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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A NONHOMOGENEOUS NEUMANN BOUNDARY PROBLEM

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Abstract. We consider a nonlinear Neumann elliptic equation driven by a p-Laplacian-type operator which is not homogeneous in general. For such an equation the energy functional does not need to be coercive, and we use suitable variational methods to show that the problem has at least two distinct, nontrivial smooth solutions. Our formulation incorporates strongly resonant equations.

Keywords: Palais-Smale condition, noncoercive functional, second deformation theorem.

Mathematics Subject Classification: 35J20, 35J60.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear Neumann problem:

$$\begin{cases} -\text{div}\, a\left(\nabla u(z)\right) = f\big(z, u(z)\big) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial n_a} = (a(\nabla u(z)), n(z))_{\mathbb{R}^N}$, with $n(\cdot) = (n_1(\cdot), \ldots, n_N(\cdot))$ the outward unit normal vector on $\partial\Omega$. Here $a = (a_i)_{i=1}^N \colon \mathbb{R}^N \to \mathbb{R}^N$ is a continuous, strictly monotone map on which we impose certain conditions (see Section 2) to obtain a p-Laplacian type operator, which unifies several important differential operators. Similar conditions are studied widely in the literature (see Damascelli [5], Montenegro [17], Motreanu-Papageorgiou [18]). Also, $f(z, \zeta)$ is a Carathéodory function, i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta)$ is continuous.

The aim of this work is to prove existence and multiplicity results for problem (1.1), when the energy functional of the problem is noncoercive. In fact, our hypotheses on the reaction f incorporates into our framework equations which are strongly resonant at infinity. Such problems are of special interest, since they exhibit a partial lack of compactness. Such a result has been obtained in Gasiński-Papageorgiou [11] for a Neumann problem driven by the p-Laplacian operator. Here we manage to employ the methods used in [11] for a much more general class of differential operators, taking advantage of some of the properties of the p-Laplacian, which this class possesses.

For the problems in which the energy functional is coercive we refer to Gasiński-Papageorgiou [12, 14] (for the Dirichlet boundary value problem) and to [13, 15] (for Neumann boundary value problems) or to Drábek-Kufner-Nicolsi [7] and Drábek-Milota [8].

2. MATHEMATICAL BACKGROUND AND THE SETTING OF THE NONHOMOGENEOUS OPERATOR

In this paper we will denote by $(\cdot,\cdot)_{\mathbb{R}^n}$ the scalar product in $\mathbb{R}^N$ and by $|\cdot|$ - the norm given by this scalar product. Also $\|\cdot\|$ denotes the norm in the Sobolev space $W^{1,p}(\Omega)$. We will assume that $1 < p < \infty$. We will use the notation of the Sobolev critical exponent

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N. \end{cases}$$

For the convenience of the reader, we present below the main mathematical tools which will be needed in the proofs of our results.

Theorem 2.1 (Theorem 5.2.10 of [9])**.** *Let X be a Banach space and let X^* be its topological dual. Suppose $\phi \in C^1(X)$ is bounded below and satisfies the Palais-Smale condition at level $c := \inf_X \phi$, i.e.*

every sequence $\{x_n\}_n \subseteq X$ such that $\phi(x_n) \longrightarrow c$ and $\phi'(x_n) \longrightarrow 0$ in X^* admits a strongly convergent subsequence.

Then there exists $x_0 \in X$, such that $c = \phi(x_0)$.

To formulate the next result, for $\phi \in C^1(X)$ and $c \in \mathbb{R}$, we define the following sets:

$$\begin{aligned} \phi^c &:= \big\{x \in X: \ \phi(x) \leq c\big\}, \\ K_\phi &:= \big\{x \in X: \ \phi'(x) = 0\big\}, \\ K_\phi^c &:= \big\{x \in K_\phi: \ \phi(x) = c\big\}. \end{aligned}$$

Theorem 2.2 (Theorem 5.1.13 of [9] (Second Deformation Theorem))**.** *If $\phi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leq +\infty$, ϕ satisfies the Palais-Smale condition for every $c \in [a, b)$, ϕ has no critical values in (a, b) and $\phi^{-1}(\{a\})$ contains at most a finite number of critical points of ϕ, then there exists a homotopy $h\colon [0,1] \times (\phi^b \setminus K_\phi^b) \longrightarrow \phi^b$ such that*

(a) $h\big(1, \phi^b \setminus K_\phi^b\big) \subseteq \phi^a$;
(b) $h(t,x) = x$ *for all* $t \in [0,1]$, *all* $x \in \phi^a$;
(c) $\phi\big(h(t,x)\big) \le \phi\big(h(s,x)\big)$ *for all* $t, s \in [0,1]$, $s \le t$, *all* $x \in \phi^b \setminus K_\phi^b$.

