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SEMI-MARKOV RELIABILITY MODEL OF TWO DIFFERENT UNITS COLD STANDBY SYSTEM

ABSTRACT

A semi-Markov stochastic process is used for solving in a reliability problem in the paper. The problem concerns of two different component cold standby system and a switch. To obtain the reliability characteristic and parameters of the system we construct so called an embedded semi-Markov process in the process describing operation process of the system. In the model the conditional time to failure of the system is represented by a random variable denoting the first passage time from the given state to the specified subset of states. We apply theorems of the Semi-Markov processes theory concerning the conditional reliability functions to calculate the reliability function and mean time to failure of the system. Often an exact reliability function of the system by using Laplace transform is difficult to calculate, frequently impossible. The semi-Markov processes perturbation theory, allows to obtain an approximate reliability function of the system in that case.

Key words:
reliability, Semi-Markov process, cold standby system.

INTRODUCTION

A model presented here is an extension of the models that have been considered by Barlow and Proshan [1], Brodi and Pogosian [3], Koroluk and Turbin [8] and Grabski [5, 6]. This model was presented in conference of ASMDA 2017 in London. Abstract of the presentation is located in Book of Abstracts [2].

We assume that the system consists of one operating unit A, the stand-by unit B that may have different probability distributions of the lifetimes. We suppose that there is an unreliable switch in the system which is used at the moment of

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the working unit failure. A discrete state space and continuous time stochastic process describes work of the system in reliability aspect. To obtain the reliability characteristics and parameters of the system we construct so called *an embedded semi-Markov process* in this process by defining the renewal kernel of that one. Construction of the renewal kernel is an important first step in solving the problem. This method was presented in [3, 5, 6]. The conditional time to failure of the system is described by a random variable that means the first passage time from the given state to the specified subset of states. To obtain the conditional reliability functions of the system we use appropriate system of integral equations. Passing to the Laplace transforms we get system of linear equations for transforms. The solution are Laplace transforms of the conditional reliability functions of the system. Applying property of Laplace transform we compute the mean time to failure of the system. Very often calculating an exact reliability function of the system by using Laplace transform is a complicated matter but there is a possibility to apply the theorem of the theory of the Semi-Markov processes perturbation [4, 7, 8] to obtain an approximate reliability function of the system. We use Pavlov and Ushakov [9] concept of the perturbed SM process, which is presented in [4] by Gertsbakh.

DESCRIPTION AND ASSUMPTIONS

We assume that the system consists of one operating unit A, the stand-by unit B and a switch. We assume that a lifetime of a basic operating unit is represented by a random variable ζ_A, with distribution given by a probability density function (PDF) $f_A(x), x \geq 0$. When the operating unit fails, the spare B is immediately put in motion by the switch. The failed unit is renewed by a single repair facility. A renewal time of a unit A is a random variable γ_A having distribution given by a cumulative distribution function (CDF) $H_A(x) = P(\gamma_A \leq x), x \geq 0$. Lifetime of the unit B is a random variable ζ_B, with PDF $f_B(x), x \geq 0$. When unit B fails, the unit A immediately starts to work by the switch (if it is 'up') and unit B is repaired. A renewal time of the unit B is a random variable γ_B having distribution given by the CDF $H_B(x) = P(\gamma_B \leq x), x \geq 0$.

Let U be a random variable having a binary distribution

$$b(k) = P(U = k) = a^k(1-a)^{1-k}, k = 0,1, 0 < a < 1,$$

where $U = 0$, if a switch is failed at the moment of the operating unit failure, and $U = 1$, if the switch work at that moment.

The failure of the system takes place when the operating unit fails and the component that has failed sooner is not still ready to work or when both the operating unit and the switch have failed.

After failure the entire system is renewed. A renewal time of whole system is random variable with distribution given by a cumulative distribution function (CDF)

$$H(x) = P(\gamma \leq x), x \geq 0.$$

Moreover we assume that all random variables, mentioned above are mutually independent.

CONSTRUCTION OF SEMI-MARKOV RELIABILITY MODEL

To describe the operation process of the system in the aspect of reliability, we have to determine the states, the renewal kernel and initial distribution. We introduce the following states:

0 — failure of the whole system because of the switch failure;

1 — failure of the whole system because of the unit B failure during repair period of the unit A;

2 — failure of the whole system because of the unit A failure during repair period of the unit B;

3 — repair of the unit A, unit B is working;

4 — repair of the unit B, unit A is working;

5 — both unit A and unit B are 'up' and unit A is working;

6 — both unit A and unit B are 'up' and unit B is working.

