Computational Methods for Stochastic Differential Equations and Stochastic Partial Differential Equations Involving Standard Brownian and Fractional Brownian Motion

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As more applied science researchers are attempting to use Stochastic Differential Equations (SDEs) as well as Stochastic Partial Differential Equations (SPDEs) in their modeling, especially when involving Fractional Brownian Motion (fBM), one common issue appears: an exact solution cannot always be found. For cases involving SPDEs, exact solutions commonly do not exist and approximation schemes for their solution are typically still in development. Therefore, in this paper, we test various Numerical methods in solving SDEs and SPDEs with standard BM that have non-linear coefficients. In addition we extend our results to problems with fBM.

Key Words: Brownian Motion (BM), fractional Brownian Motion (fBM), SDEs, SPDEs, Numerical Approximations.

Introduction

Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) involving both Brownian Motion BM) or fractional Brownian Motion (fBM) have been becoming more prevalent in applied mathematics and modeling of various systems. Some examples of SDE modeling are, but are not limited to, finance (i.e. Black-Scholes formula), networks (i.e. data transfer in wireless communications), biology (i.e. arrhythmia, brain signaling after a stroke) etc. In many of those cases, years of research and collection of empirical data is performed in order to build an appropriate model. More often than not though, the SDE that best fits the data is an SDE that does not have a simple analytical solution. Therefore the need appears for a consistent numerical method. The applications of SPDEs, while certainly still an emerging phenomenon, include the modeling of strings or surfaces in random media (ie. Objects in moving fluids and surfaces interacting in turbulent environments), solutions to the Stochastic Navier-Stokes Equation (which describes the motion of fluids), the Stochastic Heat Equation (which describes the diffusion of heat about a uniform medium), as well as numerous applications to finance and market behavior modeling.

In chapter 2 we cover some brief preliminaries about BM, fBM, SDEs and SPDEs that are essential for the numerical approximations we intent to use. In chapter 3 we will state the three different methods tested for numerical solutions of SDEs involving BM, present the results of the three methods and identify the best. Once we derive the best method, we extend it to SDEs involving fBM and compare it to an already proposed scheme (I. Lewis). We will also extend our analysis to different methods involving multidimensional SDEs and their associated methods. In chapter 4, we consider approximation schemes for SPDEs and again test different methods, identifying the best. In chapter 5, we state our conclusions.

Preliminaries

What is Brownian Motion (BM)? The honor of the discovery of the BM belongs to the Scottish botanist Robert Brown that originally described it in 1928 [1] as he observed it in the movement of pollen particles floating in liquid. The first one to actually construct the process was the Missourian mathematician Norbert Wiener in 1923. Ergo the process itself is also referred to as Wiener Process.

Definition 2.1 The process $(B(t),\ t \geq 0)$ is a Brownian Motion (BM) if it is a process of independent Gaussian increments with zero first moment, i.e. a standard Brownian Motion over $[0,\ T]$ is a random variable that depends continuously on $t0[0,\ T]$ and satisfies [2]:

$$B(0) = 0 \tag{2.1}$$

with probability 1.

For $0 \leq s < t \leq T$, the random variable given by the increment $B(t) - B(s)$ is $N(0,\ t - s)$. (2.2)

For $0 \leq s < t < u < v \leq T$, the increments $B(v) - B(u)$ and $B(t) - B(s)$ are independent. (2.3)

Some basic properties that are easily attained by the definition above are:

$$E[B(t)] = 0, \text{ from (2.2)} \tag{2.4}$$

$$E[B(t)^2] = t, \text{ from (2.2) and (2.5)} \tag{2.5}$$

Also, for $0 \leq s < t \leq T$ we can write:

$E[B(t).B(s)] = E[B(s)^2 + B(s)(B(t) - B(s)] = s + 0.(E[B(t - s)]) = s$, that is for any $s,t \in [0,\ T]$ we have that:

$$E[B(t).B(s)] = \min(s,\ t) = s \wedge t \tag{2.6}$$

Furthermore, let $\alpha > 0$ and define $C(t) = B(\alpha t)$. Then $E[C(t)] = E[B(\alpha t)] = \alpha E[B(t)] = 0$ and $E[C(t)]^2 = E[B(\alpha t)^2] = \alpha^2 E[B(t)^2] = \alpha^2 t$.

As we are planning to discuss Stochastic Differential Equations with Brownian Motion, we feel the need to also discuss the continuity of the process. To prove continuity we refer to the Kolmogorov theorem as in [3]:

Theorem 1 (Kolmogorov's Continuity theorem)
Let $X = \{X_t\}_{t \geq 0}$ a process that for all $T > 0$ there exist $\alpha, \beta, D > 0$ such that

$E[|X_t - X_s|^{\alpha}] \leq D|t - s|^{1+\beta}$, for $0 \leq s, t \leq T$.
Then there exists a continuous version of X.

A proof of the theorem can be found in [4].

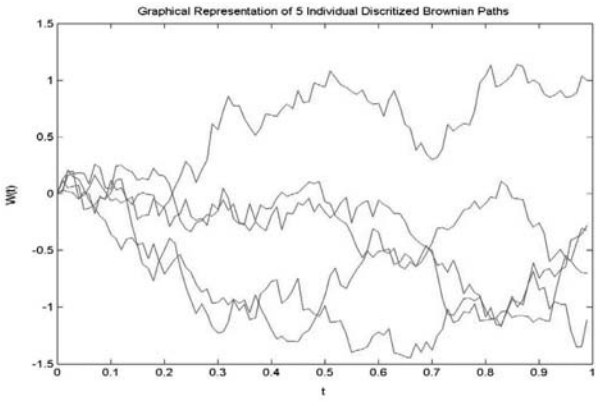


Fig. 1. Standard Brownian Motion Paths.

For Brownian Motion, it can be shown [3] that $E[|B_t - B_s|^4 = n(n+2)|t-s|^2$, which by Theorem 1 we have that B_t has a continuous version. In fact, from now we will be referring to that continuous version of B_t.

