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ANALYTICAL AND NUMERICAL SOLVING OF LINEAR NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND-ORDER WITH CHANGEABLE COEFFICIENTS BY USING CONSTANT VARIATION METHOD AND APPLICATION OF *MATHEMATICA* PROGRAM

Abstract

Introduction and aim: The paper presents the analytical and numerical algorithm of solving linear non-homogeneous equations of the second order with changeable coefficients. The aim of the work is to show the algorithms for solving equations both analytically and numerically. The additional aim is to make some graphical interpretation of solutions.

Material and methods: Some selected equations have been chosen from the subject literature. In the solutions the constant variation method has been presented.

Results: The paper presents the selected linear non-homogeneous equations of the second order with constant coefficients containing linear, homographic, logarithmic and trigonometric functions.

Conclusion: Taking into account the constant variation method it is possible to solve the second order linear non-homogeneous differential equations with changeable coefficients. Using the *Mathematica* program it is possible quickly get a solution and create its graphical interpretation.

Keywords: Ordinary differential equations, linear equations, homogeneous equations, equations of the second order, changeable coefficients, variation constant method, analytical solution, numerical solution, *Mathematica*.

(Received: 03.05.2018; Revised: 13.05.2018; Accepted: 20.05.2018)

ROZWIĄZYWANIE ANALITYCZNO-NUMERYCZNE LINIOWYCH NIEJEDNORODNYCH RÓWNAŃ RÓŻNICZKOWYCH DRUGIEGO RZĘDU O ZMIENNYCH WSPÓŁCZYNNIKACH PRZY UŻYCIU METODY WARIACJI STAŁEJ I ZASTOSOWANIU PROGRAMU MATHEMATICA

Streszczenie

Wstęp i cel: *W pracy pokazano algorytm analityczny i numeryczny rozwiązywania równań różniczkowych liniowych niejednorodnych drugiego rzędu o zmiennych współczynnikach. Celem pracy jest pokazanie algorytmu rozwiązywania równań zarówno sposobem analitycznym jak i numerycznym. Ponadto dodatkowym celem jest interpretacji graficznej rozwiązań.*

Materiał i metody: *Wybrane równania zaczerpnięto z literatury przedmiotu. W rozwiażanich równań zastosowano metodę wariacji stałej.*

Wyniki: *W pracy opracowano wybrane równania różniczkowe liniowe niejednorodne drugiego rzędu o zmiennych współczynnikach zawierających funkcje liniowe, homograficzne, logarytmiczne i trygonometryczne.*

Wniosek: *Stosując metodę uzmienniania stałej jest możliwe rozwiązywanie równań różniczkowych liniowych niejednorodnych drugiego rzędu o zmiennych współczynnikach. Wykorzystując program Mathematica można szybko uzyskać rozwiązanie oraz sporządzić jego interpretację graficzną.*

Słowa kluczowe: *Równania różniczkowe zwyczajne, równania liniowe, równania niejednorodne, równania drugiego rzędu, zmienne współczynniki, metoda wariacji stałej, rozwiązanie analityczne, rozwiązanie numeryczne, Mathematica.*

(Otrzymano: 03.05.2018; Zrecenzowano: 13.05.2018; Zaakceptowano: 20.5.2018)

1. Theoretical introduction

Definition 1.

The differential non-homogeneous linear equation of the second order with constant coefficients has the following form:

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = q(x) \tag{1}$$

where $p_1(x)$, $p_2(x)$, $q(x)$ are some continuous function in a certain interval (a, b) [2], [4].

Definition 2.

The differential homogeneous linear equation of the first order with constant coefficients has the following form:

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 \tag{2}$$

where $p_1(x)$, $p_2(x)$ are some continuous function in a certain interval (a, b) [8], [9].

Theorem 1.

The general solution of the non-homogeneous differential equation (1) is the sum of the general solution of the homogeneous differential equation (2) and the particular solution of the non-homogeneous differential equation (1) [15], [16].

We assume that it is known the first particular integral of the linear homogeneous differential equation of the second order with changeable coefficients which have the following form:

$$y_1 = y_1(x). \tag{3}$$

The second particular integral of the linear homogeneous differential equation of the second order with changeable coefficients it is possible to get from Liouville-Ostrogradzki formula [6], [11]-[13]:

$$y_2(x) = y_1(x)\int \frac{\exp[-\int p_1(x)dx]}{y_1^2(x)}dx. \tag{4}$$

The general integral of the linear homogeneous differential equation of the second order with changeable coefficients finally has the following form:

$$y(x) \equiv \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) \tag{5}$$

Where C_1 and C_2 are any real constants and the functions $y_1(x)$, $y_2(x)$ are defined by the formulae (3) and (4) [10].

The particular solution of the non-homogeneous equation (1) is found by the constant variation method. Therefore:

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x). \tag{6}$$

The unknown functions $C_1(x)$ and $C_2(x)$ in the formula (6) are determined from the following system of equations (i.e. more specifically, we designate their derivatives) [2], [6]:

$$\begin{cases} \frac{dC_1}{dx}y_1(x)+\frac{dC_2}{dx}y_2(x)=0 \\ \frac{dC_1}{dx}\frac{dy_1}{dx}+\frac{dC_2}{dx}\frac{dy_2}{dx}=q(x). \end{cases} \tag{7}$$

The system (7) can be solving by using the determinant method (provided it is a Cramer system). The main determinant (i.e. determinant of Wonks) has the following form:

$$W(x)\equiv\begin{vmatrix} y_1(x) & y_2(x) \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}=y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}\neq 0. \tag{8}$$

Thus, the determinants for the functions $\frac{dC_1}{dx}$ and $\frac{dC_2}{dx}$ have the form:

$$W_{C_1^{'}(x)}(x)\equiv\begin{vmatrix} 0 & y_2(x) \\ q(x) & \frac{dy_2}{dx} \end{vmatrix}=-q(x)y_2(x), \tag{9}$$

$$W_{C_2^{'}(x)}(x)\equiv\begin{vmatrix} y_1(x) & 0 \\ \frac{dy_1}{dx} & q(x) \end{vmatrix}=q(x)y_1(x). \tag{10}$$

