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COMPARATIVE ANALYSIS OF DYNAMIC PROPERTIES OF PARAMETRIC AND FRACTIONAL-ORDER FILTERS

The paper presents transmission models of a parametric filter with non-periodically variable parameters and a fractional-order filter. The responses of these filters on a unit step excitation have been examined as well as the dependence of filters time responses on their parameters. The obtained results have been illustrated by examples.

Keywords: parametric filter, fractional-order filter, filter dynamic models, LTV filters, dynamic properties.

1. INTRODUCTION

Linear time-varying systems are non-stationary deterministic systems with parameters variable in time. They are also called parametric systems or shortly LTV systems. Those systems are a direct generalization of classical linear and time invariant systems (LTI). Many theoretical works [1–4] as well as practical applications have been devoted to these systems. In particular, LTV systems can be applied in signal processing [5–6] especially in sampling systems, signal filtering and noise reduction [5–7] as well as current compensation in power networks [8] amplifiers [1–2], chaos generators [9], electromagnetic launchers [10], medicine devices [7]. The analysis of LTV systems can be found, among others, in [1–2]. In particular, these works concern parametric systems described by the second or higher order differential equations with time-varying parameters. In the literature, there are some methods of LTV system analysis based on transformation of such equations into equations known from the applied mathematics, e.g. into Riccati equation or Floquet equation [1] as well as Mathieu, Meissner or Hill equations [1]. These methods require the determination of the state matrix having time-varying elements [2], followed by calculation of the generalized eigenvalues and the Wronski's fundamental solution matrix. This approach is very useful in the case of stability [3] or spectral theory, but it does not usually lead to analytic solutions which are expressed in a closed form. Analytic solutions to the above mentioned differential equations exist only in some specific cases [11] which depend strictly on the parametric functions – the waveforms of

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time-varying parameters. In the theory of differential equations the fundamental solutions of the first order equation and some second order equations with time-varying coefficients can be found. On the base of Wronski's fundamental solution matrix one can determine solutions to the equations describing LTV systems [1, 4]. The main advantages of LTV system application include improving the dynamic properties of systems and reducing the transient state. The paper proposes the first order LTV system application in the low pass filter construction. The proper choice of the coefficients of the time-varying cut off frequency $\omega(t)$ allows to shorten the transient state of the system response.

One of the possibilities of passive filters realization is also the implementation of fractional-order elements L_β , C_α . Nowadays, there are a lot of works devoted to the analysis of systems with fractional-order elements, e.g. [12–13]. The most frequently used fractional-order elements in electrical engineering and electronics are supercapacitors [14] and coils with soft ferromagnetic cores [15]. Systems with fractional-order elements find numerous applications in electronics, among others, in the construction and modeling of, e.g. generators and oscillators [16], energy storage systems for electric vehicles, hybrid vehicles based on fuel cells, and models of batteries and fuel cells [17]. The fractional-order inductors are most often used in the construction of e.g. relays, loudspeakers [15] and many others. For over a dozen years, there has been a growing interest in the possibilities of realization of analog and digital fractional-order filters and analysis of their properties [18–21], e.g. their frequency characteristics [20]. Works are also being carried out on the realization of fractional-order filters, in order to replace capacitors C_α and fractional-order coils L_β by electronic active circuits, e.g. with the use of circuits with operational amplifiers [19] or with MOS transistors [21]. Potential and promising application of fractional-order electronic filters realized in such method is biomedical engineering, as well as medical and biological sciences [21].

The aim of this paper is to compare the dynamic properties of a parametric filter with non-periodically variable parameters and fractional-order filter based on their unit step response and the determination of reaching the steady state reaching by these filters, depending on their coefficients.

2. LTV FILTER MODEL

The elementary low pass LTV filter (fig.1) is described by the first order parametric differential equation in the following form:

$$\frac{dy(t)}{dt} + \omega(t)y(t) = cu(t), \quad (1)$$

where: $u(t)$, $y(t)$ – input and output signals, $\omega(t)$ – time-variable coefficient called also parametric function, c – constant gain coefficient, usually $c = \omega_0$, where ω_0 denotes cut off angular frequency of a stationary low-pass filter prototype.