Let $0 = \tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants of the states changes and $\{Y(t): t \geq 0\}$ be a random process with the state space $S = \{0, 1, 2, 3, 4, 5, 6\}$, which keeps constant values on the half-intervals $[\tau_n^*, \tau_{n+1}^*), n = 0,1, \ldots$ and it is right-continuous. This process is not semi-Markov, because a memory-less property is not satisfied for all instants of the state changes of it.

We construct a new random process in a following way. Let $0 = \tau_0$ and $\tau_1, \tau_2, \ldots$ denote *instants of the unit failures or instants of the whole system failure*.

The random process $\{X(t): t \geq 0\}$ determining following way

$$X(t) = Y(\tau_n) \text{ for } t \in [\tau_n, \tau_{n+1}), n = 0, 1, 2, \ldots$$

is the semi-Markov process. This process is called an *embedded semi-Markov process in the stochastic process* $\{Y(t): t \geq 0\}$. The Possible states changes of the process $\{X(t): t \geq 0\}$ are shown in figure 1.

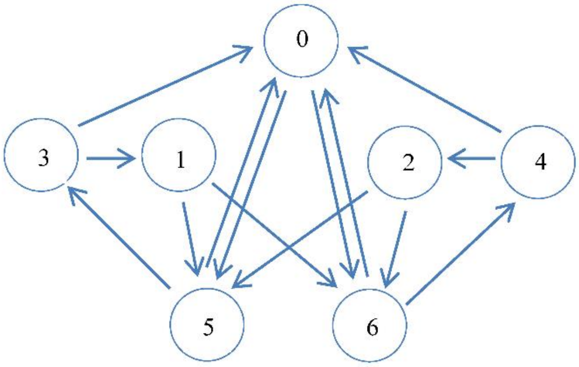

Fig. 1. Possible states changes of the process $\{X(t):\ t \geq 0\}$ [own study]

To define semi-Markov process as a model we have to determine its initial distribution and all elements of its kernel. Recall that the semi-Markov kernel is the matrix of transition probabilities of the Markov renewal process

$$Q(t) = [Q_{ij}(t): i, j \in S], \tag{1}$$

where

$$Q_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t, X(\tau_{n+1}) = j \mid X(\tau_n) = i), t \geq 0. \tag{2}$$

Let's remind that the sequence $\{X(\tau_n): n = 0,1, \dots\}$ is a *homogeneous Markov chain* with transition probabilities

$$p_{ij} = P(X(\tau_{n+1}) = j \mid X(\tau_n) = i) = \lim_{t \to \infty} Q_{ij}(t). \tag{3}$$

The function

$$G_i(t) = P(T_i \leq t) = P(\tau_{n+1} - \tau_n \leq t \mid X(\tau_n) = i) = \sum_{j \in S} Q_{ij}(t) \tag{4}$$

is the CDF distribution of so called *waiting time* T_i , denoting the time spent in state i when the successor state is unknown, the function

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t \mid X(\tau_n) = i, X(\tau_{n+1}) = j) = \frac{Q_{ij}(t)}{p_{ij}} \tag{5}$$

is the CDF of a random variable T_{ij} that is called a *holding time* of a state i, if the next state will be j. It is easy to see that

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \tag{6}$$

The semi-Markov kernel corresponding to the graph that is shown in figure 1 takes the form

$$\boldsymbol{Q}(t)=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & Q_{05}(t) & Q_{06}(t)\\ 0 & 0 & 0 & 0 & 0 & Q_{15}(t) & Q_{16}(t)\\ 0 & 0 & 0 & 0 & 0 & Q_{25}(t) & Q_{26}(t)\\ Q_{30}(t) & Q_{31}(t) & 0 & 0 & Q_{34}(t) & 0 & 0\\ Q_{40}(t) & 0 & Q_{42}(t) & Q_{43}(t) & 0 & 0 & 0\\ Q_{50}(t) & 0 & 0 & Q_{53}(t) & 0 & 0 & 0\\ Q_{60}(t) & 0 & 0 & 0 & Q_{64}(t) & 0 & 0\end{bmatrix}. \tag{7}$$

Moreover we suppose that initial distribution of the process is

$$\boldsymbol{p}(0)=[0\;0\;0\;0\;p\;q]$$

where

$$p,q>0,p+q=1.$$

Construction of the semi-Markov model consists in determining of the matrix $Q(t)$ components on the basis of assumptions. We begin from determining of the transition probabilities from the 'dawn' states.