Using Cramer formulas, we get:

$$\frac{dC_1}{dx}=\frac{W_{C_1^{'}(x)}(x)}{W(x)}, \tag{11}$$

$$\frac{dC_2}{dx}=\frac{W_{C_2^{'}(x)}(x)}{W(x)}. \tag{12}$$

Therefore

$$\frac{dC_1}{dx}=\frac{-q(x)y_2(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}, \tag{13}$$

$$\frac{dC_2}{dx}=\frac{q(x)y_1(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}. \tag{14}$$

After integration the equations (13) and (14) relative to x variable we have:

$$C_1(x)=-\int\frac{q(x)y_2(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}\,dx, \tag{15}$$

$$C_2(x)=\int\frac{q(x)y_1(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}\,dx. \tag{16}$$

The particular solution of the non-homogeneous equation (1) has the following form:

$$y(x)\equiv\hat{y}(x)=-y_1(x)\int\frac{q(x)y_2(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}\,dx+y_2(x)\int\frac{q(x)y_1(x)}{y_1(x)\frac{dy_2}{dx}-y_2(x)\frac{dy_1}{dx}}\,dx. \tag{17}$$

Finally the general solution of the non-homogeneous equation (1) has the form:

$$y(x) \equiv \tilde{y}(x) + \hat{y}(x), \tag{18}$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x) +$$
$$- y_1(x)\int \frac{q(x)y_2(x)}{y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx}}\,dx + y_2(x)\int \frac{q(x)y_1(x)}{y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx}}\,dx\,. \tag{19}$$

Where C_1 and C_2 are the real constants and the functions $y_1(x)$, $y_2(x)$ are defined by the formulae (3)-(4) and determinant of Wonks [2], [6]:

$$y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx} \neq 0. \tag{20}$$

2. Analytical and numerical solving of the second order linear non-homogeneous Differential equations with changeable coefficients using constants variation method

Example 1.

Let us consider the following non-homogeneous equation [6]:

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = \frac{3\ln^2(x)}{x^2}. \tag{21}$$

The homogeneous equation has the form:

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0. \tag{22}$$

• Analytical solution

It is clear that the first particular integral of the equation (22) is described by the formula:

$$y_1(x) = x. \tag{23}$$

The second particular integral of the homogeneous equation (22) is possible to get from Liouville-Ostrogradzki formula (4). Therefore:

$$y_2(x) = x\int \frac{\exp[-\int -\frac{3}{x}dx]}{x^2}\,dx, \tag{24}$$

$$y_2(x) = x\int \frac{\exp[3\ln(x)]}{x^2}dx = x\int \frac{\exp[\ln(x^3)]}{x^2}dx = x\int \frac{x^3}{x^2}dx = x\int x\,dx = \frac{x^3}{2}. \tag{25}$$

The general integral of the homogeneous equation (22) finally has the following form:

$$y(x) \equiv \tilde{y}(x) = C_1 x + C_2\frac{x^3}{2} \tag{26}$$

where C_1 and C_2 are any real constants. The particular solution of the non-homogeneous equation (21) is found by the constant variation method. Therefore:

$$y(x) = C_1(x)\cdot x + C_2(x)\cdot\frac{x^3}{2}. \tag{27}$$

Unknown functions $C_1(x)$ and $C_2(x)$ in (27) are determined from the following system:

$$\begin{cases} \dfrac{dC_1}{dx}\cdot x+\dfrac{dC_2}{dx}\dfrac{x^3}{2}=0 \\ \dfrac{dC_1}{dx}\cdot 1+\dfrac{dC_2}{dx}\dfrac{3x^2}{2}=\dfrac{3\ln^2(x)}{x^2}. \end{cases} \tag{28}$$

We solve the system (28) using Cramer method. The main determinant has the form:

$$W(x)\equiv\begin{vmatrix} x & \dfrac{x^3}{2} \\ 1 & \dfrac{3x^2}{2} \end{vmatrix}=\frac{3x^3}{2}-\frac{x^3}{2}=x^3\neq 0. \tag{29}$$

Thus, the determinants for the functions $\dfrac{dC_1}{dx}$ and $\dfrac{dC_2}{dx}$ have the form:

$$W_{C_1^{'}(x)}(x)\equiv\begin{vmatrix} 0 & \dfrac{x^3}{2} \\ \dfrac{3\ln^2(x)}{x^2} & \dfrac{3x^2}{2} \end{vmatrix}=-\frac{x^3}{2}\cdot\frac{3\ln^2(x)}{x^2}=-\frac{3}{2}x\ln(x), \tag{30}$$

$$W_{C_1^{'}(x)}(x)\equiv\begin{vmatrix} x & 0 \\ 1 & \dfrac{3\ln^2(x)}{x^2} \end{vmatrix}=x\cdot\frac{3\ln^2(x)}{x^2}=\frac{3\ln^2(x)}{x}. \tag{31}$$

Using Cramer formulae, we get:

$$\frac{dC_1}{dx}=-\frac{3}{2}\cdot\frac{x\ln(x)}{x^3}=-\frac{3}{2}\cdot\frac{\ln^2(x)}{x^2}, \tag{32}$$

$$\frac{dC_2}{dx}=\frac{3\ln^2(x)}{x^3}\cdot\frac{1}{x}=\frac{3\ln^2(x)}{x^4}. \tag{33}$$

