Amplitude value error of periodic signal as a two-dimensional random variable

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1. Problem formulation

Algorithms used in frequency analysis of periodic signals require application of a time window in order to limit the number of samples. In this paper Dirichlet window will be compared:

$$w(n) = 1 \quad dla \quad n = 0,1,\ldots,N-1 \tag{1}$$

With the so called symmetrical Dirichlet window [2]:

$$ws(n) = \begin{cases} 1 & dla \quad n = 1,2,\ldots,N-1 \\ 0.5 & dla \quad n = 0 \ oraz \ n = N \end{cases} \tag{2}$$

It is assumed that the aim of the frequency analysis of a periodic signal is to determine amplitude values of individual harmonics as well as corresponding frequency values. A tool used in such a situation is the discrete Fourier transform. In reality the signal period T is unknown – sampling is incoherent. The relation between the sampling period T_s and the unknown signal period T is given by [1]:

$$NT_s = (M + a)T \tag{3}$$

where: M, N – whole numbers.

The desynchronization parameter a is a random variable of the values $a \in (-0.5, 0.5)$.

The process of signal sampling also entails the presence of the unknown phase shift between the moment of the initiation of sampling and the phase of an analyzed signal. The value of this shift corresponds to a random variable $t_0 \in (0, T)$.

In the paper, for a harmonic signal with given parameters, simulation studies were conducted which were aimed at determining the amplitude error of subsequent harmonics. The error is a two-dimensional random variable determined on a plane (and). Computer simulations were conducted by performing averaging with respect to particular random variables and determining marginal distributions.

In order to visualize the error, in paper [1] the averaging with respect to the variable t_0 was conducted and the histogram of the error in relation to the parameter $a \in (-0.5, 0.5)$ was presented.

In this paper a different order of averaging was considered, which resulted in the error histogram in the function of random variable

$$t_0 \in (0,T) \tag{4}$$

Furthermore, the amplitude errors resulting from Dirichlet window application (1) and the symmetric Dirichlet window application (2) were compared. In the latter case, an analytical form of the error was determined.

2. DFT for Dirichlet and symmetric Dirichlet windows

As an example the following sampled harmonic signal will be examined:

$$y(n) = \sin(\omega \cdot t_n + \psi) \tag{5}$$

Taking into account:

$$t_n = t_0 + n \cdot T_s\,;\; \omega = \frac{2\pi}{T} \;;\; M=1$$

And equations (3) and (4), the relation (5) obtains the following form:

$$x(n) = \sin(2\pi\,\tau + 2\pi(1+a)\frac{n}{N} + \psi) \tag{6}$$

where: $\tau = \frac{t_0}{T}$; $\tau \in (0,1)$.

The signal (6) is a two-dimensional random variable defined on a plate (a *and* τ).

For the signal (6) the discrete Fourier transform will be determined (DFT):

$$X_k = \frac{2}{N}\sum_{n=0}^{N-1} w(n)x(n)\exp\left(\frac{j2\pi kn}{N}\right) \;;\; k = 0.1,...,\frac{N}{2} \tag{7}$$

where: k – number of a subsequent harmonic.

From equations (1), (6) and (7), we obtain:

$$X_k = \frac{\sin(\pi a)}{N}\,\frac{\cos\left(A - \frac{2\pi(1+a)}{N}\right) - \cos(A)\cos\left(\frac{2\pi k}{N}\right) + j\sin(\left(\frac{2\pi k}{N}\right)\cos(A)}{\sin\left(\frac{\pi(1+a-k)}{N}\right)\sin\left(\frac{\pi(1+a+k)}{N}\right)}$$

where:

$$A = 2\pi\tau + \pi a + \psi \tag{8}$$

In the case of symmetric Dirichlet window, DFT is derived from equation:

$$XS_k = \frac{2}{N}\sum_{n=0}^{N} ws(n)x(n)\exp\left(\frac{j2\pi kn}{N}\right) \;;\; k = 0.1,...,\frac{N}{2} \tag{9}$$

By applying equations (2), (6) and (9) we obtain:

$$XS_k = \frac{\sin(\pi a)}{N} \frac{\sin(A)\sin\left(\frac{2\pi(1+a)}{N}\right) + j\sin\left(\frac{2\pi k}{N}\right)\cos(A)}{\sin\left(\frac{\pi(1+a-k)}{N}\right)\sin\left(\frac{\pi(1+a+k)}{N}\right)} \tag{10}$$

where: A is determined by equation (8).