As one of the intentions is to investigate numerical approximations of Stochastic Differential Equations, the next natural step is to briefly discuss integration in terms of dB_t. Though there are multiple approaches in various research papers, we are interested in the one shown by D.J. Higham in [2] as in it is more suitable for numerical approximations. Another side benefit of the approach above is that it provides an interesting connection to Classical Riemann calculus. As such, recall the left end--point Riemann sum representation of the Riemann integral given by

$$\int_0^T f(t)dt = lim_{\Delta t \to 0} \sum_0^{N-1} f(t_j)\Delta t, \quad \text{where } \Delta t = t_{j+1} - t_j, \tag{2.7}$$

or using the midpoint

$$\int_0^T f(t)dt = \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(\tfrac{t_{j+1}-t_j}{2})\Delta t \tag{2.8}$$

First we set $f(t) \equiv B(t)$. Therefore we have from (2.7) that

$$\begin{aligned} &\sum_{j=0}^{N-1} B(t_j)[B(t_{j+1}) - B(t_j)] = \\ &= \tfrac{1}{2}\sum_{j=0}^{N-1}\{B(t_{j+1})^2 - B(t_j)^2 - [B(t_{j+1}) - \\ &- B(t_j)]^2\} = \tfrac{1}{2}\{B(T)^2 - B(0)^2 - \\ &- \sum_{j=0}^{N-1}[B(t_{j+1}) - B(t_j)]^2\}, \end{aligned} \tag{2.9}$$

by telescoping series. The second term drops off as it is equal to zero. For the third term, we have that:

$$\begin{aligned} &E[\sum_{j=0}^{N-1}(B(t_{j+1}) - B(t_j))^2 = \\ &= \sum_{j=0}^{N-1} E\left[\left(B(t_{j+1}) - B(t_j)\right)\right]^2 = \\ &= \sum_{j=0}^{N-1}(t_{j+1} - t_j) = T \end{aligned} \tag{2.10}$$

Also, the variance of the third term is of . Therefore by applying limits on both sides of (2.9) we get

$$\int_0^T B(t)sB(t) = \tfrac{1}{2}B(T)^2 - \tfrac{1}{2}T, \tag{2.11}$$

which is the Ito Integral.

By following a similar logic on (2.8) we get

$$\int_0^T B(t)odB(t) = \tfrac{1}{2}B(T)^2, \tag{2.12}$$

which is the Stratonovich Integral.

As explained by Oksendal in [3], even though the two integrals look to be different, the choice of which one to be used is really a matter depending on what properties the user is interested in. The more general and usual choice of usually looking into the Ito Integral is due to the fact that it is not looking into the future, which is a property we care for in Biology. Also Stratonovich is handled better under transformations

and especially on SDEs on manifolds. On the other hand, the Ito integrals are martingales, therefore gaining a computational advantage.

As with classical calculus, we could not possibly apply the above approach every time we need to calculate a stochastic integral. The biggest breakthrough in Stochastic Calculus could possibly be due to Kiyoshi Ito.

Lemma 2.1 (Ito's Lemma)[3]

Let X_t be an Ito process given by

$$dX_t = udt + vdB_t \tag{2.13}$$

Let $g(t,x)\epsilon C^2([0,\infty)\times\mathbb{R})$. Then $Y_t = g(t,X_t)$ is again an Ito process and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \\ + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2, \tag{2.14}$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the rules

$$dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0 \\ \text{and } dB_t\cdot dB_t = t \tag{2.15}$$

The Ito Lemma, or otherwise known as the Ito formula, is the equivalent of a change of variable formula. One could fairly easily notice from the structure of the formula that it stems from a Taylor series expansion to the second partial derivative in terms of the stochastic process.

As an example, we would like to confirm the result (2.12), i.e. evaluate $\int_0^t B_s dB_s$. Therefore we set $X_t = B_t$ and $g(t,x) = \frac{1}{2}x^2$. Then $g(t,B_t) = \frac{1}{2}B_t^2$ and by Ito's formula we get

$$d\left(\frac{1}{2}B_t^2\right) = dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB_t + \\ + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB_t)^2 = B_t dB_t + \frac{1}{2}(dB_t)^2 = \\ = B_t dB_t + \frac{1}{2}dt, \tag{2.16}$$

which leads to the same answer as (2.12), namely

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \tag{2.17}$$

Preliminaries for fractional Brownian Motion (fBM)

Our investigation will not be limited to the Brownian Motion and to SDEs/SPDEs with BM. We are interested in extending our results to the fractional Brownian motion as well. According to [6], the process has been defined in 1940 by Kolmogorov in [7] and its properties, i.e. self similarity and long term dependence, were developed by Mandelbrot and Van Ness in [8]. Another important contributor was the British hydrologist Harold Edwin Hurst [9]. In his studies on the Nile River, he observed through 800 years worth of empirical data, that the water levels had a long term dependency and self similarity. To describe that dependency, he estimated a parameter, let us call H, based on his data.

Definition 2.2 — We define a Gaussian process ($B^H(t)$, $t \geq 0$) with continuous sample paths as a standard fractional Brownian Motion (fBM) with Hurst parameter H ∈ (0, 1) if it satisfies:

$$E[B^H(t)] = 0 \tag{2.18}$$

$$E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \tag{2.19}$$

for all $s,t \in \mathbb{R}_+$.

By simply looking at expression (2.19), it is obvious that we should consider a trichotomy on the value of the power in the right hand side, more particularly at the value $H = \frac{1}{2}$:

For $H = \frac{1}{2}$, $E[B^H(t)B^H(s)] = \min(t,s)$, therefore $B^{1/2}(t)$ is the standard B.M. (2.20)

For $H > \frac{1}{2}$ the increments are positively correlated (2.21)

For $H < \frac{1}{2}$ the increments are negatively correlated (2.22)

As we mentioned above, two very important properties of fBM are self similarity and long term dependence.