Both sides of above equations we integrate relative to x variable:

$$\int\frac{dC_1}{dx}dx=-\frac{3}{2}\int\frac{\ln^2(x)}{x^2}dx, \tag{34}$$

$$\int\frac{dC_2}{dx}dx=3\int\frac{\ln^2(x)}{x^4}dx. \tag{35}$$

Let us determine the integral:

$$\begin{aligned} &-\frac{3}{2}\int\frac{\ln^2(x)}{x^2}dx=\left\langle \begin{matrix} u=\ln^2(x) \\ v'=x^{-2} \end{matrix}\middle| \begin{matrix} u'=\dfrac{2\ln(x)}{x} \\ v=\int x^{-2}dx=-x^{-1} \end{matrix}\right\rangle=-\frac{3}{2}\left[-\frac{\ln^2(x)}{x}+2\int\frac{\ln(x)}{x^2}\right]= \\ &=\frac{3}{2}\frac{\ln^2(x)}{x}-3\int\frac{\ln(x)}{x^2}=\left\langle \begin{matrix} u=\ln(x) \\ v'=x^{-2} \end{matrix}\middle| \begin{matrix} u'=x^{-1} \\ v=\int x^{-2}dx=-x^{-1} \end{matrix}\right\rangle=\frac{3}{2}\frac{\ln^2(x)}{x}-3\left[-\frac{\ln(x)}{x}+\int x^{-2}dx\right]= \\ &=\frac{3}{2}\frac{\ln^2(x)}{x}+\frac{3\ln(x)}{x}+\frac{3}{x}=\frac{3}{2}\left[\frac{\ln^2(x)}{x}+\frac{2\ln(x)}{x}+\frac{2}{x}\right] \end{aligned} \tag{36}$$

Let us determine the integral:

$$3\int\frac{\ln^2(x)}{x^4}dx=\left\langle \begin{matrix} u=\ln^2(x) \\ v'=x^{-4} \end{matrix}\middle| \begin{matrix} u'=\dfrac{2\ln(x)}{x} \\ v=\int x^{-4}dx=-\dfrac{1}{3x^3} \end{matrix}\right\rangle=3\left[-\frac{\ln^2(x)}{3x^3}+\frac{2}{3}\int\frac{\ln(x)}{x^4}\right]=$$

$$=-\frac{\ln^2(x)}{x^3}+2\int\frac{\ln(x)}{x^4}=\left\langle \begin{matrix} u=\ln(x) \\ v'=x^{-4} \end{matrix}\middle| \begin{matrix} u'=x^{-1} \\ v=\int x^{-4}dx=-\dfrac{1}{3x^3} \end{matrix}\right\rangle=-\frac{\ln^2(x)}{x^3}+2\left[-\frac{\ln(x)}{3x^3}+\frac{1}{3}\int\frac{dx}{x^4}\right]= \tag{37}$$

$$=-\frac{\ln^2(x)}{x^3}-\frac{2}{3}\frac{\ln(x)}{x^3}+\frac{2}{3}\int\frac{dx}{x^4}=-\frac{\ln^2(x)}{x^3}-\frac{2}{3}\frac{\ln(x)}{x^3}-\frac{2}{9}\frac{1}{x^3}=-\frac{1}{9}\left[\frac{9\ln^2(x)}{x^3}+\frac{6\ln(x)}{x^3}+\frac{2}{x^3}\right]$$

After integration (34) and (35) we have:

$$C_1(x)=\frac{3}{2}\left[\frac{\ln^2(x)}{x}+\frac{2\ln(x)}{x}+\frac{2}{x}\right], \tag{38}$$

$$C_2(x)=-\frac{1}{9}\left[\frac{9\ln^2(x)}{x^3}+\frac{6\ln(x)}{x^3}+\frac{2}{x^3}\right]. \tag{39}$$

The particular solution of the non-homogeneous equation (21) has the following form:

$$y(x)\equiv\hat{y}(x)=\frac{3}{2}\left[\frac{\ln^2(x)}{x}+\frac{2\ln(x)}{x}+\frac{2}{x}\right]\cdot x-\frac{1}{9}\left[\frac{9\ln^2(x)}{x^3}+\frac{6\ln(x)}{x^3}+\frac{2}{x^3}\right]\cdot\frac{x^3}{2}, \tag{40}$$

$$y(x)\equiv\hat{y}(x)=\frac{3}{2}\ln^2(x)+3\ln(x)+3-\frac{1}{2}\ln^2(x)-\frac{1}{3}\ln(x)-\frac{1}{9}=\ln^2(x)+\frac{8}{3}\ln(x)+\frac{26}{9}, \tag{41}$$

$$y(x)=\frac{1}{9}[9\ln^2(x)+24\ln(x)+26]. \tag{42}$$

Finally, the general solution of the non-homogeneous equation (21) has the formula:

$$y(x)\equiv\tilde{y}(x)+\hat{y}(x)=C_1x+C_2\frac{x^3}{2}+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26]. \tag{43}$$

$$y(x)=C_1x+C_2\frac{x^3}{2}+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26]. \tag{44}$$

where C_1 and C_2 are the real constants.

- **Numerical solution**

In numerical analysis we take into account the solution (44) where C[1]=C[2]=1, C[1]=C[2]=2, C[1]=C[2]=3, C[1]=C[2]=4; [1], [3], [5], [7], [14]. Thus

$$y(x)=x+\frac{x^3}{2}+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26], \tag{45}$$

$$y(x)=2x+x^3+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26], \tag{46}$$

$$y(x)=3x+\frac{3}{2}x^3+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26], \tag{47}$$

$$y(x)=4x+2x^3+\frac{1}{9}[9\ln^2(x)+24\ln(x)+26]. \tag{48}$$

Program 1. (Mathematica 7.0)

```
In[1]:= DSolve[y''[x] – (3/x)*y'[x] + (3/x^2)*y[x] == (3(Log[x])^2)/x^2 y[x], x] /.
        {C[1]→1,C[2]→0.5*1},{C[1]→2,C[2]→0.5*2},{C[1]→3,C[2]→0.5*3},{C[1]→4,C[2]→0.5*4}}
        Plot[Evaluate[y[x] /. %], {x, 0, 1}, Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},PlotRange → {0, 12}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]
Out[1] =
```

$\{\{\{y[x]\to x+0.5x^3+\frac{1}{9}(26+24Log[x]+9Log[x]^2)\}\}, \quad \{\{y[x]\to 2x+1.x^3+\frac{1}{9}(26+24Log[x]+9Log[x]^2)\}\},$