Theorem 2.3 (Theorem 1.7 of [16]). *Let* $h\colon \mathbb{R}_+ \to \mathbb{R}$ *be a* C^1*-function satisfying*

$$\delta < \frac{th'(t)}{h(t)} \le c_0 \quad \textit{for all} \quad t > 0$$

with some constants $\delta > 0$, $c_0 > 0$. *We define*

$$\widehat{H}(\xi) = \int_0^\xi h(t)\mathrm{d}t.$$

By $W^{1,\widehat{H}}(\Omega)$ *we denote the class of functions which are weakly differentiable in the set* Ω *with*

$$\int_\Omega \widehat{H}(|\nabla u|)\mathrm{d}z < \infty.$$

Let $\alpha \le 1$, Λ, Λ_1, M_0 *be positive constants and let* $\Omega \subseteq \mathbb{R}^N$ *be an open set. Suppose that* $A = (A_1, \dots, A_N)\colon \Omega \times [-M_0, M_0] \times \mathbb{R}^N \to \mathbb{R}^N$ *is differentiable,* $B\colon \Omega \times [-M_0, M_0] \times \mathbb{R}^N \to \mathbb{R}$ *is a Carathéodory function and functions* A, B *satisfy the following conditions:*

$$(\nabla A(z_1, \xi_1, y)x, x))_{\mathbb{R}^N} \ge \frac{h(|y|)}{|y|}|x|^2, \quad y \ne 0_N, \tag{2.1a}$$

$$\Big|\frac{\partial}{\partial y_j} A_i(z, \xi, y)\Big| \le \Lambda \frac{h(|y|)}{|y|}, \quad y \ne 0_N, \tag{2.1b}$$

$$|A(z_1, \xi_1, y) - A(z_2, \xi_2, y)| \le \Lambda_1 \left(1 + h(|y|)\right) \left(|z_1 - z_2|^\alpha + |\xi_1 - \xi_2|^\alpha\right), \tag{2.1c}$$

$$|B(z_1, \xi_1, y)| \le \Lambda_1(1 + h(|y|)|y|) \tag{2.1d}$$

for all $z_1, z_2 \in \Omega$, $\xi_1, \xi_2 \in [-M_0, M_0]$ *and* $x, y \in \mathbb{R}^N$. *Then any* $W^{1,\widehat{H}}(\Omega)$ *solution* u *of*

$$\mathrm{div} A(z, u, \nabla u) + B(z, u, \nabla u) = 0 \tag{2.2}$$

in Ω *with* $|u| \le M_0$ *in* Ω *is in* $C^{1,\beta}(\Omega)$ *for some positive* β *depending on* α, Λ, δ, c_0, N.

Next, let us recall some basic spectral properties of the Neumann p-Laplacian, which will be useful in the multiplicity results. We say that a number $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W^{1,p}(\Omega))$ if the problem

$$-\mathrm{div}(|\nabla u(z)|^{p-2}\nabla u(z)) = \lambda |u(z)|^{p-2}u(z) \quad \text{for a.a. } z \in \Omega, \quad \frac{\partial u}{\partial n_p} = 0 \quad \text{on} \quad \partial\Omega$$

admits a nontrivial solution $u \in W^{1,p}(\Omega)$, which we call an eigenfunction corresponding to λ. Here $\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}(\nabla u, n)_{\mathbb{R}^N}$. It is well known that all eigenvalues of

$(-\Delta_p, W^{1,p}(\Omega))$ are nonnegative and the smallest eigenvalue $\lambda_0 = 0$ is isolated and simple (see Gasiński-Papageorgiou [9]). There are several variational characterizations of the first nontrivial eigenvalue $\lambda_1 > \lambda_0 = 0$ (see for example [1]). The most convenient for our purposes is the following.

Proposition 2.4. *Let $1 < p < \infty$ and define*

$$C(p) = \Big\{ u \in W^{1,p}(\Omega) : \int_\Omega \big|u(z)\big|^{p-2} u(z) \mathrm{d}z = 0 \Big\}.$$

Then

$$\lambda_1 = \min_{u \in C_p \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}.$$

Moreover, for all $u \in C_p$ we have the following Poincaré-Wirtinger inequality:

$$\lambda_1 \|u\|_p^p \leq \|\nabla u\|_p^p. \tag{2.3}$$

Throughout this paper, the hypotheses on $a(y)$ are the following:

H(a): $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is such that $a(y) = a_0(|y|)y$ for any $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto t a_0(t)$ strictly increasing on $(0, \infty)$;

(ii) there exist some constants $\delta, c_0, c_1, c_2, c_3 > 0$, $q \in (1, p)$ and a function $h \in C^1(0, \infty)$ satisfying

$$\delta < \frac{t h'(t)}{h(t)} \leq c_0 \quad \text{for all} \quad t > 0, \tag{2.4}$$

$$c_1 t^{p-1} \leq h(t) \leq c_2 (t^{q-1} + t^{p-1}) \quad \text{for all} \quad t > 0,$$

such that

$$|\nabla a(y)| \leq c_3 \frac{h(|y|)}{|y|} \quad \text{for all} \quad y \in \mathbb{R}^N \setminus \{0\};$$

(iii) for all $y, \xi \in \mathbb{R}^N$ such that $y \neq 0$ we have

$$(\nabla a(y) \xi, \xi))_{\mathbb{R}^N} \geq \frac{h(|y|)}{|y|} |\xi|^2;$$

(iv) the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone, i.e.

$$(a(x) - a(y), x - y)_{\mathbb{R}^N} > 0 \quad \text{for all} \quad x, y \in \mathbb{R}^N, x \neq y.$$

We have the following properties of the map $a(y)$.