According to (2) and (3) we have

$$\begin{gathered}Q_{05}(t)=Q_{15}(t)=Q_{25}(t)=p\,H(t)\,;\\ Q_{06}(t)=Q_{16}(t)=Q_{26}(t)=q\,H(t).\end{gathered} \tag{8}$$

Transition probability from the state 3 we calculate the following way:

$$\begin{gathered}Q_{30}(t)=P(U=0,\zeta_B\le t)=(1-a)F_B(t);\\ Q_{31}(t)=P(U=1,\zeta_B\le t,\,\gamma_A>\zeta_B)=a\iint_{C_{31}} f_B(x)dx\,dH_A(y),\end{gathered} \tag{9}$$

where

$C_{31}=\{(x,y): x\le t, y>x\}$

and finally

$$Q_{31}(t)=a\int_0^t f_B(x)[1-H_A(x)]dx. \tag{10}$$

Similarly

$$Q_{34}(t)=P(U=1,\zeta_B\le t,\gamma_A<\zeta_B)=a\iint_{C_{34}} f_B(x)dx\,dH_A(y),$$

where

$C_{34} = \{(x,y): x \leq t, y < x\}$.

Hence

$$Q_{34}(t) = a \int_0^t f_B(x)\, H_A(x) dx \tag{11}$$

in a similar way we get

$$Q_{40}(t) = P(U = 0, \zeta_A \leq t) = (1-a)F_A(t); \tag{12}$$

$$Q_{42}(t) = P(U = 1, \zeta_A \leq t,\ \gamma_B > \zeta_A) = a \int_0^t f_A(x)[1 - H_B(x)]dx; \tag{13}$$

$$Q_{43}(t) = P(U = 1, \zeta_A \leq t, \gamma_B < \zeta_A) = a \int_0^t f_A(x)\, H_B(x)dx; \tag{14}$$

$$Q_{50}(t) = P(U = 0, \zeta_A \leq t) = (1-a)F_A(t); \tag{15}$$

$$Q_{53}(t) = P(U = 1, \zeta_A \leq t) = a\, F_A(t); \tag{16}$$

$$Q_{60}(t) = P(U = 0, \zeta_B \leq t) = (1-a)F_B(t); \tag{17}$$

$$Q_{64}(t) = P(U = 1, \zeta_B \leq t) = a\, F_B(t). \tag{18}$$

All elements of the kernel $Q(t)$ have been defined, hence the semi-Markov process $\{X(t): t \geq 0\}$ describing the reliability of the cold standby system is constructed.

For all states we need to calculate the transition probabilities of the embedded Markov chain and also distributions of the waiting and holding times. Applying (3), (7)–(18) we can determine the transition probabilities matrix of the embedded Markov chain $\{X(\tau_n): n = 0,1, \dots\}$

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & 0 & 0 & p & q \\ p_{30} & p_{31} & 0 & 0 & p_{34} & 0 & 0 \\ p_{40} & 0 & p_{42} & p_{43} & 0 & 0 & 0 \\ 1-a & 0 & 0 & a & 0 & 0 & 0 \\ 1-a & 0 & 0 & 0 & a & 0 & 0 \end{bmatrix}, \tag{19}$$

where

$$p_{30} = 1 - a, p_{31} = a \int_0^\infty f_B(x)[1 - H_A(x)]dx,\ p_{34} = a \int_0^\infty f_B(x) H_A(x)dx;$$

$$p_{40} = 1 - a\ , p_{42} = a \int_0^\infty f_A(x)[1 - H_B(x)]dx, p_{43} = a \int_0^\infty f_A(x) H_B(x)dx.$$

Using formula (4) and equalities (8)-(18) we obtain CDF's of the waiting times for the states $i \in S$.

$$G_0(t) = Q_{05}(t) + Q_{06}(t) = p\,H(t) + q\,H(t) = H(t); \tag{20}$$

$$G_1(t) = Q_{15}(t) + Q_{16}(t) = p\,H(t) + q\,H(t) = H(t); \tag{21}$$

$$G_2(t) = Q_{25}(t) + Q_{26}(t) = p\,H(t) + q\,H(t) = H(t); \tag{22}$$

$$\begin{gathered} G_3(t) = Q_{30}(t) + Q_{31}(t) + Q_{34}(\mathrm{t}) = \\ = (1-a)F_B(t) + a\int_0^t f_B(x)[1 - H_A(x)]dx + a\int_0^t f_B(x)\,H_A(x)dx = F_B(t); \end{gathered} \tag{23}$$

$$\begin{gathered} G_4(t) = Q_{40}(t) + Q_{42}(t) + Q_{43}(\mathrm{t}) = \\ = (1-a)F_A(t) + a\int_0^t f_A(x)[1 - H_B(x)]dx + a\int_0^t f_A(x)\,H_B(x)dx = F_A(t); \end{gathered} \tag{24}$$

$$G_5(t) = Q_{50}(t) + Q_{53}(t) = (1-a)F_A(t) + a\,F_A(t) = F_A(t); \tag{25}$$

$$G_6(t) = Q_{60}(t) + Q_{64}(t) = (1-a)F_B(t) + a\,F_B(t) = F_B(t). \tag{26}$$

Applying the equality (5) and (8)-(19) we calculate CDF's of the holding times.