Definition 2.3 A process $X(t)$, $t \geq 0$) is said to be self similar with parameter H if for each $a > 0$

$$(X(at), t \geq 0) \overset{L}{\rightarrow} (a^H X(t), t \geq 0) \tag{2.23}$$

It is fairly easy to see that for the process ($B^H(at)$, $t \geq 0$) we can write

$$E[(B^H(at)B^H(as)] = \frac{1}{2}\{(at)^{2H} + (as)^{2H} - \\ - |at-as|^{2H}\} = a^{2H}(\frac{1}{2})\{(t^{2H} + s^{2H} - \\ - |t-s|^{2H}\}]$$

Therefore fBM is a self similar process with parameter H and

$$B^H(t) \sim |a|^H B^H(t) \tag{2.24}$$

Also, regarding long range dependence, let $r(n) = E[B^H(1)(B^H(n+1) - B^H(n)]$.

Then for $H \in (\frac{1}{2}, 1)$

$$\sum_{n=1}^{\infty} r(n) = +\infty \tag{2.25}$$

and therefore the process is long range dependent.

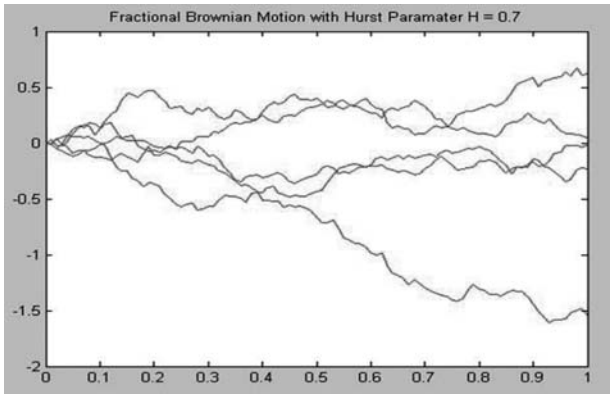


Fig. 2. Fractional Brownian Motion Paths with H = 0.7.

Also, we are interested in the following theorem as a tool for solving SDEs involving fBM:

Theorem 2.1 if $f: \mathbb{R} \longrightarrow \mathbb{R}$ is C^2 with derivatives to order two, then

$$f\left(B^H(T)\right) - f\left(B^H(0)\right) = \int_0^T f'\left(B^H(s)\right)\delta B^H(s) + \\ + H\int_0^T s^{2H-1} f''(B^H(s))ds \text{ a.s.} \quad (2.26)$$

If we let $H = \frac{1}{2}$ then we have the usual Ito formula.

Numerical Approximation and Simulations (SDEs)

The main scope of our work is to develop tolls and methods that can be used to numerically represent Brownian Motion paths, fractional Brownian Motion paths and SDEs with either BM or fBM. The purpose of simulating the first two is so that we can use them as inputs in the SDEs in both cases of actual explicit solutions and numerical approximations. The purpose to simulate SDEs comes as we can approximate numerically their solutions in cases where an explicit solution cannot be found. We will start by defining our error measurement formulas.

Definition 3.1 (Error formulas)
Let $(X_1, X_2, \dots, X_n)$ be the actual values of X and $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$ the numerical approximated values of X at time points $t_1 < t_2 < \cdots < t_n$. Then

$$E_{abs} = \sum_{J=1}^{N} \left| X_j - \hat{X}_j \right| \text{ is the absolute error,} \quad (3.1)$$

$$E_{rel} = \sum_{J=1}^{N} \left| \frac{X_j - \hat{X}_j}{X_j} \right| \text{ is the relative error, and} \quad (3.2)$$

$$E_{ave} = \sum_{J=1}^{N} \left| \frac{X_j - \hat{X}_j}{N} \right| \text{ is the average error} \quad (3.3)$$

We use different forms of error measurements so that we are susceptible to misleading results.

Next we deal with our approach to simulate the different processes. The basic and common principle is to discretize the process as we are using Matlab. Starting with the standard Brownian Motion, we use its properties, i.e. the fact that it is a Gaussian process whose increments follow a normal distribution with mean 0 and variance equal to the time-step. Therefore we use a build-in random number generator that provides us with a $N(0, 1)$ and we scale by $\sqrt{\Delta t}$, where Δt is the time-step. For our work we considered equidistant partitions, i.e. $\Delta t = \frac{T}{N}$, where T is the stopping time and N is the number of time-steps desired. Also, we usually investigate our processes on $t \in [0, 1]$ in order to reduce as much complexity and cost on the program. As expected, we produce different paths of the Brownian Motion even if we preserve all the constants (Fig. 1). Though the author's original code was successful, the code suggested in [2] by Higham is slimmer and very efficient.

We also employ the properties of the fractional Brownian motion in order to simulate its paths. The following steps are needed [10]:

1) Form an N × N matrix A whose entries are given by (2.19), i.e the covariance of the process.
2) Evaluate the square root of A using the Cholesky decomposition method.
3) Generate a 1 × N vector v whose entries are from a standard Gaussian distribution
4) Apply $\sqrt{A}$ to v.

A sample of five fBM paths with parameter H = 0.7 can be seen in Fig. 2.

As we now have tools to simulate both BM and fBM, we proceed to discuss the approximations of SDEs. We start by investigating three methods for Stochastic Differential Equations involving standard Brownian Motion as defined in [5]. The best performing method will be applied to SDEs/SPDEs with fractional Brownian Motion. So, the task is to approximate the stochastic process $X = \{X_t, t_0 \le t \le T\}$ satisfying the SDE:

$$dX_t = a(t, X_t)d_t + b(t, X_t)dW_t \text{ on } t_0 \le t \le T \\ \text{and initial value } X_{t_0} = X_0 \quad (3.4)$$

For simplicity purposes we set $a(t, X_t) = \alpha X_t$ and $b(t, X_t) = \beta X_t$. So we get

$$dX_t = \alpha X_t d_t + \beta X_t dW_t. \quad (3.5)$$

Applying the Ito formula to (3.5) we have that

$$X_t = X_0 \exp\left\{\left(\alpha - \frac{1}{2}\beta^2\right)t + \beta W_t\right\} \quad (3.6)$$

We now introduce the three methods for approximating SDEs.