Proposition 2.5. *If hypotheses $H(a)$ hold, then:*

(a) *the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone, i.e.*

$$(b - a(y), x - y)_{\mathbb{R}^N} > 0 \ \Rightarrow b = a(x) \quad \text{for all} \quad y \in \mathbb{R}^N;$$

(b) *there exists $c_4 > 0$ such that for all $y \in \mathbb{R}^N$*

$$|a(y)| \le c_4(|y|^{q-1} + |y|^{p-1}); \tag{2.5}$$

(c) *for all $y \in \mathbb{R}^N$ we have*

$$(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p. \tag{2.6}$$

Let $G_0(t) := \int_0^t a_0(s)s\mathrm{d}s$ and

$$G(y) := G_0(\|y\|), \quad y \in \mathbb{R}^N.$$

Then G is strictly convex, $G(0) = 0$ and $\nabla G(y) = a(y)$ for $y \in \mathbb{R}^N \backslash \{0\}$. Moreover, for all $y \in \mathbb{R}^N$ we have

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_4(|y|^q + |y|^p). \tag{2.7}$$

Example 2.6. The following maps satisfy hypotheses $H(a)$:

(i) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p-Laplacian operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad u \in W^{1,p}(\Omega).$$

(ii) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$. This map corresponds to the (p,q)-differential operator defined by

$$\Delta_p u + \Delta_q u, \quad u \in W^{1,p}(\Omega).$$

(iii) $a(y) = (1+|y|^2)^{(p-2)/2}y$ with $1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |\nabla u|^2)^{(p-2)/2}\nabla u), \quad u \in W^{1,p}(\Omega).$$

Remark 2.7. The hypotheses $H(a)$ unify the operators from Example 2.6 (i)–(iii), which are widely examined due to their applications in physics (see, for example, [3, 6]). The motivation for this kind of hypotheses comes from the regularity theorem of Lieberman (Theorem 2.3): in the case $A(z, \xi, y) = a(y)$, $B(z, \xi, y) = f(z, \xi)$, one obtains the following form of the assumptions (2.1):

$$(\nabla a(y)x, x)_{\mathbb{R}^N} \ge \frac{h(|y|)}{|y|}|x|^2, \quad y \ne 0_N, \tag{2.8a}$$

$$\Big|\frac{\partial}{\partial y_j}a_i(y)\Big| \le \Lambda\frac{h(|y|)}{|y|}, \quad y \ne 0_N, \tag{2.8b}$$

$$|f(z, \xi)| \le \Lambda_1(1 + h(|y|)|y|). \tag{2.8c}$$

Thus, assuming on the map a hypotheses $H(a)$(ii)–(iii), we guarantee that for a suitable reaction term f, all weak solutions of problem (1.1) actually have locally Hölder continuous first derivatives.

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u(z)), \nabla v(z))_{\mathbb{R}^N}\, \mathrm{d}z, \quad u, v \in W^{1,p}(\Omega), \tag{2.9}$$

where $\langle \cdot, \cdot\rangle$ denotes duality brackets for $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$.

Observe that by Proposition 2.5, the proof of Proposition 3.1 in Gasiński-Papageorgiou [10] remains valid for hypotheses $H(a)$(i)–(iv). Thus we have the following result.

Proposition 2.8. *The nonlinear map $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ defined by* (2.9) *is bounded, continuous and of type $(S)_+$, i.e., if*

$$u_n \longrightarrow u \quad \textit{weakly in } W^{1,p}(\Omega)$$

and

$$\limsup_{n\to+\infty} \big\langle A(u_n),\ u_n - u\big\rangle \le 0,$$

then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

To deal with the boundary condition in problem (1.1), we use appropriate function space framework, due to Casas-Fernández [4].

3. EXISTENCE THEOREM

In this section we will prove an existence theorem for some version of problem (1.1), whose particular case will be used for the multiplicity result in the next section. Namely, we consider the following nonlinear Neumann problem:

$$\begin{cases} -\mathrm{div} a\,(\nabla u(z)) = f\big(z, u(z)\big) + h(z) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $h \in L^\infty(\Omega)$ is such that

$$\int_\Omega h(z)\mathrm{d}z = 0 \tag{3.2}$$

and on f we will impose some hypotheses.