$$F_{05}(t) = F_{15}(t) = F_{25}(t) = F_{06}(t) = F_{16}(t) = F_{26}(t) = H(t); \tag{27}$$

$$F_{30}(t) = F_B(t), F_{31}(t) = \frac{\int_0^t f_B(x)[1-H_A(x)]dx}{\int_0^\infty f_B(x)[1-H_A(x)]dx}, F_{34}(t) = \frac{\int_0^t f_B(x)H_A(x)dx}{\int_0^\infty f_B(x)H_A(x)dx}; \tag{28}$$

$$F_{40}(t) = F_A(t), F_{42}(t) = \frac{\int_0^t f_A(x)[1-H_B(x)]dx}{\int_0^\infty f_A(x)[1-H_B(x)]dx}, F_{43}(t) = \frac{\int_0^t f_A(x)H_B(x)dx}{\int_0^\infty f_A(x)H_B(x)dx}; \tag{29}$$

$$F_{50}(t) = F_{53}(t) = F_A(t),\ F_{60}(t) = F_{64}(t) = F_B(t). \tag{30}$$

RELIABILITY CHARACTERISTICS

Assume that evolution of a system reliability is describe by a finite states space S semi-Markov process $\{X(t): t \geq 0\}$. Elements of a set S represent the reliability states of the system. Let S_+ consists of the functioning states (up states) and S_- contains all the failed states (down states). The subset S_+ and S_- form a partition of S , i.e., $S = S_+ \cup S_-$ and $S_+ \cap S_- = \emptyset$. Suppose that $i \in S_+$ is an initial state of the process. The conditional reliability function is defined by

$$R_i(t) = P(\forall u \in [0,t], X(u) \in S_+ | X(0) = i), i \in S_+. \tag{31}$$

Let $S_- = D$, and $S_+ = D'$. From the Chapman-Kolmogorov property of a two dimensional Markov chain $\{(X(\tau_n), \tau_n): n = 0,1,2, \dots\}$, we obtain

$$R_i(t) = 1 - G_i(t) + \sum_{j \in D'} \int_0^t R_j(t-u) dQ_{ij}(u), i \in D'. \quad (32)$$

Passing to the Laplace transform we get

$$\tilde{R}_i(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j \in D'} \tilde{q}_{ij}(s) \tilde{R}_j(s), i \in D', \quad (33)$$

where $\tilde{R}_j(s) = \int_0^\infty e^{-st} R_j(t) dt$.

The matrix form of the equation system is

$$(I - q_{D'}(s)) R(s) = W_{D'}(s), \quad (34)$$

where

$R(s) = [\tilde{R}_i(s): i \in D']^T, \quad W_{D'}(s) = [\frac{1}{s} - \tilde{G}_i(s): i \in D']^T$

are one column matrices, and

$q_{D'}(s) = [\tilde{q}_{ij}(s): i, j \in D'], \quad I = [\delta_{ij}: i, j \in D']$

are square matrices. Note that

$$\tilde{G}_i(s) = \frac{1}{s} \sum_{j \in D'} \tilde{q}_{ij}(s).$$

Elements of the matrix $\tilde{R}(s)$ are the Laplace transforms of the conditional reliability functions. We obtain the reliability functions $R_i(t), i \in D'$ by inverting the Laplace transforms $\tilde{R}_i(s), i \in D'$.

Now the equation (33) takes the form

$$\begin{bmatrix} 1 & -\tilde{q}_{34}(s) & 0 & 0 \\ -\tilde{q}_{43}(s) & 1 & 0 & 0 \\ -\tilde{q}_{53}(s) & 0 & 1 & 0 \\ 0 & -\tilde{q}_{64}(s) & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_3(s) \\ \tilde{R}_4(s) \\ \tilde{R}_5(s) \\ \tilde{R}_6(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \tilde{F}_B(s) \\ \frac{1}{s} - \tilde{F}_A(s) \\ \frac{1}{s} - \tilde{F}_A(s) \\ \frac{1}{s} - \tilde{F}_B(s) \end{bmatrix}. \quad (35)$$