Definition 3.2 (Euler Method)
For $t_0 < t_1 < t_2 < \ldots < t_n = T$ on the interval $[t_0, T]$, the Euler approximation is a continuous time stochastic process $Y = \{Y_t, t_0 \leq t \leq T\}$ satisfying the iterative scheme:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta B_t \tag{3.7}$$

More specifically in our case that we wish to apply the method to (3.6), we get:

$$Y_{n+1} = Y_n + \alpha Y_n \Delta t + \beta Y_n \Delta B_t \tag{3.8}$$

Definition 3.3 (Heun Method)
For $t_0 < t_1 < t_2 < \ldots < t_n = T$ on the interval $[t_0, T]$, the Heun method is satisfying the iterative scheme:

$$\begin{aligned} Y_{n+1} = Y_n + \frac{1}{2}a\big(\tilde{Y}_n + Y_n\big)\Delta t + \\ + \frac{1}{2}\beta\big(\tilde{Y}_n + Y_n\big)\Delta B_t, \end{aligned} \tag{3.9}$$

where

$$\tilde{Y}_n = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta B_t \tag{3.10}$$

More specifically in our case that we wish to apply the method to (3.6), we get:

$$\begin{aligned} Y_{n+1} = Y_n + \frac{1}{2}a\big(\tilde{Y}_n + Y_n\big)\Delta t + \\ + \frac{1}{2}\beta\big(\tilde{Y}_n + Y_n\big)\Delta B_t, \end{aligned} \tag{3.11}$$

where

$$\tilde{Y}_n = Y_n + \alpha Y_n \Delta t + \beta Y_n \Delta B_t \tag{3.12}$$

The principle behind the Heun method is very much alike to the Euler one, with the difference that instead of the process being evaluated at the endpoints, the trapezoid rule is being used.

Definition 3.4 (Milstein Method)
For $t_0 < t_1 < t_2 < \ldots < t_n = T$ on the interval $[t_0, T]$, the Milstein approximation is a continuous time stochastic process $Y = \{Y_t, t_0 \leq t \leq T\}$ satisfying the iterative scheme:

$$\begin{aligned} Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta B + \\ + \frac{1}{2}b(Y_n)b'(Y_n)[(\Delta B)^2 - \Delta t] \end{aligned} \tag{3.13}$$

More specifically in our case that we wish to apply the method to (3.6), we get:

$$\begin{aligned} Y_{n+1} = Y_n + \alpha Y_n \Delta t + \beta Y_n \Delta B + \\ + \frac{1}{2}\beta(Y_n)\beta'(Y_n)[(\Delta B)^2 - \Delta t] \end{aligned} \tag{3.14}$$

The Milstein method is in a sense an "evolutionary" form of the Euler method. The basic difference is that one extra term is included in the method. Another important remark is that the Ito-Taylor expansion is used in order to derive this method, therefore providing an order 1.0 strong Taylor scheme. Next we compare the three methods with the actual solution graphically.

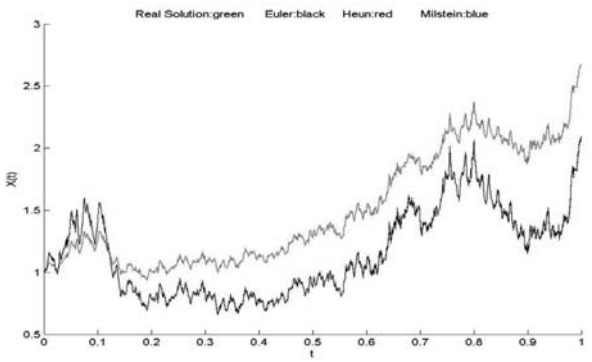


Fig. 3. Simulations for N=1000 and $\alpha > \frac{1}{2}\beta^2$.

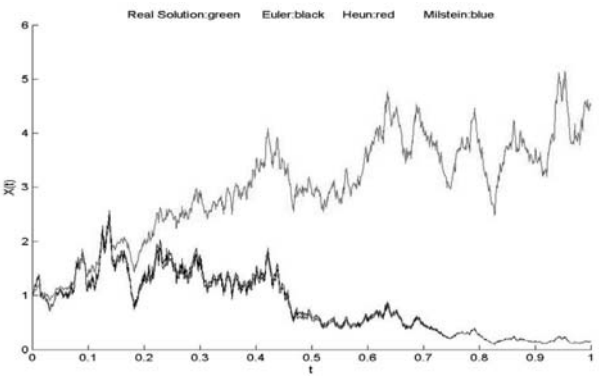


Fig. 4. Simulations for $\alpha < \frac{1}{2}\beta^2$.

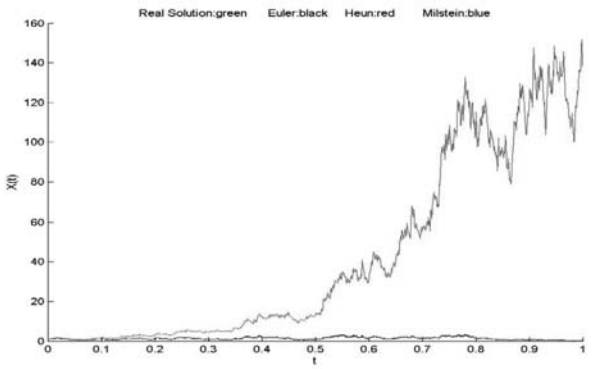


Fig. 5. Simulations for $\alpha = \frac{1}{2}\beta^2$.

As shown by graphs 3–5 we get the idea that the Heun method is not appropriate for SDEs whatsoever. In fact, the scheme seems to diverge once BM is involved. Therefore it is completely abandoned for our purposes. In comparing the two remaining methods, even though both seem to follow the actual solution, the Milstein scheme seems to have a much smaller deviation from the actual solution (Tables 1–2). The result is not surprising as both Euler and Milstein can be derived by applying the Taylor polynomial expansion to the SDE, with the

Table 1. Table of Absolute Errors.

Euler	Heun	Milstein
4.8317	372.3806	0.6604
12.0207	278.0391	0.9094
26.0531	679.4561	1.9465
10.3954	179.0899	**0.5445**
13.6615	321.4044	1.0420
15.8134	321.9179	1.2279
9.2588	404.3833	1.0448
22.5554	577.9528	3.8917
40.3136	798.9531	3.4242
13.9642	378.7190	0.7851

Table 2. Table of Relative Errors.