In our work we will consider the following direct sum decomposition of the Sobolev space $W^{1,p}(\Omega)$

$$W^{1,p}(\Omega) = \mathbb{R} \oplus V,$$

where

$$V = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u(z)\mathrm{d}z = 0 \right\}.$$

Hence, every $u \in W^{1,p}(\Omega)$ admits a unique decomposition

$$u = r_u + \widehat{u} \quad \text{with } r_u \in \mathbb{R} \text{ and } \widehat{u} \in V.$$

Due to Poincaré-Wirtinger inequality (see for example Gasiński-Papageorgiou [9, p. 84]), there exists a constant $c_0(N,p) > 0$ such that for every $v \in V$ we have

$$\|v\|_p \leq c_0(N,p)\|\nabla v\|_p. \tag{3.3}$$

In particular, $\|\nabla(\cdot)\|_p$ is an equivalent norm on V.

Before stating the existence result, let us consider the following auxiliary problem:

$$\begin{cases} -\mathrm{div} a\left(\nabla u(z)\right) = h(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Let $\psi\colon V \longrightarrow \mathbb{R}$ be the C^1-functional, defined by

$$\psi(v) = \int_\Omega G(\nabla v(z))\mathrm{d}z - \int_\Omega h(z)v(z)\mathrm{d}z \quad \text{for all} \quad v \in V.$$

Proposition 3.1. *Problem* (3.4) *has a solution* $v_0 \in V \cap C^1(\overline{\Omega})$*, which is a minimizer of* ψ*.*

Proof. As $L^p(\Omega) \subseteq L^1(\Omega)$, there exists a constant $\hat{c}_1 > 0$ such that

$$\|v\|_1 \leq \hat{c}_1\|v\|_p \quad \text{for all} \quad v \in V. \tag{3.5}$$

Thus, by virtue of the Poincaré-Wirtinger inequality (see (3.3) and recall that $\|\nabla(\cdot)\|_p$ is an equivalent norm on V) and Propostion 2.5 (see (2.7)), we see that ψ is coercive:

$$\psi(v) \geq \frac{c_1}{p(p-1)}\|\nabla v\|_p^p - \hat{c}_1\|h\|_\infty\|v\|_p \geq \frac{c_1}{p(p-1)}\|\nabla v\|_p^p - \hat{c}_1 c_0(N,p)\|\nabla v\|_p.$$

Also, using (2.7), we see easily that ψ is sequentially weakly lower semicontinuous (observe that G is strictly convex). Hence, by the Weierstrass theorem, we can find $v_0 \in V$ such that

$$\psi(v_0) = \inf\left\{\psi(v) : \ v \in V\right\},$$

so

$$\psi'(v_0) = 0 \quad \text{in} \quad V^*.$$

This implies

$$\left\langle A(v_0), v\right\rangle = \int_\Omega h(z)v(z)\mathrm{d}z \quad \text{for all} \quad v \in V. \tag{3.6}$$

For $y \in W^{1,p}(\Omega)$, let us define

$$v(z) = y(z) - \frac{1}{|\Omega|_N} \int_\Omega y(z)\mathrm{d}z.$$

Then $v \in V$. Thus, from (3.6) and (3.2) we have

$$\langle A(v_0), y \rangle = \int_\Omega h(z)y(z)\mathrm{d}z.$$

As $y \in W^{1,p}(\Omega)$ was arbitrary, we obtain

$$A(v_0) = h \quad \text{in} \quad W^{1,p}(\Omega)^*.$$

This implies that $v_0 \in V$ solves (3.4) with $v_0 \in L^\infty(\Omega)$ (see, for example, Gasiński-Papageorgiou [10, pp. 860–861]).

By the regularity theorem of Lieberman (Theorem 2.3), we have

$$v_0 \in V \cap C^1(\overline{\Omega}). \qquad \square$$

The hypotheses on the reaction term f are the following:

H(f)₁: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)_+$, $c_5 > 0$, $r \in (p, p^*)$ such that

$$|f(z,\zeta)| \le a(z) + c_5|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R};$$

(ii) there exists $\xi \in L^1(\Omega)$ such that

$$F(z,\zeta) \le \xi(z) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $F(z,\zeta) = \int_0^\zeta f(z,s)\mathrm{d}s$;

(iii) there exists $c_6 \in \mathbb{R} \setminus \{0\}$ such that

$$\int_\Omega F(z,c_6)\mathrm{d}z > 0.$$

Example 3.2. The following function satisfies hypotheses $H(f)_1$ (for the sake of simplicity we drop the z-dependence):

$$f(\zeta) = \begin{cases} \dfrac{\pi}{2\mathrm{e}}|\zeta|^{p-2}\zeta & \text{if} \quad |\zeta| \le 1, \\[2ex] \dfrac{\pi}{2}\exp(-|\zeta|)\sin\left(\dfrac{\pi}{2}\zeta\right) + \operatorname{sgn}(\zeta)\exp(-|\zeta|)\cos\left(\dfrac{\pi}{2}\zeta\right) & \text{if} \quad |\zeta| > 1. \end{cases}$$