3. Results of a numerical experiment

Applying random number generators with uniform distribution, 4 million iterations of signal (6) were carried out generating M = 2000 values of parameter a and M = 2000 values of relative time τ. In accordance with equation (8) the DFT was determined and subsequently the errors of amplitudes of particular harmonics were also determined:

$$b(a,\tau)_k = \begin{Bmatrix} |X_k| & dla & k \neq 1 \\ |X_k| - 1 & dla & k = 1 \end{Bmatrix} \tag{11}$$

Error $b(a,\tau)_k$ is a two-dimensional random variable determined on a plate (a,τ) where: $a \in (-0.5, 0.5); \tau \in (0,1)$. While conducting averaging with respect to a (τ) marginal distributions are obtained – one-dimensional random variables.

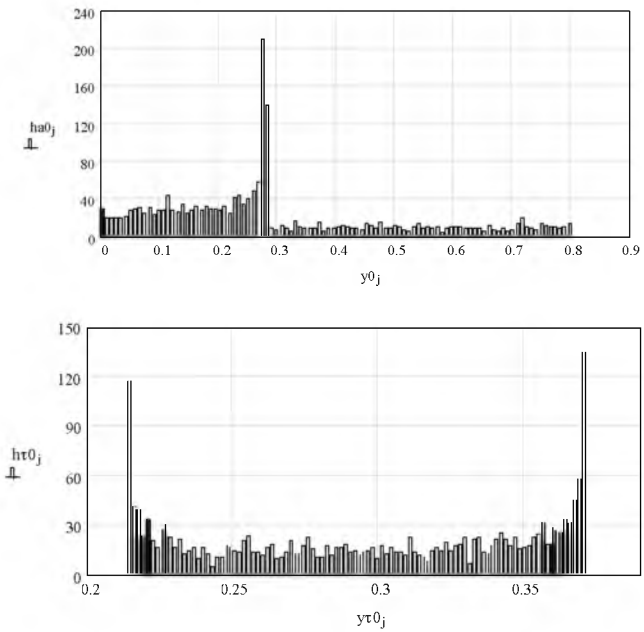


Fig. 1a. Histograms of errors of amplitude value: ha determined for a varying parameter a ($y0$); $h\tau$ calculated for varying time τ ($y\tau0$). DFT results for Dirichlet window – mean value