$\{\{y[x]\to 3x+1.5x^3+\frac{1}{9}(26+24Log[x]+9Log[x]^2)\}\}, \{\{y[x]\to 4x+2.x^3+\frac{1}{9}(26+24Log[x]+9Log[x]^2)\}\}$

Out[2] = = Graphics =

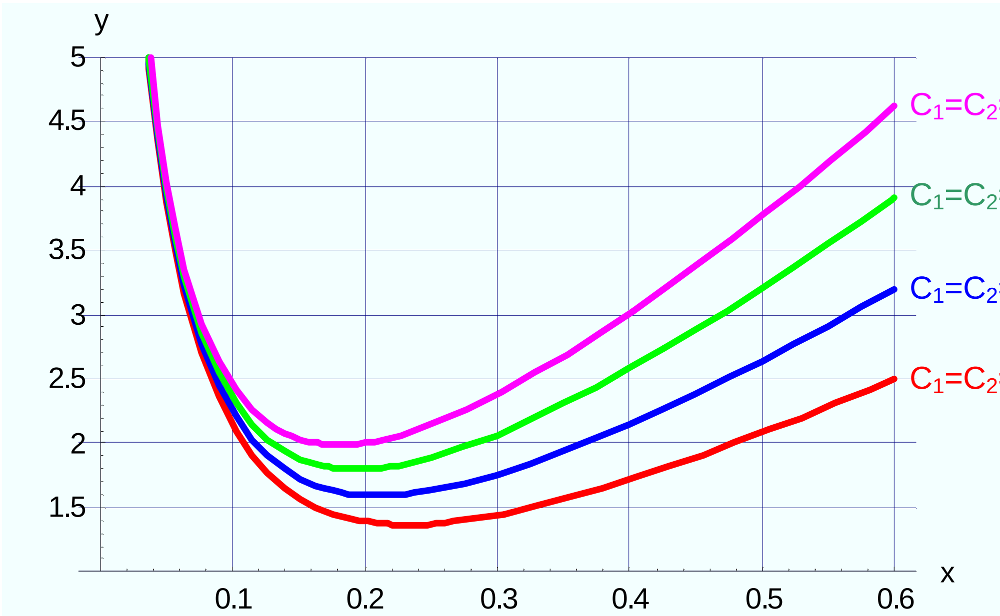

Fig. 1. Graphs of the functions (45)-(48), some solutions of the second order linear non-homogeneous differential equation (21) with changeable coefficients and for constants: $C_1=C_2=1$, $C_1=C_2=2$, $C_1=C_2=3$, $C_1=C_2=4$

Source: Program and graphs in Mathematica elaborated by the Authors

Example 2.

Let us consider the following non-homogeneous equation [6]:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}y=\frac{\sin[2\ln(x)]}{x^2}. \tag{49}$$

The homogeneous equation has the form:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}y=0. \tag{50}$$

• **Analytical solution**

The first particular integral of the homogeneous equation (50) is described by the formula:

$$y_1(x)=\cos[\ln(x)]. \tag{51}$$

Let us proof that the function (51) is the first particular integral of the equation (50):

$$\frac{dy_1}{dx} = -\frac{\sin[\ln(x)]}{x}, \tag{52}$$

$$\frac{d^2y_1}{dx^2} = \frac{-x\cos[\ln(x)] + x\sin[\ln(x)]}{x^3}, \tag{53}$$

Let us put the functions (51) and (52) into the equation (50):

$$\begin{aligned} &\frac{-x\cos[\ln(x)] + x\sin[\ln(x)]}{x^3} - \frac{1}{x}\frac{\sin[\ln(x)]}{x} + \frac{1}{x^2}\cos\ln(x)] = \\ &= -\frac{\cos[\ln(x)]}{x^2} + \frac{\sin[\ln(x)]}{x^2} - \frac{\sin[\ln(x)]}{x^2} + \frac{\cos[\ln(x)]}{x^2} = 0, \quad 0 = 0. \end{aligned} \tag{54}$$

The above result means that the function (51) is the first particular integral of the equation (50).

The second particular integral of the homogeneous equation (50) is possible to get from Liouville-Ostrogradzki formula (4). Therefore:

$$y_2(x) = \cos[\ln(x)]\int \frac{\exp(-\int \frac{dx}{x})}{\cos^2[\ln(x)]}dx, \tag{55}$$

$$\begin{aligned} &y_2(x) = \cos[\ln(x)]\int \frac{\exp[-\ln(x)]}{\cos^2[\ln(x]}dx = \cos(\ln(x))\int \frac{\exp[\ln(x^{-1})]}{\cos^2[\ln(x]}dx = \\ &= \cos[\ln(x)]\int \frac{dx}{x\cos^2[\ln(x]}dx = \left\langle \begin{matrix} \ln(x) = t \\ \frac{1}{x}dx = dt \end{matrix} \right\rangle = \cos[\ln(x)]\int \frac{dt}{\cos^2(t)} = \cos[\ln(x)]\cdot \mathrm{tg}(t) = \\ &= \cos[\ln(x)]\cdot \mathrm{tg}[\ln(x)] = \cos[\ln(x)]\cdot \frac{\sin[\ln(x)]}{\cos[\ln(x)]} = \sin[\ln(x)]. \end{aligned} \tag{56}$$

$$y_2(x) = \sin[\ln(x)]. \tag{57}$$

The general integral of the homogeneous equation (50) finaly has the following form:

$$y(x) \equiv \tilde{y}(x) = C_1\cos[\ln(x)] + C_2\sin[\ln(x)]. \tag{58}$$

where C_1 and C_2 are any real constants.

The particular solution of the non-homogeneous equation (49) is found by the constant variation method.