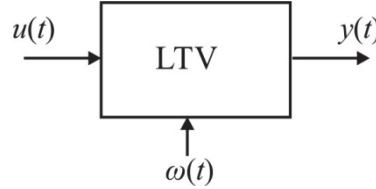


Fig.1. Model of the first order LTV filter

The $\omega(t)$ is a parametric function and it can be interpreted as a cut off angular frequency which depends on time. Here, the exponential waveform of the parameter has been assumed

$$\omega(t) = (\omega_0 + Ce^{-\gamma t}) \mathbf{1}(t), \quad \omega_0 > 0, C \in R, \gamma \in R^+, \quad (2)$$

where: ω_0 – cut off angular frequency of a stationary prototype, C – maximum or minimum value of $\omega(t)$, γ – variability rate, $\mathbf{1}(t)$ – unit step.

Another method of LTV system expression in the time domain is a parametric convolution with an impulse response as a kernel [10]:

$$y(t) = \int_0^t h(t, \tau) u(\tau) d\tau, \quad (3)$$

where: $h(t, \tau)$ - system impulse response.

It is worth to notice, that in contrast to classical stationary systems, the impulse response of an LTV system is a function of two variables - time and the moment of switching the input signal to the system.

The solution to parametric differential equation (1) is well known [11] and given by:

$$y(t) = y_0(t) e^{-\alpha(t)} + \int_0^t e^{-\alpha(t)} e^{\alpha(\tau)} u(\tau) d\tau, \quad (4)$$

thus

$$\alpha(t) = \int_0^t \omega(\tau) d\tau = \omega_0 t + \frac{C}{\gamma} (1 - e^{-\gamma t}). \quad (5)$$

For zero initial condition equation (4) can be written as:

$$y(t) = \int_0^t e^{-\omega_0(t-\tau)} e^{\frac{C}{\gamma}(\exp(-\gamma t) - \exp(-\gamma \tau))} u(\tau) d\tau. \quad (6)$$

From the comparison of equation (6) and the parametric convolution expressed by (2) it can be concluded that the impulse response of the considered system is given as:

$$h(t, \tau) = \omega_0 e^{-\omega_0(t-\tau)} e^{\frac{C}{\gamma}(\exp(-\gamma t) - \exp(-\gamma \tau))}. \quad (7)$$

Representing a part of statement (7) by its functional series one gets:

$$e^{-\frac{C}{\gamma} \exp(-\gamma t)} \cong \sum_{k=0}^n (-1)^k \left(\frac{C}{\gamma} \right)^k \frac{e^{-k\gamma t}}{k!}. \quad (8)$$

In the next step, using formulae (6) and (8), one can express the output signal as:

$$y(t) = \left[\omega_0 \int_0^t e^{\frac{C}{\gamma} \exp(-\gamma t)} e^{-\omega_0(t-\tau)} \sum_{k=0}^N (-1)^k \left(\frac{C}{\gamma} \right)^k \frac{e^{-\gamma k \tau}}{k!} u(\tau) d\tau \right] \mathbf{1}(t). \quad (9)$$

The equation (9) describes in the system response to any excitation, for example unit step response (for $u(t)=\mathbf{1}(t)$):

$$y(t) = \left[e^{\frac{C}{\gamma} \exp(-\gamma t)} e^{-\omega_0 t} \sum_{k=0}^N \frac{(-1)^k}{k!} \left(\frac{C}{\gamma} \right)^k \frac{e^{t(\omega_0 - \gamma k)} - 1}{\omega_0 - \gamma k} \right] \mathbf{1}(t) \quad (10)$$

Elementary first-order LTV systems described in this work allow to construct more complex higher order filters, which are stable, if the elementary sections are stable [3].

3. FRACTIONAL-ORDER FILTER MODEL

The model of the corresponding fractional-order filter is shown in fig. 2. It consists of a series resistance R and a supercapacitor, modeled as an ideal fractional-order capacitance C_α .