The solution is

$$\tilde{R}_3(s) = \frac{\tilde{q}_{34}(s)\left(1 - s\tilde{F}_A(s)\right) + (1 - s\tilde{F}_B(s))}{s(1 - \tilde{q}_{34}(s)\, \tilde{q}_{43}(s))}; \quad (36)$$

$$\tilde{R}_4(s) = \frac{\tilde{q}_{43}(s)\left(1 - s\tilde{F}_B(s)\right) + (1 - s\tilde{F}_A(s))}{s(1 - \tilde{q}_{34}(s)\, \tilde{q}_{43}(s))}; \quad (37)$$

$$\tilde{R}_5(s) = \frac{a\left(1-s\,\tilde{F}_B(s)\right) + a\,\tilde{q}_{34}(s)\left(1-s\,\tilde{F}_A(s)\right)+(1-s\,\tilde{F}_A(s))(1-\tilde{q}_{34}(s)\,\tilde{q}_{43}(s))}{s(1-\tilde{q}_{34}(s)\tilde{q}_{43}(s))}; \tag{38}$$

$$\tilde{R}_6(s) = \frac{a\left(1-s\,\tilde{F}_A(s)\right) +a\,\tilde{q}_{43}(s)\left(1-s\,\tilde{F}_B(s)\right)+(1-s\tilde{F}_B(s))(1-\tilde{q}_{34}(s)\,\tilde{q}_{43}(s))}{s(1-\tilde{q}_{34}(s)\,\tilde{q}_{43}(s))}. \tag{39}$$

The Laplace transform of unconditional reliability function of the system is

$$\tilde{R}(s) = p\,\tilde{R}_5(s) + q\tilde{R}_6(s). \tag{40}$$

A conditional means to failure of the system we can calculate using equalities

$$E(\,\Theta_i) = \lim_{s\to 0^+} \tilde{R}(s), s \in (0,\infty). \tag{41}$$

Therefore, from (38), (39) and (40) we obtain

$$E(\Theta_5) = \frac{a\,E(T_3)+ap_{34}\,E(T_4)+E(T_5)-p_{34}p_{43}\,E(T_5)}{1-p_{34}p_{43}} = E(\zeta_A) + \frac{a\,E(\zeta_B)+ap_{34}\,E(\zeta_A)}{1-p_{34}p_{43}}; \tag{42}$$

$$E(\Theta_6) = \frac{a\,E(T_4)+ap_{43}\,E(T_3)+E(T_6)-p_{34}p_{43}\,E(T_6)}{1-p_{34}p_{43}} = E(\zeta_B) + \frac{a\,E(\zeta_A)+ap_{34}\,E(\zeta_B)}{1-p_{34}p_{43}}. \tag{43}$$

According to (40), (41) and (42) we get the mean time to failure of the system.

$$E(\,\Theta) = p\,E(\zeta_A) + q\,E(\zeta_B) + p\,a\frac{E(\zeta_B)+p_{34}\,E(\zeta_A)}{1-p_{34}p_{43}} + q\,a\frac{E(\zeta_A)+p_{34}\,E(\zeta_B)}{1-p_{34}p_{43}}, \tag{44}$$

where

$$p_{34} = a\int_0^\infty f_B(x)\,H_A(x)dx, p_{43} = a\int_0^\infty f_A(x)\,H_B(x)dx. \tag{45}$$

AN APPROXIMATE RELIABILITY FUNCTION

In general case calculating an exactly reliability function of the system by means of Laplace transforms is a complicated matter. Finding an approximate reliability function of that system is possible by using results from the theory of semi-Markov processes perturbations. The perturbed semi-Markov processes are defined in different ways by different authors. We introduce Pavlov and Ushakov concept of the perturbed semi-Markov process presented by Gertsbakh [4].

Let $D' = S - D$ be a finite subset of states and D be at least countable subset of S. Suppose $\{X(t): t \geq 0\}$ is SM process with the state space $S = D \cup D'$ and the kernel $\mathrm{Q}(t) = [Q_{ij}(t): i,j \in S]$, the elements of which have the form

$$Q_{ij}(t) = p_{ij}F_{ij}(t). \tag{46}$$

Assume that

$$\varepsilon_i = \sum_{j \in D} p_{ij} \quad \text{and}\ p_{ij}^0 = \frac{p_{ij}}{1-\varepsilon_i}, i, j \in D'. \qquad (47)$$

Let us notice that $\sum_{j \in D'} p_{ij}^0 = 1$.

A semi-Markov process $\{X(t): t \geq 0\}$ *with the discrete state space* S *defined by the renewal kernel* $Q(t) = [\, p_{ij} F_{ij}(t): i, j \in S\,]$*, is called the perturbed process with respect to SM process* $\{X^0(t): t \geq 0\}$ *with the state space* D' *defined by the kernel*

$$Q^0(t) = [\, p_{ij}^0 F_{ij}(t): i, j \in D'\,]. \qquad (48)$$

We quote our version of I. V. Pavlov and I. A. Ushakov [9] theorem. The random variable $\Theta_{iD} = \inf\{t: X(t) \in D \mid X(0) = i\}, i \in D'$ denotes the first passage time from the state $i \in D'$ to the subset D. The function $G_i^0(t) = \sum_{j \in D'} Q_{ij}^0(t)$ denotes CDF of the waiting time in the state $i \in D'$. The number $m_i^0 = \int_0^\infty x\, dG_i^0(t), i \in D'$, is the expected value of the waiting time in state i for the process $\{X^0(t): t \geq 0\}$. Denote the stationary distribution of the embedded Markov chain in SM process $\{X^0(t): t \geq 0\}$ by $\pi^0 = [\pi_i^0: i \in D']$. Let

$$\varepsilon = \sum_{i \in D'} \pi_i^0 \varepsilon_i \quad \text{and}\ m^0 = \sum_{i \in D} \pi_i^0 m_i^0. \qquad (49)$$

We are interested in the limiting distribution of the random variable $\Theta_{iD}, i \in D'$. We will quote a theorem, which can be found in the monograph [6] on page 72.