Therefore:

$$y(x) = C_1(x)\cdot\cos[\ln(x)] + C_2(x)\cdot\sin[\ln(x)]. \tag{59}$$

Unknown functions $C_1(x)$ and $C_2(x)$ in (59) are determined from the following system:

$$\begin{cases} \frac{dC_1}{dx}\cdot\cos[\ln(x)] + \frac{dC_2}{dx}\cdot\sin[\ln(x)] = 0 \\ \frac{dC_1}{dx}\cdot\left\{\frac{-\sin[\ln(x)]}{x}\right\} + \frac{dC_2}{dx}\cdot\left\{\frac{\cos[\ln(x)]}{x}\right\} = \frac{\sin[2\ln(x)]}{x^2}. \end{cases} \tag{60}$$

The system (60) can be solved by using Cramer method. The main determinant has the form:

$$W(x) \equiv \begin{vmatrix} \cos[\ln(x)] & \sin[\ln(x)] \\ -\dfrac{\sin[\ln(x)]}{x} & \dfrac{\cos[\ln(x)]}{x} \end{vmatrix} = \frac{\cos^2[\ln(x)]}{x} + \frac{\sin^2[\ln(x)]}{x} = \frac{1}{x}, \quad x \neq 0. \tag{61}$$

Thus, the determinants for the functions $\frac{dC_1}{dx}$ and $\frac{dC_2}{dx}$ have the form:

$$W_{C_1^{'}(x)}(x) \equiv \begin{vmatrix} 0 & \sin[\ln(x)] \\ \dfrac{\sin[2\ln(x)]}{x^2} & \dfrac{\cos[\ln(x)]}{x} \end{vmatrix} = -\frac{\sin[\ln(x)] \cdot \sin[2\ln(x)]}{x^2}, \tag{62}$$

$$W_{C_1^{'}(x)}(x) \equiv \begin{vmatrix} \cos[\ln(x)] & 0 \\ -\dfrac{\sin[\ln(x)]}{x} & \dfrac{\sin[2\ln(x)]}{x^2} \end{vmatrix} = \frac{\cos[\ln(x)] \cdot \sin[2\ln(x)]}{x^2}. \tag{63}$$

Using Cramer formulae, we get:

$$\frac{dC_1}{dx} = -\frac{2\sin^2[\ln(x)] \cdot \cos[\ln(x)]}{x}, \tag{64}$$

$$\frac{dC_2}{dx} = \frac{2\sin[\ln(x)] \cdot \cos^2[\ln(x)]}{x}. \tag{65}$$

Both sides of above equations we integrate relative to x variable:

$$\int \frac{dC_1}{dx} dx = -2\int \frac{\sin^2[\ln(x)] \cdot \cos[\ln(x)]}{x} dx, \tag{66}$$

$$\int \frac{dC_2}{dx} dx = 2\int \frac{\sin[\ln(x)] \cdot \cos^2[\ln(x)]}{x} dx. \tag{67}$$

Let us determine the integral:

$$\begin{aligned} &-2\int \frac{\sin^2[\ln(x)] \cdot \cos[\ln(x)]}{x} dx = \left\langle \begin{matrix} \ln(x) = t \\ \dfrac{dx}{x} = dt \end{matrix} \right\rangle = -2\int \sin^2(t) \cdot \cos(t)\, dt = \left\langle \begin{matrix} \sin(t) = p \\ \cos(t)dt = dp \end{matrix} \right\rangle = \\ &= -2\int p^2\, dp = -\frac{2}{3}p^3 = -\frac{2}{3}\sin^3(t) = -\frac{2}{3}\sin^3[\ln(x)]. \end{aligned} \tag{68}$$

Let us determine the integral:

$$\begin{aligned} &2\int \frac{\sin[\ln(x)] \cdot \cos^2[\ln(x)]}{x} dx = \left\langle \begin{matrix} \ln(x) = t \\ \dfrac{dx}{x} = dt \end{matrix} \right\rangle = 2\int \sin(t) \cdot \cos^2(t)\, dt = \left\langle \begin{matrix} \cos(t) = p \\ -\sin(t)dt = dp \end{matrix} \right\rangle = \\ &= -2\int [-\sin(t) \cdot \cos^2(t)\, dt = 2\int p^2\, dp = -\frac{2}{3}p^3 = -\frac{2}{3}\cos^3(t) = -\frac{2}{3}\cos^3[\ln(x)]. \end{aligned} \tag{69}$$

After integration in (66) and (67) we have:

$$C_1(x) = -\frac{2}{3}\sin^3[\ln(x)], \tag{70}$$

$$C_2(x) = -\frac{2}{3}\cos^3[\ln(x)]. \tag{71}$$

The particular solution of the non-homogeneous equation (49) has the following form:

$$y(x) \equiv \hat{y}(x) = -\frac{2}{3}\sin^3[\ln(x)]\cdot\cos[\ln(x)] - \frac{2}{3}\cos^3[\ln(x)]\cdot\sin[\ln(x)] = \tag{72}$$

$$= -\frac{2}{3}\sin[\ln(x)]\cdot\cos[\ln(x)]\cdot\left\{\sin^2[\ln(x)] + \cos^2[\ln(x)]\right\} = -\frac{2}{3}\sin[\ln(x)]\cdot\cos[\ln(x].$$

$$y(x) \equiv \hat{y}(x) = -\frac{1}{3}\sin[2\ln(x)]. \tag{73}$$

Finally, the general solution of the non-homogeneous equation (49) has the formula:

$$y(x) \equiv \tilde{y}(x) + \hat{y}(x) = C_1\cos[\ln(x)] + C_2\sin[\ln(x)] - \frac{1}{3}\sin[2\ln(x)], \tag{74}$$

$$y(x) = C_1\cos[\ln(x)] + C_2\sin[\ln(x)] - \frac{1}{3}\sin[2\ln(x)] \tag{75}$$

where C_1 and C_2 are the real constants.