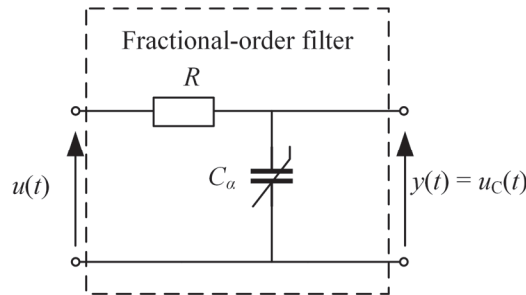


Fig. 2. Model of the fractional-order passive RC_α filter

Model of the fractional-order capacitor, in Laplace domain, is given by a formula:

$$Z_C(s) = \frac{1}{s^\alpha C_\alpha}, \quad \alpha, \beta \in \mathbb{R}, \quad (11)$$

where: C_α – pseudocapacitance, α – fractional-order parameter (dimensionless). It should be noted that the name pseudocapacitance results from the unit of this quantity. It is not farad, as in the case of classic capacitance, but $F/s^{(1-\alpha)}$.

The analyzed filter is described by the differential equation of a fractional order, by the time derivative of the signal $y(t)$:

$$\frac{d^\alpha y(t)}{dt^\alpha} + \frac{1}{RC_\alpha} y(t) = \frac{1}{RC_\alpha} u(t) . \quad (12)$$

The solution of the above fractional-order differential equation is possible using the Laplace transform method, due to the fact that the analyzed system is linear. Using the Laplace transform of a fractional-order derivative defined in the Caputo sense [22]:

$$\mathcal{L}\{ {}^C_0 D_t^\alpha f(t) \} = s^\alpha F(s) - \sum_{k=0}^n s^{\alpha-k-1} f^{(k)}(0) , \quad (13)$$

for zero initial condition, the Laplace transform of the output voltage of the fractional-order RC_α filter is described by a formula:

$$Y(s) = U_C(s) = \frac{1}{RC_\alpha} U(s) \frac{1}{\left(s^\alpha + \frac{1}{RC_\alpha} \right)} . \quad (14)$$

Laplace transform of impulse response $H(s)$ is described by the formula:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{RC_\alpha} \frac{1}{\left(s^\alpha + \frac{1}{RC_\alpha} \right)} . \quad (15)$$

It can be proved [22], that the inverse transform of a function:

$$G(s) = \frac{k! s^{\nu-\mu}}{(s^\nu \mp a)^{k+1}} , \quad (16)$$

can be described by the following relation:

$$\mathcal{L}^{-1}\{G(s)\} = t^{\nu k + \mu - 1} E_{\nu, \mu}^{(k)}(\pm at^\nu) , \quad (17)$$

where: $E_{\nu, \mu}^{(k)}(\pm at^\nu)$ – classic k -th order two-parameter Mittag-Leffler function derivative described by the following series:

$$E_{\nu, \mu}^{(k)}(\pm at^\nu) = \sum_{k=0}^{\infty} \frac{(\pm at^\nu)^k}{\Gamma(\nu k + \mu)} . \quad (18)$$

Hence, in specific case, for $\alpha = \nu = \mu$, $k = 0$, the formula (18) takes the form:

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^\alpha + a} \right\} = t^{\alpha-1} E_{\alpha, \alpha}(-at^\alpha) . \quad (19)$$

Using the formula (19), the convolution and Laplace transform linearity theorems, the relationship (15) can be written in time domain in a general form for arbitrary type of excitation:

$$y(t) = \left(\frac{1}{RC_\alpha} \sum_{k=0}^{\infty} \left(-\frac{1}{RC_\alpha} \right)^k \frac{1}{\Gamma(\alpha(k+1))} \int_0^t u(t-\tau) \tau^{\alpha(k+1)-1} d\tau \right) \mathbf{1}(t), \quad (20)$$

where: $\Gamma(\alpha(k+1))$ – special gamma Euler function.