Theorem 1. *If the embedded Markov chain defined by the matrix of transition probabilities* $P = [\, p_{ij}: i, j \in S]$ *satisfies following conditions*:

- $f_{iA} = P(\Delta_D < \infty | X(0) = i) = 1, i \in D', \Delta_D = \min\{n: X(\tau_n) \in D\}$;
- $\underset{i \in D}{\forall}\ \mu_{iD} = \sum_{n=1}^{\infty} n f_{iD}(n) < \infty$;
- $\underset{c>0}{\exists}\ \underset{i,j \in S}{\forall}\ 0 < E(T_{ij}) \leq c$,

then

$$\lim_{\varepsilon \to 0} P(\varepsilon \Theta_{iD} > x) = e^{-\frac{x}{m^0}}, \qquad (50)$$

where $\pi^0 = [\pi_i: i \in D']$ is the unique solution of the linear system of equations

$$\pi^0 = \pi^0\, P^{\,0}, \pi^0\, \mathbf{1} = 1. \qquad (51)$$

The considered SM process $\{X(t): t \geq 0\}$ with the state space $S = \{0,1,2,3,4,5,6\}$ we can assume to be the perturbed process with respect to the SM process $\{X^0(t): t \geq 0\}$ with the state space $D' = \{3,4,5,6\}$ and the kernel

$$Q^0(t) = \begin{bmatrix} 0 & Q_{34}^0(t) & 0 & 0 \\ Q_{43}^0(t) & 0 & 0 & 0 \\ Q_{53}^0(t) & 0 & 0 & 0 \\ 0 & Q_{64}^0(t) & 0 & 0 \end{bmatrix}, \tag{52}$$

where

$Q_{34}^0(t) = p_{34}^0 F_{34}(t), Q_{43}^0(t) = p_{43}^0 F_{43}(t),\ Q_{53}^0(t) = p_{53}^0 F_{53}(t);$

$Q_{64}^0(t) = p_{64}^0 F_{64}(t).$

From (4), (7) and (52) we obtain

$$p_{34}^0 = 1, p_{43}^0 = 1, p_{53}^0 = 1, p_{64}^0 = 1.$$

The transition matrix of the embedded Markov chain of SM process $\{X^0(t): t \geq 0\}$ is

$$P^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Taking under consideration presented above the CDF's $F_{ij}(t)$, $t \geq 0$, we get

$$Q_{34}^0(t) = F_{34}(t) = \frac{\int_0^t f_B(x)\, H_A(x)\, dx}{\int_0^\infty f_B(x) H_A(x)\, dx}; \tag{53}$$

$$Q_{43}^0(t) = F_{43}(t) = \frac{\int_0^t f_A(x)\, H_B(x)\, dx}{\int_0^\infty f_A(x) H_B(x)\, dx}; \tag{54}$$

$$Q_{50}^0(t) = F_{50}(t) = F_A(t), Q_{60}^0(t) = F_{60}(t) = F_B(t)\,.$$

From (19) and (47) we have

$$\varepsilon_3 = p_{30} + p_{31} = 1 - a \int_0^\infty f_B(x) H_A(x) dx; \tag{55}$$

$$\varepsilon_4 = p_{40} + p_{42} = 1 - a \int_0^\infty f_A(x) H_B(x) dx; \tag{56}$$

$$\varepsilon_5 = p_{50} = 1 - a, \varepsilon_6 = p_{60} = 1 - a.$$

From the system of equations

$[\pi_1^0\ \pi_2^0\ \pi_3^0\ \pi_4^0\ ]\ P^0 = [\pi_1^0\ \pi_2^0\ \pi_3^0 \pi_4^0], \pi_1^0 + \pi_2^0 + \pi_3^0 + \ \pi_4^0\ = 1$

we get

$\pi_3^0 = 0.5, \pi_4^0 = 0.5, \pi_5^0 = 0\ , \pi_6^0 = 0\ .$

From (47), (48), (50)–(56) and from the presented above theorem it follows that for small ε