In this case the potential function F is given by

$$F(\zeta) = \begin{cases} \dfrac{\pi}{2\mathrm{e}p}|\zeta|^p & \text{if} \quad |\zeta| \le 1, \\[2ex] \dfrac{\pi}{2\mathrm{e}p} - \exp(-|\zeta|)\cos\left(\dfrac{\pi}{2}\zeta\right) & \text{if} \quad |\zeta| > 1. \end{cases}$$

Let $\varphi\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional, given by

$$\varphi(u) = \int\limits_\Omega G(\nabla u(z))\mathrm{d}z - \int\limits_\Omega F\big(z, u(z)\big)\mathrm{d}z - \int\limits_\Omega h(z)u(z)\mathrm{d}z, \quad u \in W^{1,p}(\Omega)$$

(the energy functional for (3.1)). For the unique decomposition $u = r_u + v_u$ with $r_u \in \mathbb{R}$ and $v_u \in V$ of any $u \in W^{1,p}(\Omega)$, we have

$$\psi(u) = \psi(v_u) - \int\limits_\Omega F\big(z, u(z)\big)\mathrm{d}z, \quad u \in W^{1,p}(\Omega) \tag{3.7}$$

(see (3.2)).

Remark 3.3. Hypotheses $H(f)_1$ incorporate problems which are strongly resonant at infinity (see Bartolo-Benci-Fortunato [2]). As a consequence, we encounter a partial lack of compactness in terms of the Palais-Smale condition, i.e. the energy functional ϕ does not satisfy this condition at any level $c \in \mathbb{R}$. Thus we need to specify the interval, in which the Palais-Smale condition is satisfied.

In what follows, let $v_0 \in V \cap C^1(\overline{\Omega})$ be a solution of problem (3.4), which exists by Proposition 3.1, and also

$$\beta := \int\limits_\Omega \limsup_{|\zeta| \to +\infty} F(z, \zeta)\mathrm{d}z < \infty \tag{3.8}$$

(see hypothesis $H(f)_1$(ii)).

Proposition 3.4. *If hypotheses $H(a)$ and $H(f)_1$ hold and*

$$c < \psi(v_0) - \beta \in (-\infty, +\infty],$$

then φ satisfies the Palais-Smale condition at level c.

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$\varphi(u_n) \longrightarrow c < \psi(v_0) - \beta \tag{3.9}$$

and

$$\varphi'(u_n) \longrightarrow 0 \quad \text{in } W^{1,p}(\Omega)^*. \tag{3.10}$$

As the sequence $\{\varphi(u_n)\}_n$ is convergent in $W^{1,p}(\Omega)$, we have that it is bounded by some constant $M_1 > 0$.

Recall that there exists unique $r_n \in \mathbb{R}$, $v_n \in V$ such that

$$u_n = r_n + v_n \quad \text{for all} \quad n \geq 1. \tag{3.11}$$

First we will show that the sequence $\{v_n\}_n$ is bounded in $W^{1,p}(\Omega)$. Indeed, by (2.7) (see Proposition 2.5), hypothesis $H(f)_1$(ii), (3.5) and the Poincaré-Wirtinger inequality (3.3), for $\hat{c}_2 := \hat{c}_1 c_0(N,p) > 0$ we have that

$$\begin{aligned} M_1 \geq \varphi(u) &\geq \frac{c_1}{p(p-1)} \|\nabla v_n\|_p^p - \int_\Omega F(z,u_n)\mathrm{d}z - \int_\Omega h v_n \mathrm{d}z \\ &\geq \frac{c_1}{p(p-1)} \|\nabla v_n\|_p^p - \|\xi\|_1 - \hat{c}_2 \|\nabla v_n\|_p \quad \text{for all} \quad n \geq 1. \end{aligned}$$

This implies that there exist some constants $M_2 > 0$, $\hat{c}_3 > 0$ such that

$$\hat{c}_3 \|\nabla v_n\|_p \leq M_2 \quad \text{for all} \quad n \geq 1$$

(recall that $p > 1$). Thus

$$\text{the sequence } \{v_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded}$$

(recall that $\|\nabla(\cdot)\|_p$ is an equivalent norm on $V \subseteq W^{1,p}(\Omega)$).

So, by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} v_n &\longrightarrow \widehat{v} \quad \text{weakly in } W^{1,p}(\Omega), \\ v_n &\longrightarrow \widehat{v} \quad \text{in } L^p(\Omega), \\ v_n(z) &\longrightarrow \widehat{v}(z) \quad \text{for almost all } z \in \Omega. \end{aligned}$$

In particular, for almost all $z \in \Omega$ there exists $m(z) \geq 0$ such that

$$\big|v_n(z)\big| \leq m(z) \quad \text{for almost all } z \in \Omega, \text{ all } n \geq 1, \tag{3.12}$$

with $m \in L^p(\Omega)$.