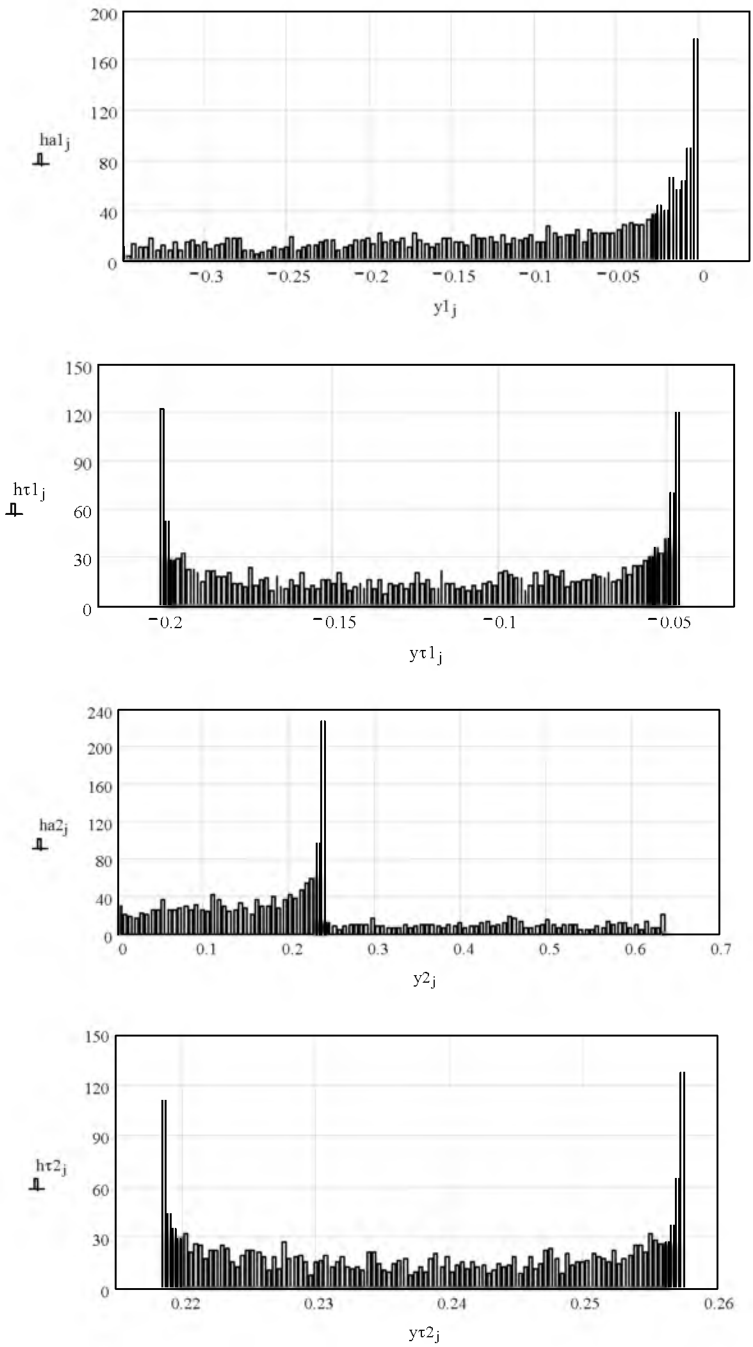


Fig. 1b. Histograms of errors of amplitude value. DFT results for Dirichlet window – the first and the second harmonic

Figure 1 presents histograms of values of amplitude errors h0 (mean value), h1 (first harmonic), h2 (second harmonic) - the results refer to the window of the length N = 16. Histograms ha correspond to varying desynchronization parameter a – values $b(a,\tau)_k$ were initially averaged with respect to time τ. Histograms $h\tau$ correspond to varying values of time τ - averaging with respect to parameter a.