Euler	Heun	Milstein
3.4319	20.5666	0.6359
6.1103	23.7858	1.1628
1.3683	28.6385	0.8478
1.4295	74.8322	1.5371
2.8765	11.2171	0.4262
1.4639	14.8793	0.6616
3.2527	10.0544	0.4610
5.2890	23.4771	0.5525
4.9824	29.2045	1.6498
5.3197	16.8437	0.6747

difference that the Milstein scheme is of higher order. The one main concern usually with higher order schemes, is the how computationally expensive it can be. Truth is though, that even a standard home computer can easily run the programs in matter of seconds. As such, we further test the Milstein scheme against the actual solutions of two more non-linear SDEs, namely:

$$dX_t = \alpha^2 X_t(1 + X_t^2)dt - \alpha(1 + X_t^2)dW_t, \quad (3.15)$$

that has as an explicit solution

$$X_t = \cot\left(\alpha W_t + \operatorname{arccot}(X_0)\right), \quad (3.16)$$

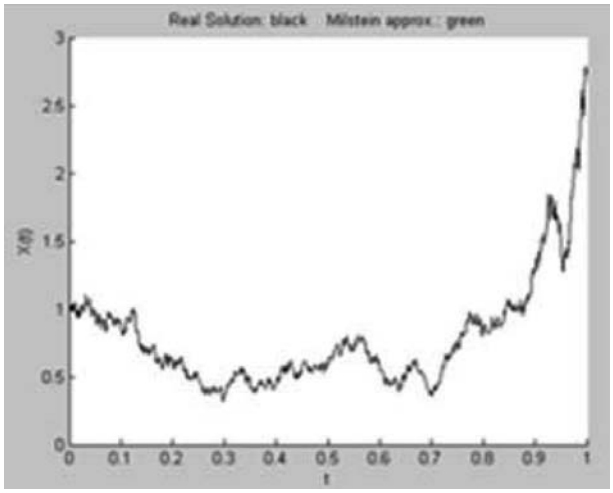
Also we test the SDE

$$dX_t = -\frac{1}{2}\alpha^2 X_t dt - \alpha\sqrt{1 - X_t^2}dW_t, \quad (3.17)$$

whose solution is

$$X_t = \cos\left(\alpha W_t + \arccos(X_0)\right), \quad (3.18)$$

Our next step is to extend our results to provide a method that works in SDEs with fBM. We also compare numerically our method with an N-step method suggested by Ian Lewis in [6]. As with the Milstein method for SDEs involving Brownian Motion, we apply the Taylor polynomial to the general form of SDE with fBM. Our result and suggested method is given by:



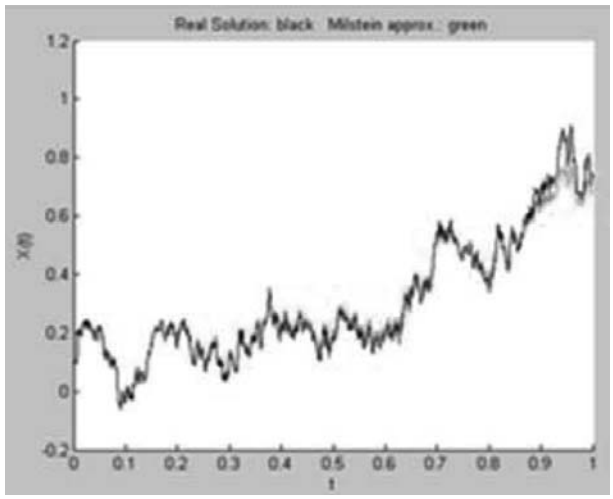


Fig. 6–7. Simulations of SDEs using the Milstein Method.

Table 3–4 Errors of SDEs using the Milstein Method.

Trial	Average Error
1	0.0009
2	0.0019
3	0.0013
4	0.0002
5	0.0001
6	0.0006
7	0.0004

Trial	Average Error
1	0.0238
2	0.0090
3	0.0098
4	0.0201
5	0.0316
6	0.0055
7	0.0060

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\left(\mathrm{B}^{\mathrm{H}}_{\mathrm{t}_{n+1}} - \mathrm{B}^{\mathrm{H}}_{\mathrm{t}_n}\right) + \frac{1}{2}\mathrm{b}(\mathrm{Y_n})\mathrm{b}'(\mathrm{Y_n})[\left(\mathrm{B}^{\mathrm{H}}_{\mathrm{t}_{n+1}} - \mathrm{B}^{\mathrm{H}}_{\mathrm{t}_n}\right)^2 - (\mathrm{t}^{2\mathrm{H}}_{n+1} - \mathrm{t}^{2\mathrm{H}}_{n})] \quad (3.19)$$

One remark for our method is that if we set we get expression (3.13) which is the Milstein method for SDEs involving standard Brownian Motion.

Proof:
The Milstein Scheme for standard Brownian motion can be produced by adding the term to the Euler method. In similar approach we have

$$Y_{n+1} = Y_n + a\Delta t + b(Y_n) + \mathrm{b}(\mathrm{Y_n})\mathrm{b}'(\mathrm{Y_n})I_{(1,1)} \quad (3.20)$$

Evaluating the last term we have:

$$\begin{aligned} I_{(1,1)} &= \int_{t_n}^{t_{n+1}} \int_0^{s2} dB^H_{s1} dB^H_{s2} = \\ &= \int_{t_n}^{t_{n+1}} B^H_{s2} dB^H_{s2} = \\ &= [\left(\mathrm{B}^{\mathrm{H}}_{\mathrm{t}_{n+1}} - \mathrm{B}^{\mathrm{H}}_{\mathrm{t}_n}\right)^2 - (\mathrm{t}^{2\mathrm{H}}_{n+1} - \mathrm{t}^{2\mathrm{H}}_{n})] \end{aligned} \quad (3.21)$$