Claim. *The sequence $\{u_n\}_n \subseteq W^{1,p}(\Omega)$ is bounded.*

Aiming for a contradiction, suppose that, passing to a subsequence if necessary, we have

$$\|u_n\| \longrightarrow +\infty.$$

Using the unique decomposition (3.11), (3.12) and the fact, that the sequence $\{v_n\}_n \subseteq W^{1,p}(\Omega)$ is bounded, we can easily see that this implies

$$\big|u_n(z)\big| \longrightarrow +\infty \quad \text{for almost all} \quad z \in \Omega,$$

which leads us to the following contradiction:

$$\begin{aligned} \psi(v_0) - \beta > c = \liminf_{n\to\infty} \varphi(u_n) &= \liminf_{n\to\infty} \left(\psi(v_n) - \int_\Omega F(z,u_n(z))\mathrm{d}z \right) \\ &\geq \psi(v_0) - \int_\Omega \limsup_{n\to\infty} F(z,u_n(z))\mathrm{d}z = \psi(v_0) - \beta. \end{aligned}$$

Here we have used the Fatou lemma for functions bounded from below and the fact that v_0 is a minimizer of ψ (see Proposition 3.1). This proves the claim.

As the sequence $\{u_n\}_n \subseteq W^{1,p}(\Omega)$ is bounded, it admits a weakly convergent subsequence. So, passing to a subsequence if necessary, we can find $u \in W^{1,p}(\Omega)$ such that

$$u_n \longrightarrow u \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.13}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega) \tag{3.14}$$

(recall that $r \in (p, p^*)$).

We want to show that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$ by the use of Proposition 2.8. For this purpose, let us first notice that using our assumption (3.10), we can find a sequence $\{\varepsilon_n\}_n \subseteq (0,\infty)$ such that

$$\left| \langle A(u_n), w \rangle - \int_\Omega f(z, u_n(z)) w(z) \mathrm{d}z - \int_\Omega h(z) y(z) \mathrm{d}z \right| \leq \varepsilon_n \|w\| \quad \text{for all} \quad w \in W^{1,p}(\Omega),$$

with $\varepsilon_n \searrow 0$. Setting $w = u_n - u \in W^{1,p}(\Omega)$ we obtain

$$\begin{aligned} &\left| \langle A(u_n), u_n - u \rangle - \int_\Omega f(z, u_n(z))(u_n - u)(z) \mathrm{d}z - \int_\Omega h(z)(u_n - u)(z) \mathrm{d}z \right| \\ &\qquad \leq \varepsilon_n \|u_n - u\| \quad \text{for all} \quad n \geq 1. \end{aligned} \tag{3.15}$$

As $h \in L^\infty(\Omega) \subseteq L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, from (3.13) we have

$$\int_\Omega h(z)(u_n - u)(z) \mathrm{d}z \longrightarrow 0. \tag{3.16}$$

Also, by $H(f)_1$(ii) and (3.14), we obtain

$$\begin{aligned} \left| \int_\Omega f(z, u_n(z))(u_n - u)(z) \mathrm{d}z \right| &\leq \|u_n - u\|_r \left(\int_\Omega |f(z, u_n(z))|^{r'} \mathrm{d}z \right)^{1/r'} \\ &\leq \|u_n - u\|_r \left(\int_\Omega a(z)^{r/(r-1)} \mathrm{d}z + c_5 \|u_n\|_r^r \right)^{1/r'} \\ &\leq M_3 \|u_n - u\|_r \end{aligned} \tag{3.17}$$

with some constant $M_3 > 0$ and $r' > 1$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Therefore, if in (3.15) we pass to the limit as $n \to +\infty$, using (3.16), (3.17) and (3.14) we obtain

$$\lim_{n \to +\infty} \langle A(u_n),\ u_n - u \rangle = 0.$$

Thus, by Proposition 2.8, we have that

$$u_n \longrightarrow u \quad \text{in} \quad W^{1,p}(\Omega)$$

and so we have proven that φ satisfies the Palais-Smale condition at any level $c < \psi(v_0) - \beta$. $\square$

Having this compactness result, we are ready to state an existence theorem for problem (1.1).

Theorem 3.5. *Let $v_0 \in V$ be a minimizer of ψ (see Proposition 3.1). If hypotheses $H(a)$, $H(f)_1$ hold and*

$$\beta = \int\limits_{\Omega} \limsup_{|\zeta|\to+\infty} F(z,\zeta)\mathrm{d}z < \int\limits_{\Omega} F(z,v_0)\mathrm{d}z,$$

then problem (3.1) admits a nontrivial solution $u^* \in C^1(\overline{\Omega})$.

Proof. We are going to apply Theorem 2.1 to our problem. From hypothesis $H(f)_1$(ii), we have that φ is bounded below (see (3.7)).

Set

$$m_\varphi := \inf\left\{\varphi(u) : \ u \in W^{1,p}(\Omega)\right\} > -\infty. \tag{3.18}$$

As

$$-\infty < m_\varphi \le \varphi(v_0) = \psi(v_0) - \int\limits_{\Omega} F(z,v_0)\mathrm{d}z < \psi(v_0) - \beta,$$

by Proposition 3.4, we have that φ satisfies the Palais-Smale condition at level m_φ.