• Numerical solution

In numerical analysis we take into account the solution (75) where C[1]=C[2]=1, C[1]=C[2]=2, C[1]=C[2]=3, C[1]=C[2]=4; [1], [3], [5], [7], [14]. Hence:

$$y(x) = \cos[\ln(x)] + \sin[\ln(x)] + \frac{1}{3}\sin[2\ln(x)], \tag{76}$$

$$y(x) = 2\cos[\ln(x)] + 2\sin[\ln(x)] + \frac{1}{3}\sin[2\ln(x)], \tag{77}$$

$$y(x) = 3\cos[\ln(x)] + 3\sin[\ln(x)] + \frac{1}{3}\sin[2\ln(x)], \tag{78}$$

$$y(x) = 4\cos[\ln(x)] + 4\sin[\ln(x)] + \frac{1}{3}\sin[2\ln(x)]. \tag{79}$$

Program 2. (Mathematica 7.0)

In[1]:= DSolve[y''[x] + (1/x)*y'[x] + (1/x^2)*y[x] == (Sin[2*Log[x])/x^2 y[x], x] /.
{{C[1] → 1, C[2] → 1},{C[1] → 2, C[2] → 2},{C[1] → 3, C[2] → 3},{C[1] → 4, C[2] → 4}}
Plot[Evaluate[y[x] /. %], {x, 0.01, 1.5}, Background → RGBColor[0.95,0,1],
PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
{RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
{RGBColor[1,0,1], Thickness[0.009]}},PlotRange → {–6, 6}, AxesOrigin → {0, 0},
AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]

Out[1] =

{{{y[x]→Cos[Log[x]]+Sin[Log[x]]- $\frac{1}{6}$ (-6Cos[Log[x]]Sin[Log[x]]-Cos[3Log[x]]Sin[Log[x]]+Cos[Log[x]]Sin[3log[x]])}},
{{y[x]→2Cos[Log[x]]+2Sin[Log[x]]- $\frac{1}{6}$ (-6Cos[Log[x]]Sin[Log[x]]-Cos[3Log[x]]Sin[Log[x]]+Cos[Log[x]]Sin[3log[x]])}},
{{y[x]→3Cos[Log[x]]+3Sin[Log[x]]- $\frac{1}{6}$ (-6Cos[Log[x]]Sin[Log[x]]-Cos[3Log[x]]Sin[Log[x]]+Cos[Log[x]]Sin[3log[x]])}},
{{y[x]→4Cos[Log[x]]+4Sin[Log[x]]- $\frac{1}{6}$ (-6Cos[Log[x]]Sin[Log[x]]-Cos[3Log[x]]Sin[Log[x]]+Cos[Log[x]]Sin[3log[x]])}}}

Out[2] = = Graphics =

Let us proof the following identity:

$$\frac{1}{3}\sin[2\ln[x]] = -\frac{1}{6}\{-6\cos[\ln(x)]\sin[\ln(x)] - \cos[3\ln(x)]\sin[\ln(x)] + \cos[\ln(x)]\sin[3\ln(x)]\}. \quad (80)$$

Let us consider the right side of the identity (80):

$$P = -\frac{1}{6}\{-6\cos[\ln(x)]\sin[\ln(x)] + \cos[\ln(x)]\sin[3\ln(x)] - \sin[\ln(x)]\cos[3\ln(x)]\} =$$

$$= -\frac{1}{6}\{-6\cos[\ln(x)]\sin[\ln(x)] + \sin[3\ln(x) - \ln(x)]\} = \cos[\ln(x)]\sin[\ln(x)] - \frac{1}{6}\sin[2\ln(x)] = \quad (81)$$

$$= \cos[\ln(x)]\sin[\ln(x)] - \frac{1}{6}\sin[2\ln(x)] = \cos[\ln(x)]\sin[\ln(x)] - \frac{1}{3}\sin[\ln(x)]\cos[\ln(x)] =$$

$$= \frac{2}{3}\sin[\ln(x)]\cos[\ln(x)] = \frac{1}{3}\sin[2\ln(x)].$$

Above fact means that the identity (80) is true. It means that program 2 in *Mathematica* shows the answer in according with the analytical form $\frac{1}{3}\sin[2\ln[x]]$ in formula [76]-[79].

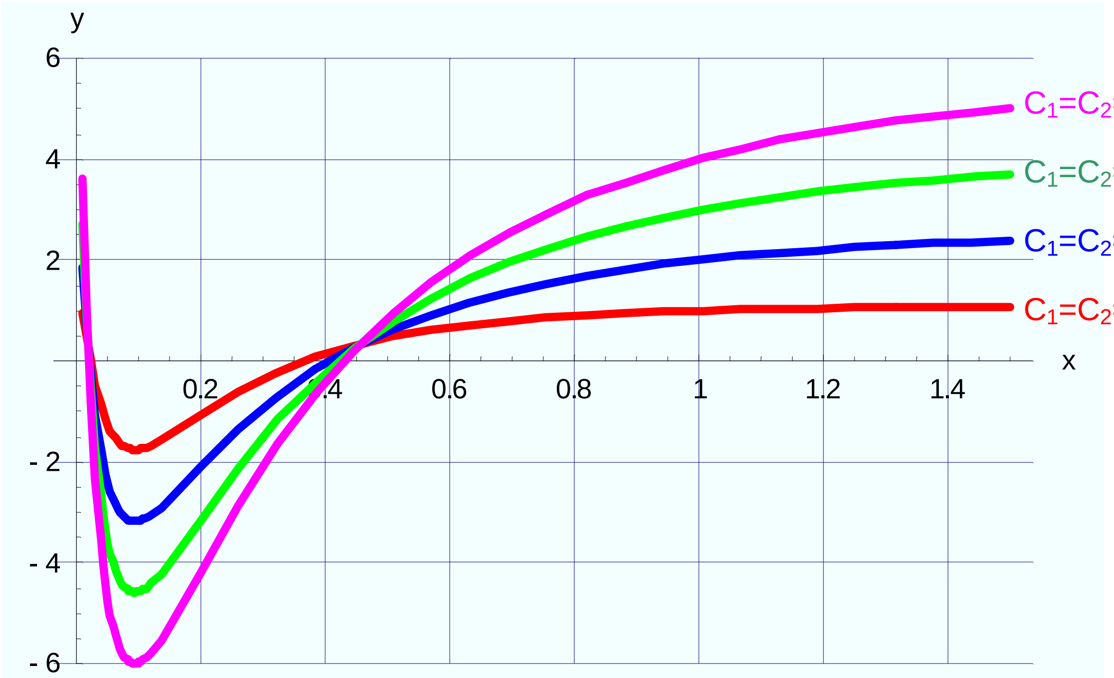

Fig. 2. Graphs of the functions (76)-(79), some solutions of the second order linear non-homogeneous differential equation (49) with changeable coefficients and for constants: $C_1=C_2=1$, $C_1=C_2=2$, $C_1=C_2=3$, $C_1=C_2=4$

Source: Program and graphs in Mathematica elaborated by the Authors

<u>Example 3</u>.