Substituting back in (3.20) we get

$$\begin{aligned} Y_{n+1} &= Y_n + a(Y_n)\Delta t + b(Y_n)\left(\mathrm{B}^{\mathrm{H}}_{\mathrm{t}_{n+1}} - \mathrm{B}^{\mathrm{H}}_{\mathrm{t}_n}\right) \\ &+ \frac{1}{2}\mathrm{b}(\mathrm{Y_n})\mathrm{b}'(\mathrm{Y_n})[\left(\mathrm{B}^{\mathrm{H}}_{\mathrm{t}_{n+1}} - \mathrm{B}^{\mathrm{H}}_{\mathrm{t}_n}\right)^2 - \\ &- (\mathrm{t}^{2\mathrm{H}}_{n+1} - \mathrm{t}^{2\mathrm{H}}_{n})] \blacksquare \end{aligned} \quad (3.22)$$

For the numerical simulation, we consider the SDE

$$dX_t = AX_t dt + CX_t dB^H_t \text{ with } X_0 = 1 \quad (3.23)$$

Its solution is given by

$$X_t = \exp\{At - \frac{1}{2}C^2 t^{2H} + CB^H_t\} \quad (3.24)$$

Next we run a comparison of the extended Milstein scheme to the actual solution of the SDE with $H = 0.7$. The outcome is very encouraging.

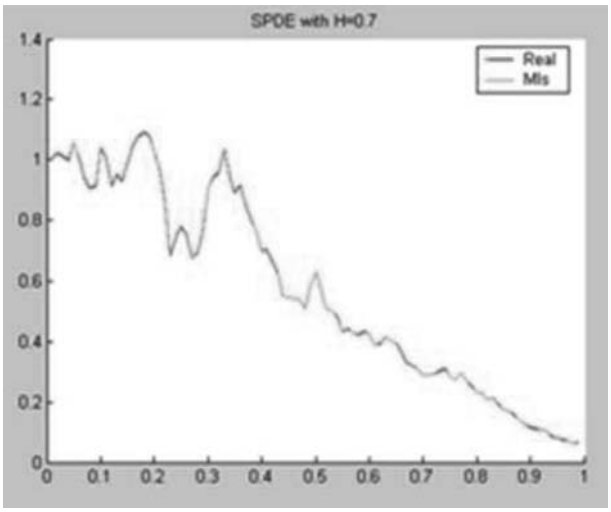


Fig. 8. SDE with fBM using the extended Milstein Method.

Table 5. Average Errors.

Trial	Average Error
1	0.0010
2	0.0037
3	0.0074
4	0.0031
5	0.0014
6	0.0046
7	0.0041

In a head to head comparison with the method suggested in [6], we resulted in an absolute error of zero. After further investigation it seems that the two schemes are in fact the same scheme. The main difference is that the suggested method in this paper is a much simpler expression and not dependent on summations of triple integrals.

Multidimensional SDEs

An extension of these principals which will allow our methods a significant increase in capability and depth is to examine some numerical simulations of stochastic dynamical systems. Broadly speaking, a dynamical system is an n-dimensional system of coupled differential equations. These systems can exhibit non-linear behavior and the numerical study of stochastic variants allows an empirical study of the effects of small variations on the critical points and manifolds of such systems. In order to investigate the behavior of these systems, we require an updated modeling architecture.

Consider the previous formulation of an SDE, modified such that

$$dX_t = A(t, X_t)d_t + B(t, X_t)dW_t \quad (3.25)$$

where $A \in R^n \times \theta \to R^n, B \in R^n \times \theta \to R^{n\times n}$ and W is also n-dimensional. This formulation (made simpler by the choice of nxn diagonal noise wherein each state component is perturbed only by the corresponding noise component) gives rise to an n-dimensional Milstein method approximation [5, 348] with the usual assumption that $t_0 < t_1 < t_2 < \ldots < t_n = T$ on the interval $[t_0, T]$.

Definition 3.5 — Multidimensional Milstein Method

$$y^k_{n+1} = y^k_n + hA^k + B^{k,k}\Delta W^k_n + \frac{1}{2}B^{k,k}\frac{\partial B^{k,k}}{\partial X^k}\left(\left(\Delta W^k_h\right)^2 - h\right);\ y^k_0 = x^k_0 \quad (3.26)$$

for the k^{th} component of A and B and n^{th} time step of the solution.

We note also, stating without proof (interested parties can see [5, 350]) that our expanded multidimensional Milstein method also converges with order $\gamma = 1$, or, more accurately stated,

$$E\left(\left|x_T - y_T^k\right|\right) \leq Ck^{\gamma} \tag{3.27}$$

where the constant C does not depend on k.

Due to the general non-linearity of our system(s) and consequently its lack of an explicit reachable solution, no measurement of absolute error can be made. We must therefore consider a new architecture for measuring the efficacy of our algorithm. An adaptation of the approach used by Davie & Gaines in [13], likewise used in the SPDE analysis below, is proposed. We generate several independent Brownian sheets and discretize each using variable numbers of space steps each sharing a common factor. Each discretization will then share a number of space points with each other. Summing over the squared differences of the approximated values, we define:

Definition 3.6 — Squared Difference Convergence Formula

$$S_i = \sum_{j=1}^{m}\sum_{k=1}^{n}(u_{j,k}^{i} - u_{j,k}^{i+1})^2 \tag{3.28}$$

where n is the number of shared points, and m is the number of simulated independent Brownian sheets for each i[th] discretization. In each case, we can examine the ratios of successive values of S_i, ie S_2/S_1, etc. which should be approximately equal to the order of convergence for our method. One advantage of this convergence analysis is that it is easily generalizable to different time-discretization methods. We therefore repeat our analysis to include both

Definition 3.7 — Multidimensional Euler Method

$$y_{n+1}^k = y_n^k + hA^k + B^{k,k}\Delta W_h^k \; ; \; y_0^k = x_0^k \tag{3.29}$$

which converges with a rate of $\gamma = \frac{1}{2}$ (?? — Kloeden & Platen — 341).

In order to examine this method, we choose a non-linear two-dimensional dynamical system, represented in the deterministic sense by

$$\begin{cases} \dot{x} = y \\ \dot{y} = x(1-x^2) + y \end{cases} \tag{3.30}$$

We can gain an intuitive appreciation of the qualities of such a system by studying the phase-plane plot of its (deterministic) behavior (plot by MATLAB's standard ode solver ode45 which is based on a well-known fourth-order Runge-Kutta scheme) presented in fig. 9.

