v(t)|^{p(t)} \Delta t < \infty
\end{aligned}$$

with function $q : \mathbb{T} \to \mathbb{R}$ such that p, q are conjugative on $\mathbb{T}$. Consequently the Lebesgue Dominated Convergence Theorem implies the hemicontinuity of L. □

Theorem 5.9. *If the function f in problem* (5.1) *does not depend upon u, i.e.*

$$f(t, u^\sigma(t)) = f_2(t),$$

for $t \in \mathbb{T}$ and $f_2 \in L^\alpha(\mathbb{T})$ with $\alpha = \frac{p^-}{p^- - 1}$, then problem (5.1) *has a unique weak solution.*

Proof. According to embedding (4.5) and Theorem 3.18, we obtain that if $f_2 \in L^{\alpha}(\mathbb{T})$ with $\alpha = \frac{p^-}{p^- - 1}$, then the functional $J : X \to \mathbb{R}$ defined by

$$J(v) = \int_{\mathbb{T}} f_2(t) v^{\sigma}(t) \Delta t$$

for $v \in X$, is linear and continuous on X. By Theorems 5.4, 5.5 and 5.8, the operator L defined in (5.4) is a homeomorphism and problem (5.1) has a unique weak solution. □

Acknowledgements
The Authors would like to thank Professor Marek Galewski for suggesting us the topic of this research.

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Received: June 5, 2018.
Revised: December 23, 2018.
Accepted: February 25, 2019.