$$R(t) = P(\Theta_{iD} > t)\ =\ P(\varepsilon\Theta_{iD} > \varepsilon t) \approx \exp\left[-\frac{\varepsilon}{m^0}\ t\right], t \geq 0, \tag{57}$$

where

$$\varepsilon = 0.5(\varepsilon_3 + \varepsilon_4) = 1 - 0.5\ a\left(\int_0^\infty\ f_B(x)H_A(x)dx + \int_0^\infty\ f_A(x)H_B(x)dx\right) \tag{58}$$

and

$$m^0 = 0.5(m_3^0 + m_4^0) = 0.5\frac{\int_0^\infty xf_B(x)H_A(x)dx}{\int_0^\infty f_B(x)H_A(x)dx} + 0.5\frac{\int_0^\infty xf_A(x)H_B(x)dx}{\int_0^\infty f_A(x)H_B(x)dx}. \tag{59}$$

From the shape of the parameter ε it follows that we can apply this formula only if the numbers $P(\gamma_B \geq \zeta_A)$, $P(\gamma_A \geq \zeta_B)$ denoting probabilities of the components failure during the repair periods of an earlier failed components are small.

EXAMPLE

We assume that random variables ζ_A, ζ_B, denoting the lifetimes of units A and B, have exponential distributions defined by PDF's

$$f_A(x) =\ \alpha_A e^{-\alpha_A x}, f_B(x) =\ \alpha_B e^{-\alpha_B x}, x \geq 0;$$

$$\alpha_A > 0, \alpha_B > 0.$$

The repair times of the failed units which are represented by the random variables γ_A, γ_B have Erlang distributions with PDF

$$h_A(x) = \mu_A^2\ x\ e^{-\mu_A x}, h_B(x) = \mu_B^2\ x\ e^{-\mu_B x}, x \geq 0;$$

$$\mu_A > 0, \mu_B > 0.$$

Now, the functions that are elements of the semi-Markov kernel (7), are given by the following equalities.

$$Q_{05}(t) = Q_{15}(t) = Q_{25}(t) = p\,H(t)\,;$$

$$Q_{06}(t) = Q_{16}(t) = Q_{26}(t) = q\,H(t), \text{ where } p, q > 0, p + q = 1;$$

$$Q_{30}(t) = P(U = 0, \zeta_B \le t) = (1 - a)(1 - e^{-\alpha_B t});$$

$$Q_{31}(t) = a \int_0^t \alpha_B e^{-\alpha_B x}(1 + \mu_A\, x\, e^{-\mu_A x})dx =$$

$$= a \frac{\alpha_B}{(\alpha_B + \mu_A)^2}\left[\alpha_A + 2\mu_B - (\alpha_B + \alpha_B \mu_A\, t + \mu_A\,(2 + \mu_A\, t))e^{-(\alpha_B + \mu_A)\,t}\right];$$

$$Q_{34}(t) = a \int_0^t \alpha_B e^{-\alpha_B x}(1 - (1 + \mu_A\, x)\, e^{-\mu_A x})dx;$$

$$Q_{40}(t) = P(U = 0, \zeta_A \le t) = (1 - a)(1 - e^{-\alpha_A t});$$

$$Q_{42}(t) = P(U = 1, \zeta_A \le t,\, \gamma_B > \zeta_A) =$$

$$= a \frac{\alpha_A}{(\alpha_A + \mu_B)^2}\left[\alpha_B + 2\mu_A - (\alpha_A + \alpha_A \mu_B\, t + \mu_B\,(2 + \mu_B\, t))e^{-(\alpha_A + \mu_B)\,t}\right];$$

$$Q_{43}(t) = P(U = 1, \zeta_A \le t, \gamma_B < \zeta_A) =$$

$$= a \int_0^t \alpha_A e^{-\alpha_A x}(1 - (1 + \mu_B\, x\,)e^{-\mu_B x})dx;$$

$$Q_{50}(t) = P(U = 0, \zeta_A \le t) = (1 - a)(1 - e^{-\alpha_A t});$$

$$Q_{53}(t) = P(U = 1, \zeta_A \le t) = a(1 - e^{-\alpha_A t});$$

$$Q_{60}(t) = P(U = 0, \zeta_B \le t) = (1 - a)(1 - e^{-\alpha_B t});$$

$$Q_{64}(t) = P(U = 1, \zeta_B \le t) = a\,(1 - e^{-\alpha_B t}).$$

From assumption and (44), (45) we obtain the mean time to failure of the system in this case

$$E(\,\Theta) = \; p\,E(\zeta_A) + q\,E(\zeta_B) + p\,a \frac{E(\zeta_B) + p_{34}\,E(\zeta_A)}{1 - p_{34}p_{43}} + q\,a \frac{E(\zeta_A) + p_{34}\,E(\zeta_B)}{1 - p_{34}p_{43}}\,,$$

where

$$E(\zeta_A) = \frac{1}{\alpha_A}, E(\zeta_B) = \frac{1}{\alpha_B}\;;$$

$$p_{34} = a \int_0^\infty \alpha_B e^{-\alpha_B x}(1 - (1 + \mu_A\, x)\, e^{-\mu_A x})dx = \frac{(\mu_A)^2}{(\alpha_B + \mu_A)^2}\;;$$

$$p_{43} = a \int_0^\infty \alpha_A e^{-\alpha_A x}(1 - (1 + \mu_B\, x\,)e^{-\mu_B x})dx = \frac{(\mu_B)^2}{(\alpha_A + \mu_B)^2}.$$