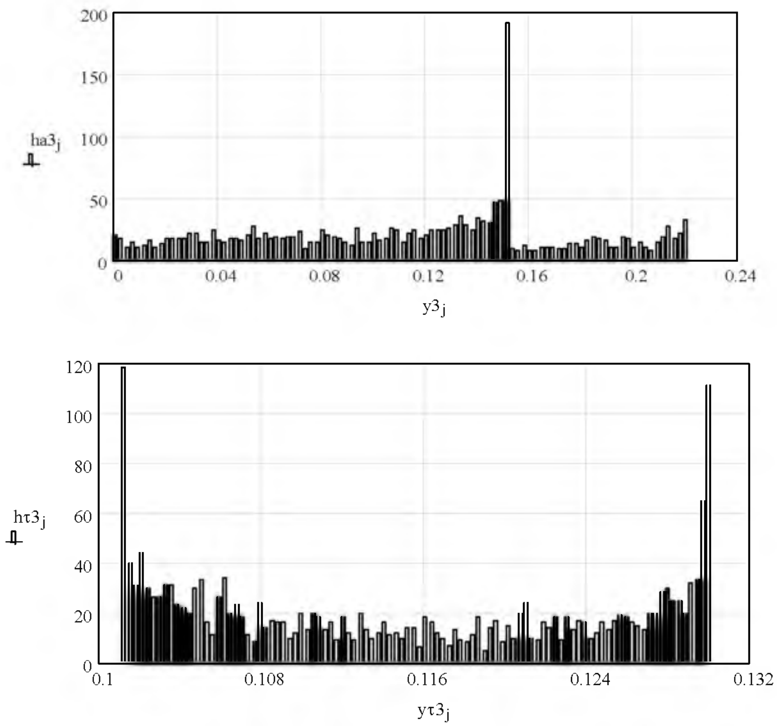


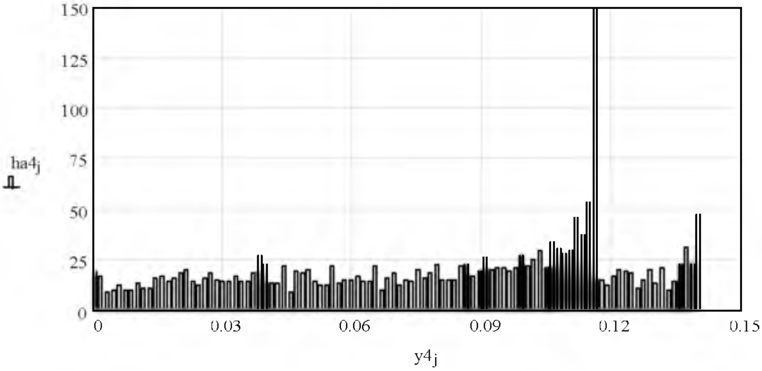
Fig. 1c. Histograms of amplitude value error. DFT results for Dirichlet window – the third harmonic

Figure 1 reveals substantial differences between the histograms ha and $h\tau$. Histogram $h\tau$ is symmetric and presents a rescaled density distribution of a harmonic signal with random phase [3, 4]. The attention should be drawn to the difference in the length of intervals including the error (values y). In the case of histogram ha the interval is substantially longer than for histogram $h\tau$ - particularly visible for the constant component and the second harmonic.

Figures 1c and 1d are the continuation of figures 1a and 1b, whereas they refer to the third and fourth harmonic. From figures1c and 1d the same conclusions can be drawn as for Fig.1a and 1b.

Analogous calculations were conducted for symmetric Dirichlet window:

$$bs(a,\tau)_k = \begin{cases} |XS_k| & dla \quad k \neq 1 \\ |XS_k| - 1 & dla \quad k = 1 \end{cases} \tag{12}$$



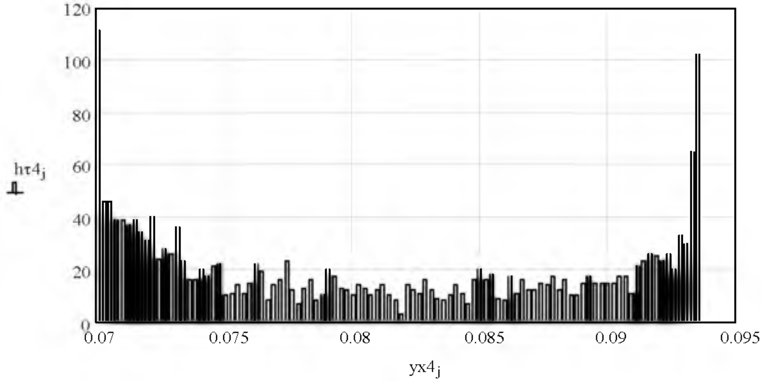


Fig. 1d. Histograms of amplitude value error. DFT results for Dirichlet window – the third harmonic

In the case of symmetric Dirichlet window the obtained histograms were similar to those presented in Fig.1a, 1b and 1c. Differences were observed with reference to Fig. 1d, i.e. for the fourth harmonic. In the case of histogram *ha* the differences are rather subtle and consist in a narrower interval of the varying value of error for histogram *hsa4 (y4`< 0.12*) than for *ha4 (y4 < 0.141).* However in the case of histogram $h\tau$ the intervals containing error values differ substantially: for $h\tau 4 \quad 0{,}070 < y\tau 4 < 0.094$ , whereas for $hs\tau 4 \quad 0{,}054 < y\tau 4\text{`} < 0.073$.