We note the critical points at x = -1, 0, 1 (unstable, saddle, and unstable points, resp.) and the stable and unstable manifolds passing through the origin. In order to convert this system to an approachable stochastic dynamical system, we add simple noise terms to each differential equation, making:

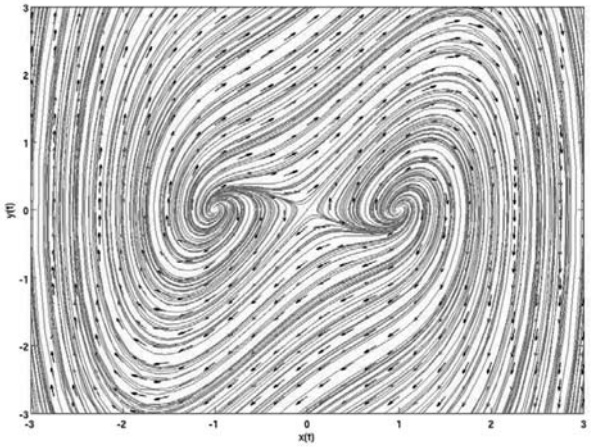


Fig. 9. System (3.30) with multiple solution curves.

$$\begin{cases} \dot{x} = y + c_1 W_t \\ \dot{y} = x(1-x^2) + y + c_2 W_t \end{cases} \tag{3.31}$$

and apply (3.26) using a Milstein discretization with N = 100 time steps and a standard Brownian Motion (H = 1/2).

Initial results are seen below, plotting fifty paths along the y space dimension [11]:

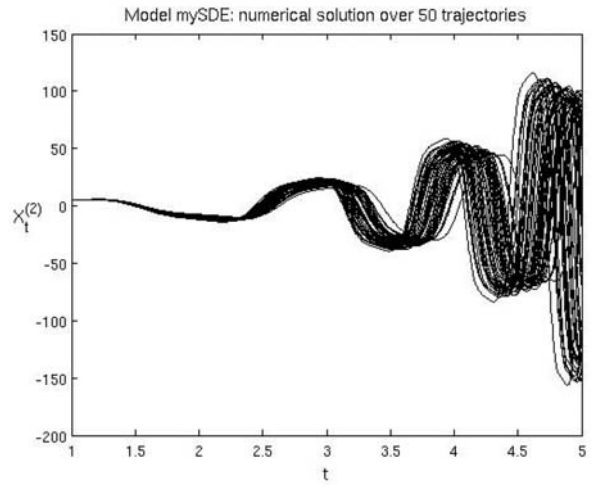


Fig. 10. Solutions for y in system (3.31) plotted for fifty independent Brownian sheets.

and with a calculated mean path (green) along with 95% empirical confidence intervals (dashed lines) and the upper and lower empirical quartiles (dotted lines) presented in fig. 11.

We broadly witness the qualitative behaviors of the deterministic model — with the solution circling around the cluster of critical points for large times. More detailed results along with stochastic phase-plane and error analysis as well as consideration of a fractional Brownian Motion to be presented in-conference.

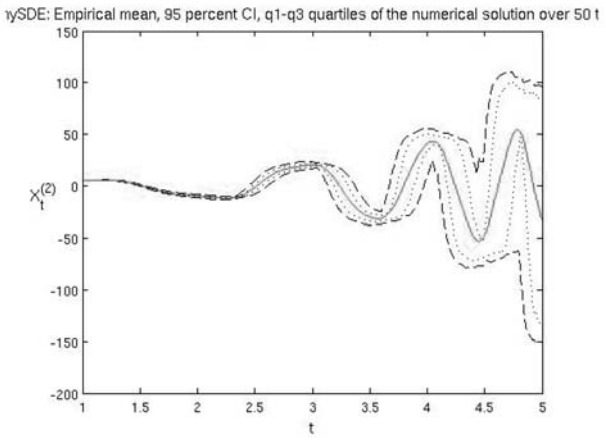


Fig. 11. Averaged solution for (3.31) with empirical quartiles and 95% CI.

Numerical Approximations and Simulations (SPDEs)

The majority of Partial Differential Equations (PDEs) are at the least difficult to solve; many exhibit non-linear behavior which makes finding analytical solutions infeasible. The numerical simulation of PDEs, therefore, offers a route towards the controlled study of the behavior of a class of equations with vast utility in industry and science. The study and simulation of SPDEs is a relatively young and developing field due in large part to the challenges involved in crossing the difficulties of simulating deterministic PDEs (a field in itself) with the complexities of stochastic calculus (interested readers are encouraged to see the excellent plenary paper [12]). PDEs are similar in structure to ODEs but may involve partial derivatives of the state variables in addition. We can therefore generalize the structure of a stochastic PDE as follows:

$$dU_t^N = [A_N U_t^N + f_N(U_t^N)dt] + \sigma_N(U_t^N)dW_t \quad (4.1)$$

where $U_t \in R^N$, f & g are N-dimensional functions, and A is an N-dimensional operator (typically involving partial derivatives of the state variables). Approaches to numerical approximation schemes for SPDEs run roughly parallel to those of SODEs, however an important distinction is that an Ito formula, as discussed above (Lem2.1) is not available. We will here restrict ourselves to consideration of SPDEs of the parabolic type with a single space variable, producing the SPDE

$$\frac{\partial}{\partial t}U_t = \frac{\partial^2}{\partial x^2}U_t + f(U_t) + \sigma(U_t)\frac{\partial^2}{\partial t \partial x}W(t,x) \quad (4.2)$$

For convenience, we assume Dirichlet boundary conditions and set our space variable to vary over the closed unit interval.