Let us consider the following non-homogeneous equation [6]:

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{4}{x^2}y = \frac{x}{2}. \quad (82)$$

The homogeneous equation has the form:

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{4}{x^2}y = 0. \quad (83)$$

• Analytical solution

It is clear that the first particular integral of the equation (83) is described by the formula:

$$y_1(x) = x^2. \tag{84}$$

The second particular integral of the homogeneous equation (83) is possible to get from Liouville-Ostrogradzki formula (4). Therefore:

$$y_2(x) = x^2 \int \frac{\exp(-\int -\frac{3}{x}dx)}{x^4} dx = x^2 \int \frac{\exp(3\int \frac{dx}{x})}{x^4} dx = x^2 \int \frac{\exp[3\ln(x)]}{x^4} dx = \tag{85}$$

$$= x^2 \int \frac{\exp[\ln(x^3)]}{x^4} dx = x^2 \int \frac{x^3}{x^4} dx = x^2 \int \frac{dx}{x} = x^2 \ln(x),$$

$$y_2(x) = x^2 \ln(x). \tag{86}$$

The general integral of the homogeneous equation (83) finally has the following form:

$$y(x) \equiv \tilde{y}(x) = C_1 x^2 + C_2 x^2 \ln(x). \tag{87}$$

where C_1 and C_2 are any real constants.

The particular solution of the non-homogeneous equation (82) is found by the constant variation method. Therefore:

$$y(x) = C_1(x) \cdot x^2 + C_2(x) \cdot x^2 \ln(x). \tag{88}$$

Unknown functions $C_1(x)$ and $C_2(x)$ in (88) are determined from the following system:

$$\begin{cases} \frac{dC_1}{dx} \cdot x^2 + \frac{dC_2}{dx} \cdot x^2 \ln(x) = 0 \\ \frac{dC_1}{dx} \cdot 2x + \frac{dC_2}{dx} \cdot [2x\ln(x) + x] = \frac{x}{2}. \end{cases} \tag{89}$$

We solve the system (89) using Cramer method. The main determinant has the form:

$$W(x) \equiv \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} = 2x^3 \ln(x) + x^3 - 2x^3 \ln(x) = x^3 \neq 0. \tag{90}$$

Thus, the determinants for the functions $\frac{dC_1}{dx}$ and $\frac{dC_2}{dx}$ have the form:

$$W_{C_1'(x)}(x) \equiv \begin{vmatrix} 0 & x^2 \ln(x) \\ \frac{x}{2} & 2x\ln(x) + x \end{vmatrix} = -\frac{x^3}{2} \ln(x), \tag{91}$$

$$W_{C_1'(x)}(x) \equiv \begin{vmatrix} x^2 & 0 \\ 2x & \frac{x}{2} \end{vmatrix} = \frac{x^3}{2}. \tag{92}$$

Using Cramer formulae, we get:

$$\frac{dC_1}{dx} = -\frac{x^3}{2} \ln(x) \cdot \frac{1}{x^3} = -\frac{1}{2} \ln(x), \tag{93}$$

$$\frac{dC_2}{dx} = \frac{x^3}{2} \cdot \frac{1}{x^3} = \frac{1}{2}. \tag{94}$$

Both sides of above equations we integrate relative to x variable:

$$\int \frac{dC_1}{dx} dx = -\frac{1}{2} \int \ln(x)\, dx = \left\langle \begin{matrix} u = \ln(x) \\ v' = 1 \end{matrix} \middle| \begin{matrix} u' = \frac{1}{x} \\ v = x \end{matrix} \right\rangle = -\frac{1}{2}\left[x \ln(x) - \int dx\right] = \\ = -\frac{1}{2}[x \ln(x) - x] = -\frac{1}{2} x[\ln(x) - 1], \tag{95}$$

$$\int \frac{dC_2}{dx} dx = \frac{1}{2} \int dx = \frac{1}{2} x. \tag{96}$$

After integration in (95) and (96) we have:

$$C_1(x) = -\frac{1}{2} x[\ln(x) - 1], \tag{97}$$

$$C_2(x) = \frac{1}{2} x. \tag{98}$$

The particular solution of the non-homogeneous equation (82) has the following form:

$$y(x) \equiv \hat{y}(x) = -\frac{1}{2} x[\ln(x) - 1] \cdot x^2 + \frac{x^3}{2} \ln(x) = -\frac{x^3}{2} \ln(x) + \frac{x^3}{2} + \frac{x^3}{2} \ln(x). \tag{99}$$

$$y(x) \equiv \hat{y}(x) = \frac{x^3}{2}. \tag{100}$$

Finally, the general solution of the non-homogeneous equation (82) has the formula:

$$y(x) \equiv \tilde{y}(x) + \hat{y}(x) = C_1 x^2 + C_2 x^2 \ln(x) + \frac{x^3}{2}, \tag{101}$$

$$y(x) = x^2[C_1 + C_2 \ln(x)] + \frac{x^3}{2} \tag{102}$$

where C_1 and C_2 are the real constants.