Therefore, we can use Theorem 2.1 to find $u^* \in W^{1,p}(\Omega)$ such that

$$\varphi(u^*) = m_\varphi.$$

Moreover, by hypothesis $H(f)_1$(iii), we have that

$$\varphi(u^*) = m_\varphi \le \varphi(c_6) < 0 = \varphi(0),$$

so

$$u^* \neq 0.$$

Also, we have

$$\varphi'(u^*) = 0,$$

so

$$A(u^*) = N_f(u^*) + h$$

and thus $u^* \in C^1(\overline{\Omega})$ (see the proof of Proposition 3.1) is a nontrivial solution of (3.1). $\square$

Remark 3.6. Observe that hypothesis $H(f)_1$(iii) is needed only to guarantee that u^* is nontrivial. (Observe that v_0 can be trivial if $h = 0$.)

4. MULTIPLICITY THEOREM

In this section we prove a multiplicity theorem for problem (1.1) (i.e., we set $h = 0$ in problem (3.1). To this end we need additional hypotheses on f:

$\boldsymbol{H(f)_2}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses $H(f)_2$(i)–(iii) are the same as $H(f)_1$(i)–(iii) and

(iv) we have

$$\beta = \int_\Omega \limsup_{|\zeta|\to+\infty} F(z,\zeta) < 0$$

and there exists $\vartheta \in L^\infty(\Omega)_+$, $\vartheta \neq 0$, such that

$$\vartheta(z) \leq \liminf_{\zeta\to 0} \frac{F(z,\zeta)}{|\zeta|^p} \text{ uniformly for almost all } z \in \Omega;$$

(v) we have

$$F(z,\zeta) \leq \widehat{\lambda}_1 \frac{c_1}{p(p-1)} |\zeta|^p \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $\widehat{\lambda}_1 > 0$ being the first nonzero eigenvalue of the negative Neumann p-Laplacian and $c_1 > 0$ as in $H(a)$(ii).

Example 4.1. The following function satisfies hypotheses $H(f)_2$ (as before, for the sake of simplicity, we drop the z-dependence):

$$f(\zeta) = \begin{cases} \widehat{\lambda}_1 \dfrac{c_1}{p-1} |\zeta|^{p-2}\zeta & \text{if} \quad |\zeta| \leq 1, \\[2ex] \dfrac{\widehat{\lambda}_1 c_1 + p - 1}{p-1} \dfrac{\zeta}{|\zeta|^{p+2}} - |\zeta|^{r-2}\zeta & \text{if} \quad |\zeta| > 1, \end{cases}$$

where $p < r < p^*$. In this case the potential function F is given by

$$F(\zeta) = \begin{cases} \widehat{\lambda}_1 \dfrac{c_1}{p(p-1)} |\zeta|^p & \text{if } |\zeta| \leq 1, \\[2ex] -\dfrac{\widehat{\lambda}_1 c_1 + p - 1}{p(p-1)} \dfrac{1}{|\zeta|^p} - \dfrac{1}{r}|\zeta|^r + \dfrac{2\widehat{\lambda}_1 \cdot c_1}{p(p-1)} + \dfrac{r+p}{rp} & \text{if } |\zeta| > 1. \end{cases}$$

Now the energy functional $\widehat{\varphi}\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ is given by

$$\widehat{\varphi}(u) = \int_\Omega G(\nabla u) \mathrm{d}z - \int_\Omega F\big(z, u(z)\big) \mathrm{d}z, \quad u \in W^{1,p}(\Omega).$$

Evidently, $\widehat{\varphi} \in C^1\big(W^{1,p}(\Omega)\big)$.

Theorem 4.2. *If hypotheses $H(a)$ and $H(f)_2$ hold, then problem* (1.1) *has at least two nontrivial smooth solutions $u_1^*, u_2^* \in C^1(\overline{\Omega})$.*

Proof. Let $v_0 \in V$ be a minimizer of ψ (see Proposition 3.1). Since we obtain problem (1.1) by setting $h = 0$ in problem (3.1), we have

$$v_0 = 0.$$

Therefore,

$$\int_\Omega F\big(z, v_0(z)\big)\mathrm{d}z = 0 > \beta$$

(see hypothesis $H(f)_2$(iv)). Thus, we can apply Theorem 3.5 to obtain one nontrivial smooth solution $u_1^* \in C^1(\overline{\Omega})$.

The existence of the second nontrivial solution will be shown via the second deformation theorem (see Theorem 2.2).