The impulse response of the filter is determined by the relation:

$$h(t) = \frac{1}{RC_\alpha} \sum_{k=0}^{\infty} \left(-\frac{1}{RC_\alpha} \right)^k \frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} \mathbf{1}(t). \quad (21)$$

For specific case of excitation, e.g. for a unit step $u(t) = \mathbf{1}(t)$, the relation (20) takes the following form:

$$y(t) = \frac{1}{RC_\alpha} t^\alpha \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{RC_\alpha} t^\alpha \right)^k}{\Gamma(\alpha(k+1)+1)} \mathbf{1}(t). \quad (22)$$

The introduced relations have been then illustrated with a simulation example and compared to the dynamic properties of a similar parametric filter with non-periodically variable parameters.

4. SIMULATION STUDIES

The first order low-pass LTV filter presented in fig. 1 with a varying parameter $\omega(t)$ has been analyzed. The variability is expressed by equation (2), assuming that the cut off angular frequency of a stationary prototype is equal $\omega_0 = 1$ rad/s.

In the first diagram the cut off angular frequency variation is presented, while in the second one – there are the filter responses. To compare, the dot line represents the step response of a classical LTI (linear time invariant) filter with a constant parameter ω_0 . It is easy to notice, that when the parametric function achieves its steady value ω_0 , the LTV system becomes an equivalent to a classic first order low-pass filter with a constant value of the cut off angular frequency.

For the fractional-order low-pass RC_α filter, as in fig. 2, the following parameters have been chosen: the resistance $R = 10 \Omega$ and a fractional-order capacitor (supercapacitor) of pseudocapacitance $C_\alpha = 0.1 \text{ F/s}^{(1-\alpha)}$, so that the filter cut off angular frequency (for the coefficient $\alpha = 1$) equals $\omega_0 = 1$ rad/s. Simulations of the filter response have been performed on the unit step voltage excitation, and the waveforms of output signal $y(t)$ have been presented in fig. 4.

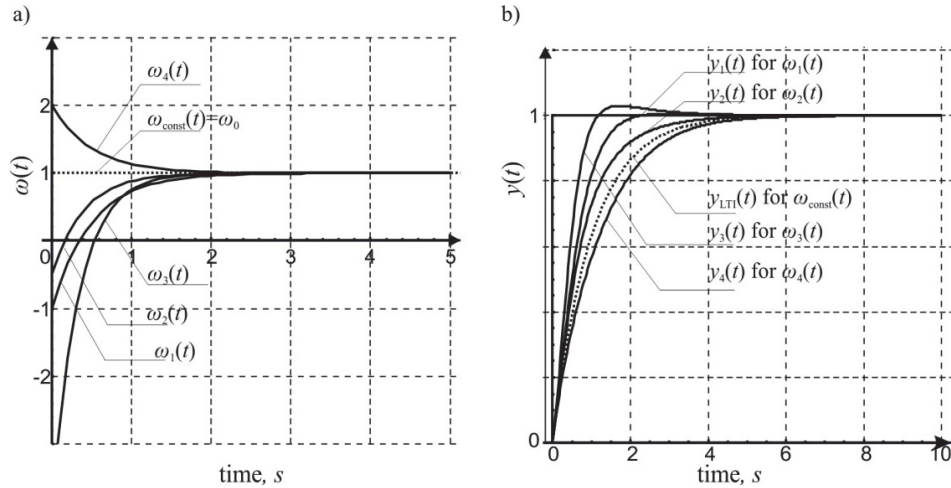


Fig. 3. The calculation results: a) waveforms of the parametric function, b) corresponding waveforms of the output signal

As it can be noticed from the waveforms from fig. 4, the value of coefficient α can shape the output response of the signal. The lower the value of parameter α is, the longer time it takes for the system to reach the set value. For $\alpha > 1$, the system response becomes oscillating, despite the fact, that there is only one reactance element included in the low-pass filter. It is caused by the fact, that the order of the differentiation is higher than 1. For $\alpha = 1$, the response of the system is described by the classic integer-order differential equation.