$$u(t,0) = u(t,1) = 0\,;\; x \in [0,1] \quad (4.3)$$

To construct a numerical algorithm with which to approximate the solution, U_t, we make use of a common technique from the study of PDEs and replace the space derivatives of U and W with their finite differences over some small space-step. The result is the system of Stochastic Ordinary Differential Equations seen below [15]

$$\begin{aligned} dU^n(t,x_j) = {} & n^2\left(U^n(t,x_{j+1}) - 2u^n(t,x_j) + \right. \\ & \left. + u^n(t,x_{j-1})\right)dt + f\left(t,x_j,u^n(t,x_j)\right)dt + \\ & n\sigma\left(t,x_j,u^n(t,x_j)\right)d(W(t,x_{j+1}) - W(t,x_j)) \end{aligned} \quad (4.4)$$

Considering the above system, we are now free to apply and examine a variety of time-discretization schemes [13]. Using the explicit Euler scheme, we arrive at:

Definition 4.1 — Explicit Euler SPDE scheme

$$\begin{aligned} u_{j,m+1} = {} & u_{j,m} + n^2 h(u_{j+1,m} + u_{j-1,m} - \\ & - 2u_{j,m}) + n\sigma(u_{j,m})W_{j,m} \end{aligned} \quad (4.5)$$

which has been found [14] to converge at a rate of . For certain applications, however, implicit schemes are preferred and so we generate the following:

Definition 4.2 — Crank-Nicholson SPDE scheme

$$\begin{aligned} u_{j,m+1} = {} & u_{j,m} + \frac{n^2 h(u_{j+1,m} + u_{j-1,m} - 2u_{j,m})}{2} + \\ & + \frac{n^2 h(u_{j+1,m+1} + u_{j-1,m+1} - 2u_{j,m+1})}{2} + nW_{j,m} \end{aligned} \quad (4.6)$$

Definition 4.3 — Backward Euler SPDE scheme

$$\begin{aligned} u_{j,m+1} = {} & u_{j,m} + n^2 h(u_{j+1,m+1} + \\ & + u_{j-1,m+1} - 2u_{j,m+1}) + nW_{j,m} \end{aligned} \quad (4.7)$$

which are each known to converge at a rate of $\gamma = \sqrt{2}$ [14].

For part of our evaluation of numerical methods, we will concentrate on the simple but canonical one-dimensional stochastic heat equation

$$\dot{u} = \frac{\partial^2 u}{\partial x^2} + \sigma(x,t,u)\dot{W}(x,t) \quad (4.8)$$

We therefore assume the operator A in the above formulation (4.1) to be the one-dimensional Laplacian and set $f = 0$. We assume an arbitrary region of $0 \le x \le 1$ with periodic boundary conditions $u(0, t) = u(1, t)$ and initial condition $u(x, 0) = \sin(\pi x)$. Again, to first attain a qualitative view of our problem, we consider the deterministic system, or equivalently, the case where $\equiv 0$:

Here we see the familiar effects of the heat equation- with the heat dissipating to a flat curve over time. Farther, we can begin to observe qualitative stochastic effects by using simple forms for

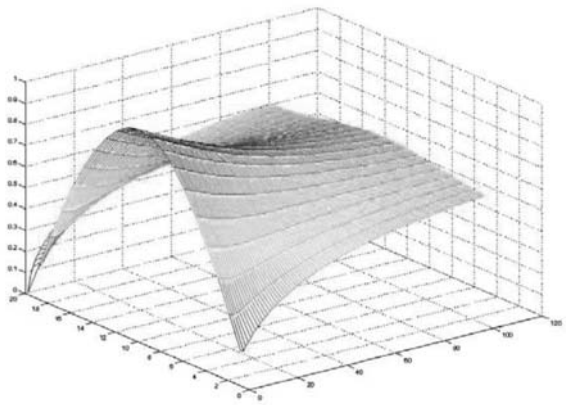


Fig. 12. The deterministic heat equation with T = 100.

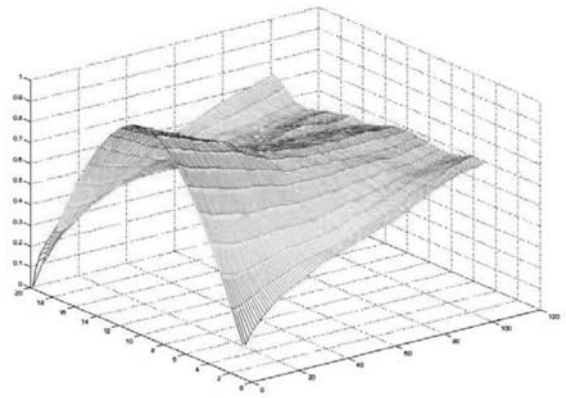


Fig. 13. The stochastic heat equation with $\sigma = .2$.

And the significantly more extreme effects when σ is increased

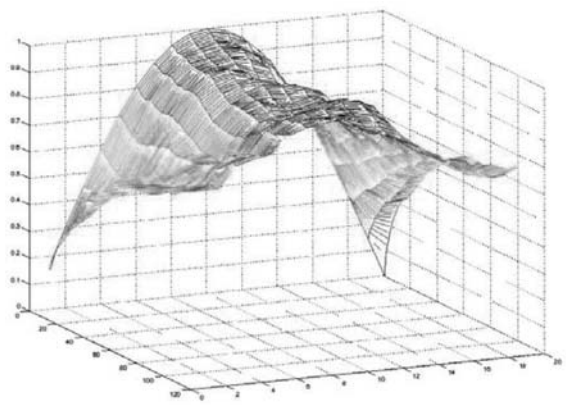


Fig. 14. The stochastic heat equation with .

We again witness the qualitative behaviors of the deterministic model- with the smoothing effects partially negated by the white noise in the system. More detailed results along with noisy initial and boundary conditions, implicit scheme implementation, and convergence order analysis (using the method outlined in (3.28) as well as consideration of fractional Brownian Motion cases to be presented in-conference.

Conclusions

We believe that our methods for simulating Brownian Motion and fractional Brownian Motion are strong due to the fact that they are derived directly from the properties of the processes. Regarding SDEs with Brownian Motion, we reject the Heun method and choose to either use either Euler or Milstein method. The Milstein method is somewhat closer to the exact solution, but the Euler method might be more appropriate for finer partitions on t. Finally we suggest that for SDEs involving fBM, the extended Milstein method should be used. Conclusions for multidimensional SDEs and SPDEs are reserved for the presentation of expanded data.

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