4. Results of numerical calculations

Using the analytical form of amplitude value error determined by means of the DFT method (equation (8) and (10)), boundary values of the error were determined:

$$B(a)_k = \int_0^1 b(a,\tau)_k \, d\tau \quad ; \quad BS(a)_k = \int_0^1 bs(a,\tau)_k \, d\tau \tag{13}$$

$$B`(\tau)_k = \int_{-0,5}^{0,5} b(a,\tau)_k \, da \quad ; \quad BS`(\tau)_k = \int_{-0,5}^{0,5} bs(a,\tau)_k \, da \tag{14}$$

Taking into account the fact that the length of the interval of integration equals 1, equations (13) and (14) determine the averaged value of error with respect to one of the parameters (a or τ).

Figure 2 presents values of these errors determined using the procedure of numerical integration provided by MathCad.

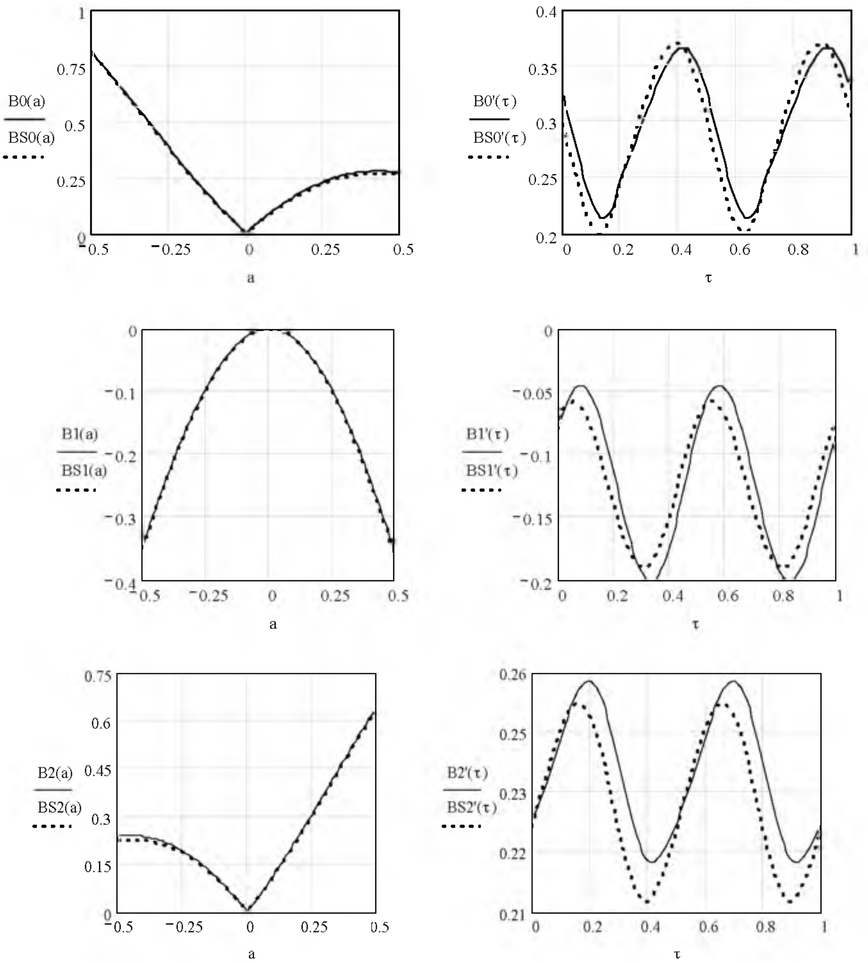


Fig. 2a. Values of amplitude errors as a function of parameter a and time τ -mean value, first and second harmonic