For

$$\alpha_A = 0{,}0002, \qquad \alpha_B = 0{,}0005, \qquad \mu_A = 0{,}06, \qquad \mu_B = 0{,}05\ \left[\frac{1}{h}\right];$$
$$p = 1, \qquad a = 0{,}968$$

we have

$$E(\Theta) = E(\zeta_A) + a\frac{E(\zeta_B)+p_{34}\,E(\zeta_A)}{1-p_{34}p_{43}} = 668862\ [h].$$

CONCLUSIONS

The reliability model of the cold standby system consist of two different units is constructed by using the concept of the embedded semi-Markov process.

Results of semi-Markov process theory allowed us to compute reliability characteristics of the system.

The Laplace transform of unconditional reliability function of the system is

$$\widetilde{R}(s) = p\,\widetilde{R}_5(s) + q\tilde{R}_6(s),$$

where the Laplace transform of conditional reliability functions $\widetilde{R}_5(s)$, $\tilde{R}_6(s)$ are given by (38) and (39).

The mean time to failure of the considered cold standby system depend on of the both components probability distribution of the lifetimes and renewal times and also on initial distribution of the process and the switch reliability

$$E(\Theta) = p\,E(\zeta_A) + q\,E(\zeta_B) + p\,a\frac{E(\zeta_B)+p_{34}\,E(\zeta_A)}{1-p_{34}p_{43}} + q\,a\frac{E(\zeta_A)+p_{34}\,E(\zeta_B)}{1-p_{34}p_{43}};$$

$$p_{34} = a\int_0^\infty f_B(x)H_A(x)\,dx, p_{43} = a\int_0^\infty f_A(x)H_B(x)\,dx.$$

If operating process starts from the state 5 with probability $p = 1$ then mean time to failure is

$$E(\Theta) = E(\zeta_A) + a\frac{E(\zeta_B)+p_{34}\,E(\zeta_A)}{1-p_{34}p_{43}}$$

This results was presented in [6].

If distributions of times to failure and renewal times of components A and B are identical: $f_A(x) = f_B(x) = f(x), H_A(x) = H_B(x) = H(x)$, we obtain result shown in [5].

$$E(\Theta) = E(\zeta) + a\,\frac{E(\zeta)}{1-c}\ \text{where}\ c = a\int_0^\infty f(x)H(x)\,dx.$$

The cold standby causes the increase of the mean time to failure $1+\frac{a}{1-c}$ times in this case.

If moreover the switch is reliable ($a=1$) we get well known result presented in [1, 3, 8].

The approximate reliability function of the system is exponential (58)

$$R(t) \approx \exp\left[-\frac{\varepsilon}{m^0}\,t\right], t \geq 0.$$

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SEMI-MARKOWSKI MODEL SYSTEMU Z REZERWĄ ZIMNĄ ZŁOŻONY Z DWÓCH RÓŻNYCH PODSYSTEMÓW

STRESZCZENIE

Do rozwiązania problemu z zakresu teorii niezawodności został zastosowany proces semi-Markowa. Problem dotyczy tak zwanego systemu z rezerwą zimną, który jest złożony z dwóch różnych podsystemów i przełącznika. Aby uzyskać charakterystyki i parametry niezawodności tego systemu, jako model funkcjonowania systemu konstruujemy proces semi-Markowa — tak zwany proces włożony w inny proces stochastyczny. W naszym modelu czas zdatności systemu jest reprezentowany przez zmienną losową oznaczającą czas pierwszego przejścia z danego stanu do określonego podzbioru stanów. W celu obliczenia funkcji niezawodności i średniego czasu do awarii systemu stosujemy twierdzenia teorii procesów semi-markowskich dotyczące warunkowej funkcji niezawodności. Najczęściej dokładna funkcja niezawodności systemu przy zastosowaniu transformaty Laplace'a jest trudna do wyliczenia. W takim przypadku teoria zaburzonych procesów semi-markowskich pozwala otrzymać przybliżoną funkcję niezawodności systemu.

Słowa kluczowe:
niezawodność, proces semi-Markowa, system z rezerwą zimną.