First, we prove that φ is negative on $\overline{B}_R \cap \mathbb{R}$ for some $R > 0$, where

$$\overline{B}_R = \big\{u \in W^{1,p}(\Omega) : \ \|u\| \le R\big\}.$$

For any fixed $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,\zeta) \ \ge \ \big(\vartheta(z) - \varepsilon\big)|\zeta|^p \quad \text{for almost all} \quad z \in \Omega, \ \text{all } |\zeta| \le \delta$$

(see hypothesis $H(f)_2$(iv)). Then

$$\varphi(r) = -\int_\Omega F(z,r)\mathrm{d}z \le \ |r|^p\left(\varepsilon|\Omega|_N - \int_\Omega \vartheta(z)\mathrm{d}z\right), \quad r \in [-\delta, \delta].$$

Thus, choosing $\varepsilon \in \big(0, \frac{1}{|\Omega|_N}\int_\Omega \vartheta(z)\mathrm{d}z\big)$, we have that $\varphi(r) < 0$, $r \in [-\delta(\varepsilon), \delta(\varepsilon)]$ and

$$\max\big\{\varphi(r) : \ r \in \overline{B}_R \cap \mathbb{R}\big\} < 0 \quad \text{for all} \quad R \in \big(0, \ \delta|\Omega|_N^{\frac{1}{p}}\big). \tag{4.1}$$

Claim. *Define*

$$\Gamma := \big\{\gamma \in C\big(\overline{B}_R \cap \mathbb{R}, \ W^{1,p}(\Omega)\big) : \ \gamma|_{\partial\overline{B}_R \cap \mathbb{R}} = id|_{\partial\overline{B}_R \cap \mathbb{R}}\big\}.$$

Then

$$\widehat{c}_R := \inf_{\gamma\in\Gamma} \max_{v \in \overline{B}_R \cap \mathbb{R}} \varphi\big(\gamma(v)\big) \ge 0 \quad \textit{for all} \quad R > 0.$$

Note that

$$\big(\partial\overline{B}_R \cap \mathbb{R}\big) \cap C(p) = \emptyset \quad \text{for all} \quad R > 0.$$

For every $u \in C(p)$ (see Proposition 2.4), we have

$$\varphi(u) \ge \frac{c_1}{p(p-1)}\|\nabla u\|_p^p - \widehat{\lambda}_1 \frac{c_1}{p(p-1)}\|u\|_p^p \ge 0 \tag{4.2}$$

(see (2.7), $H(f)_2$(v) and the Poincaré-Wirtinger inequality) and

$$\inf_{C(p)} \varphi = 0$$

(see Gasiński-Papageorgiou [9, p. 756]). For a fixed $R > 0$ and arbitrary $\gamma \in \Gamma$, let us define

$$\sigma(r) := \int_\Omega \left|\gamma(r)\right|^{p-2} \gamma(r) \mathrm{d}z, \quad r \in \overline{B}_R \cap \mathbb{R} = [-R_0, R_0],$$

with $R_0 := R|\Omega|_N^{\frac{1}{p}} > 0$. Then

$$\sigma(-R_0) < 0 < \sigma(R_0).$$

(Observe that $\partial B_R \cap \mathbb{R} = \{\pm R_0\}$.) Thus, by virtue of the Bolzano theorem, we can find $\widehat{r} \in \overline{B}_R \cap \mathbb{R}$ such that

$$\sigma(\widehat{r}) = \int_\Omega \left|\gamma(\widehat{r})\right|^{p-2} \gamma(\widehat{r}) \mathrm{d}z = 0.$$

That means

$$\gamma(\widehat{r}) \in C(p)$$

and

$$\gamma\big(\overline{B}_R \cap \mathbb{R}\big) \cap C(p) \neq \emptyset.$$

Thus, from (4.2) we obtain

$$\widehat{c}_R \ \geq \ 0 \tag{4.3}$$

(recall that $\gamma \in \Gamma$ was arbitrary). This proves the claim.

Set

$$a = \inf \varphi = \varphi(u_1^*) < 0 \quad \text{and} \quad b = \varphi(0) = 0.$$

By virtue of Proposition 3.4 and hypothesis $H(f)_2$(iv), we have that φ satisfies the Palais-Smale condition for every level $c \in [a, b]$.

To obtain a contradiction, suppose that $\{0, u_1^*\}$ are the only critical points of φ. Then

$$\varphi^{-1}(\{a\}) = \{u_1^*\}$$

and we are able to apply Theorem 2.2. Using (4.1) and the homotopy

$$\widehat{h} \colon [0,1] \times (\varphi^b \setminus K_\varphi^b) \ \longrightarrow \ \varphi^b$$

given by Theorem 2.2, we can easily produce a map

$$\gamma_0 \colon \overline{B}_R \cap \mathbb{R} \ \longrightarrow \ W^{1,p}(\Omega)$$

such that

$$\varphi\big(\gamma_0(u)\big) < 0 \quad \text{for all} \quad u \in \overline{B}_r \cap \mathbb{R},$$

which implies $\widehat{c}_R < 0$ (see (4)), in contradiction with the claim (for details see for example Gasiński-Papagriorgiou [11]).

Thus, there exists $u_2^* \in K_\varphi$, such that $u_2^* \notin \{0, u_1^*\}$. Then

$$A(u_2^*) = N_f(u_2^*).$$

That implies that $u_2^* \in C^1(\overline{\Omega})$ solves (1.1) (see the proof of Proposition 3.1). Thus we have proved the existence of two distinct, nontrivial solutions of (1.1), as desired. $\square$

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