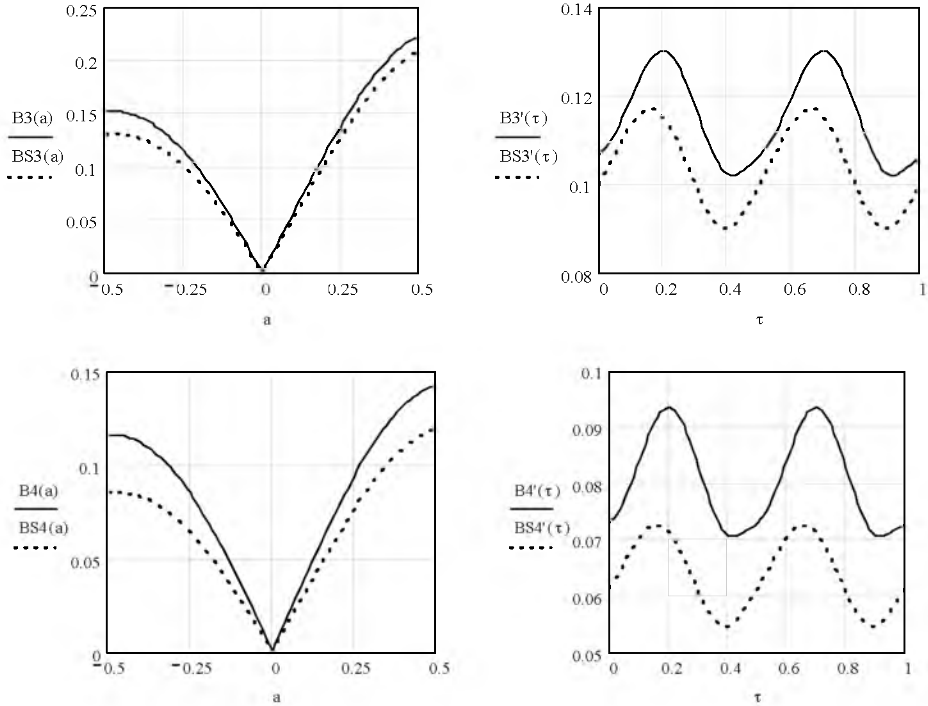


Fig. 2b. Values of amplitude errors as a function of parameter a and time τ - third and fourth harmonic

In figure2 an increasing tendency of error B(a) can be noticed accompanied by an increase in the value of the absolute value of desynchronization parameter |a|. On comparison of curves B(a) and BS(a) comparable results for k = 0,1,2 (zero component, first and second harmonic) can be observed. For the third and fourth harmonic BS(a) < B(a) can be observed, which means that symmetric Dirichlet window yields smaller values of the amplitude error.

As for errors $B'(\tau)$ and $BS'(\tau)$, it can be concluded that they contain a constant and a variable component which can be approximately modeled by a double-frequency sinusoid– two periods of the sinusoid correspond to the interval $\tau \in (0,1)$. It should be noticed that the interval of the varying value of the error $B'(\tau)$ ($BS'(\tau)$) is substantially smaller than for the error B(a) (BS(a)). Comparing the values of errors $B'(\tau)$ and $BS'(\tau)$, comparable values for k = 0 and k=1 can be observed. However for k = 2, 3, 4, $BS'(\tau) < B'(\tau)$ can be observed, which entails smaller values of the amplitude error for the second, third and fourth harmonic determined for symmetric Dirichlet window.

While conducting integration of error $B(a)_k$ ($B`(\tau)_k$) with respect to variable a (τ) we obtain the total mean error BT_k:

$$BT_k = \int_{-0.5}^{0.5} B(a)_k da = \int_0^1 B`(\tau)_k d\tau \tag{15}$$

$$BST_k = \int_{-0.5}^{0.5} BS(a)_k da = \int_0^1 BS`(\tau)_k d\tau \tag{16}$$

Values of errors BT_k and BST_k are presented in Table 1, where they are compared with empirically obtained values (BC_k i BSC_k) for 4 million simulation iterations. Table 1 shows that theoretical values fit closely to empirical ones – BT and BC as well as BST and BSC. Comparing BT and BST values it can be concluded that – apart from the first harmonic – the symmetric Dirichlet window yields smaller error than classic Dirichlet window.