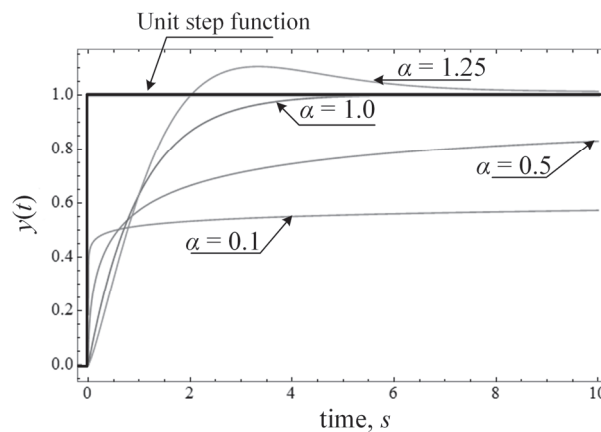


Fig. 4. The simulation results of the output signal $y(t)$ waveform for the fractional-order filter

For both simulation examples, the mean square error (23) has been adopted as a quality criterion for assessing the dynamic properties of the filters. The results of the obtained errors are summarized in table 1.

$$MSE = \frac{\int_0^t (\mathbf{1}(t) - y(t))^2 dt}{\int_0^t \mathbf{1}(t)^2 dt} \cdot 100\%. \quad (23)$$

Table 1. Mean square error of the filters response for the considered simple low-pass filters.

LTI	LTV			Fractional-order		
MSE, %	$\omega(t)$		MSE, %	α	MSE, %	
	C	γ				
5.0	$\omega_1(t)$	-2.0	2.0	3.6	0.50	8.7
	$\omega_2(t)$	-1.5	2.5	4.1	0.70	6.1
	$\omega_{\text{const}}(t)$	0.0	any	5.0	1.00	5.0
	$\omega_3(t)$	-5.0	3.0	2.9	1.25	5.6
	$\omega_4(t)$	1.0	2.0	5.8	1.50	7.7

The numerical tests presented above, show that the dynamic properties of parametric filters are clearly better than of the fractional-order filters. The mean square error is much lower for the parametric filter than for the fractional-order filter. In case of fractional-order filter, for too low value of the parameter α , the time constant becomes very big, and the signal needs very long time to reach the unit step value. However, the correctness of this conclusion for other types of excitations, more complex filter structures and other quality criteria requires further analysis.

5. SUMMARY

The aim of the paper was to compare the dynamic properties of parametric filters with non-periodic variable parameters and fractional-order filters. The comparison based on determining the filter response for unit step excitation. The numerical tests show that the dynamic properties of parametric filters are clearly better than of fractional-order filters. It should be noted, however, that the above conclusion concerns the filter response to unit step. The correctness of this conclusion for other types of excitations, more complex filter structures and other quality criteria requires further analysis.

Time-varying filter parameters and fractional-order filters as well allow to design more flexible systems for which the properties can be designed more freely. Changes in the waveform of the parameter can be used to shape the sys-

tem response. For example, according to the variability of the parametric function, the cut off frequency may be varied in time. In every period, when the parametric function achieves its steady value, the LTV system becomes an equivalent to a classic first order low-pass filter with a constant value of the cut off angular frequency.

In fractional-order filters, the coefficient α is a parameter, that can shape the output response of the signal. The lower the value of parameter α is, the longer time it takes for the system to reach the set value. For $\alpha > 1$, the system response is oscillating, despite the fact, that there is only one reactance element included in the low-pass filter. The reason is the value of derivative order in the fractional-order differential equation. The cut off angular frequency depends not only on the parameters R and C_α , but also on the fractional-order parameter α . In case $\alpha = 1$, the response of the system is described by the classic integer-order differential equation of the low-pass RC filter.

In the era of applying numerical approach to solving of almost everything, it must be reminded that the closed form solutions are important and cannot be ignored or neglected. Their importance comes from their role in understanding of qualitative features of phenomena and processes described by differential equations or impulse response functions. Moreover, even if there is no clear physical meaning of a closed form solution, it can still be used to verify the convergence and evaluate the errors of numerical algorithms as well as asymptotic and approximate analytical methods. In the case of the presented model, the numerical solution was also implemented in order to compare the analytical and the numerical results.

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