Table 1. Values of amplitude errors of subsequent harmonics

Harmonic number	BC_k	BT_k	BSC_k	BST_k
0	0.297	0.295	0.292	0.290
1	-0.121	-0.122	-0.123	-0.124
2	0.238	0.239	0.232	0.233
3	0.115	0.115	0.104	0.104
4	0.081	0.081	0.063	0.064

Figure 3 presents theoretical values of errors of particular harmonics derived from equations (15) and (16), where symbol BS refers to the symmetric window. Interesting is the fact that the error of the first harmonic is negative (B1(N) < 0; BS1(N) < 0), which means that the method understates the value of an estimated amplitude.

As far as other harmonics are concerned, BS(N) < B(N) occurs, which means that the use of the symmetric window yields slightly smaller error than the classic window.

As the length of the window increases, the error of the amplitude value approaches the constant value.

However strange it may seem, it is the consequence of the previously made assumptions, namely, desynchronization parameter $a \in (-0.5, 0.5)$ and the random variable $\tau = \frac{t_0}{T}$ is included in the interval $\tau \in (0,1)$. In reality and in accordance with equation (3), an increase of number N entails an increase of whole number M. However a signal described by equation (6) was defined for M = 1. Furthermore, equations (8) and (10) as well as (15) and (16) were determined for such a value of

M. Therefore, the generalization that the value of error gets stabilized as N increases cannot be made. Figures 3a and 3b are presented in order to compare the values of amplitude error of both the symmetric and classic windows.

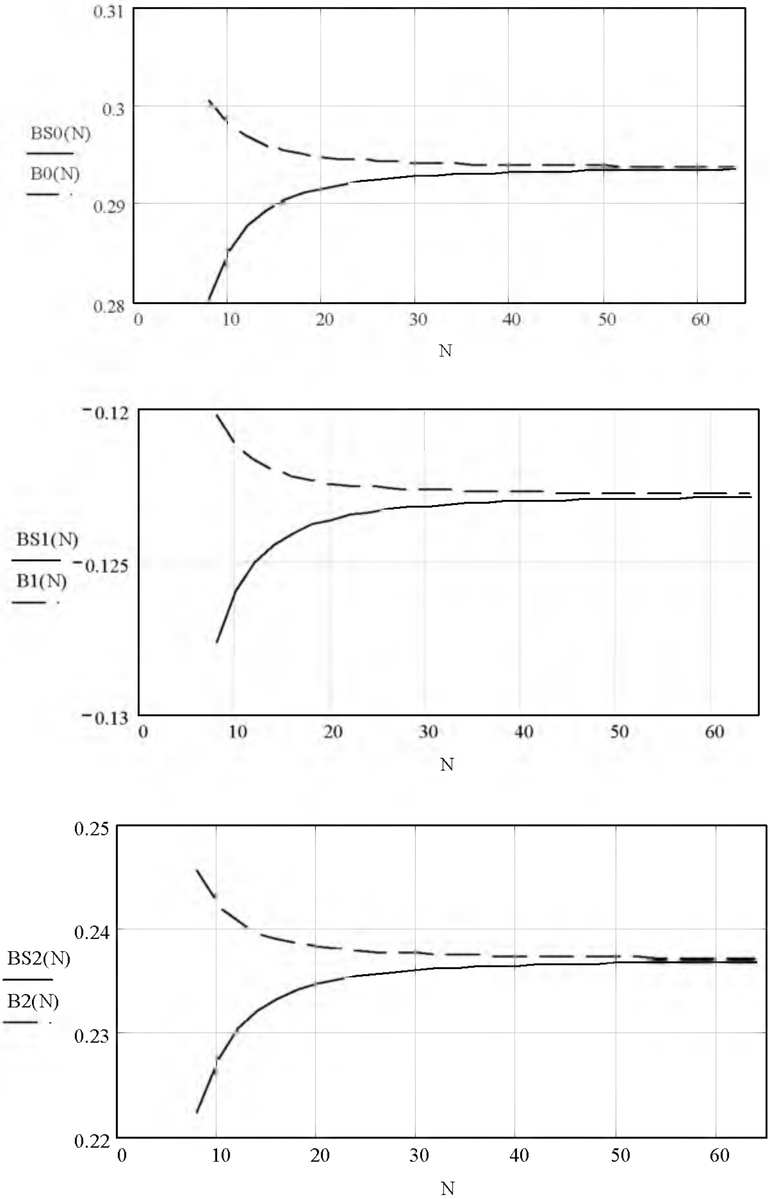


Fig. 3a. Values of amplitude errors of particular harmonics as a function of window length *N*– mean value, first and second harmonic

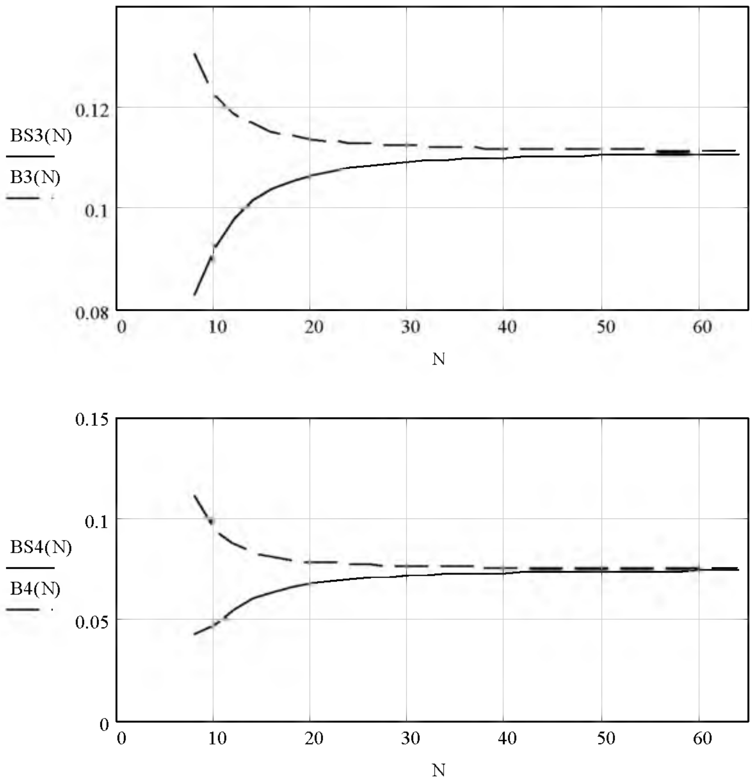


Fig. 3b. Values of amplitude errors of particular harmonics as a function of window length *N*– third and fourth harmonic

5. Summary

This paper is concerned with the error of the amplitude value of a periodic signal determined by means of the DFT induced by incoherent sampling and unknown phase shift at the moment of sampling. The error is a two – dimensional random variable dependent on desynchronization parameter $a \in (-0.5, 0.5)$ and shift $t_0 \in (0, T)$ where T- signal period. Computer simulations were conducted by performing averaging with respect to particular random variables and determining marginal distributions.

Figure 1 reveals substantial differences between the histograms ha and $h\tau$. Histogram $h\tau$ is symmetric and presents a rescaled density distribution of a harmonic signal with random phase. The attention should be drawn to the difference in the length of intervals including the error (values y). In the case of histogram ha the interval is substantially longer than for histogram $h\tau$ - particularly visible for the constant component and the second harmonic.

Figure 2 presents values of errors averaged with respect to one of the parameters (a or τ). An increasing tendency of error B(a) can be noticed accompanied by an increase in the value of the absolute value of desynchronization parameter |a|. On comparison of curves B(a) and BS(a) comparable results for k = 0,1,2 (zero component, first and second harmonic) can be observed. For the third and fourth harmonic (Fig.2b) BS(a) < B(a) and BS`($\tau$) < B`(τ) can be observed, which means that symmetric Dirichlet window yields smaller values of the amplitude error than the classic window.

The fact that theoretical values fit closely to empirical ones – BT and BC as well as BST and BSC – proves the correctness of numerical calculations (Table 1).

References

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