• Numerical solution

In numerical analysis we take into account the solution (102) where C[1]=C[2]=1, C[1]=C[2]=2, C[1]=C[2]=3, C[1]=C[2]=4; [1], [3], [5], [7], [14]. Hence:

$$y(x) = x^2[1 + \ln(x)] + \frac{x^3}{2}, \tag{103}$$

$$y(x) = x^2[2 + 2\ln(x)] + \frac{x^3}{2}, \tag{104}$$

$$y(x) = x^2[3 + 3\ln(x)] + \frac{x^3}{2}, \tag{105}$$

$$y(x) = x^2[4 + 4\ln(x)] + \frac{x^3}{2}. \tag{106}$$

Program 3. (Mathematica 7.0)

In[1]:= DSolve[y''[x] + (–3/x)*y'[x] + (4/x^2)*y[x] == (Sin[2*Log[x])/x^2 y[x], x] /.
{{C[1] → 1, C[2] → 1},{C[1] → 2, C[2] → 2},{C[1] → 3, C[2] → 3},{C[1] → 4, C[2] → 4}}
Plot[Evaluate[y[x] /. %], {x, 0.01, 1.5}, Background → RGBColor[0.95,0,1],
PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
{RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
{RGBColor[1,0,1], Thickness[0.009]}},PlotRange → {–6, 6}, AxesOrigin → {0, 0},
AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]

Out[1] = {{{y[x]→Cos[Log[x]]+Sin[Log[x]]- $\frac{1}{3}$ Sin[2Log[x]]}}, {{y[x]→2Cos[Log[x]]+2Sin[Log[x]]- $\frac{1}{3}$ Sin[2Log[x]]}},
{{y[x]→3Cos[Log[x]]+3Sin[Log[x]]- $\frac{1}{3}$ Sin[2Log[x]]}}, {{y[x]→4Cos[Log[x]]+4Sin[Log[x]]- $\frac{1}{3}$ Sin[2Log[x]]}}},

Out[2] = = Graphics =

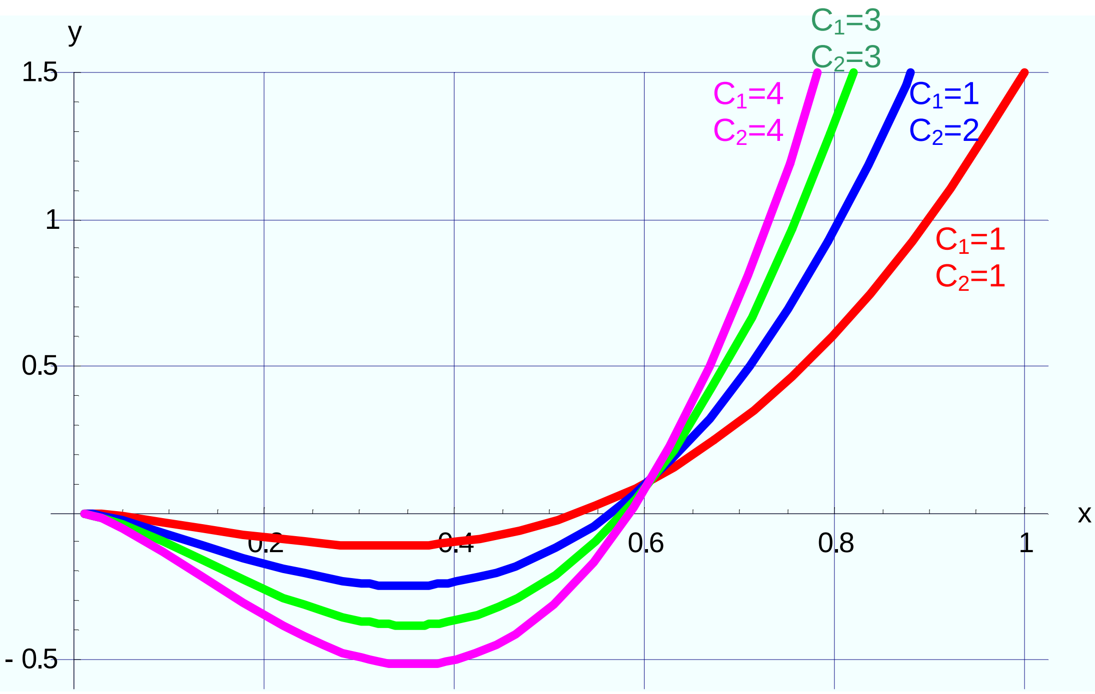

Fig. 3. Graphs of the functions (103)-(106), some solutions of the second order linear non-homogeneous differential equation (82) with changeable coefficients and for constants: $C_1=C_2=1$, $C_1=C_2=2$, $C_1=C_2=3$, $C_1=C_2=4$

Source: Program and graphs in Mathematica elaborated by the Authors

Example 4.

Let us consider the following non-homogeneous equation [6]:

$$\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\frac{1}{x^2}y=\frac{1}{x^3}. \tag{107}$$

The homogeneous equation has the form:

$$\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\frac{1}{x^2}y=0. \tag{108}$$

• **Analytical solution**

The first particular integral of the equation (108) is described by the formula:

$$y_1(x) = \frac{1}{x}. \tag{109}$$

Let us proof that the function (109) is the first particular integral of the equation (108):

$$\frac{dy_1}{dx} = \frac{-1}{x^2}, \tag{110}$$

$$\frac{d^2y_1}{dx^2} = \frac{2}{x^3}. \tag{111}$$

Let us put the functions (110) and (111) into the equation (108):

$$\frac{2}{x^3} + \frac{3}{x}\cdot\frac{-1}{x^2} + \frac{1}{x^2}\frac{1}{x} = 0 \quad \Leftrightarrow \quad \frac{2}{x^3} - \frac{3}{x^3} + \frac{1}{x^3} = 0 \quad \Leftrightarrow \quad \frac{3}{x^3} - \frac{3}{x^3} = 0 \quad \Leftrightarrow \quad 0 = 0. \tag{112}$$

The above result means that the function (109) is the first particular integral of the equation (108).

The second particular integral of the homogeneous equation (108) is possible to get from Liouville-Ostrogradzki formula (4). Therefore:

$$y_2(x) = \frac{1}{x}\int \frac{\exp(-\int \frac{3}{x}dx)}{x^{-2}}dx = \frac{1}{x}\int \frac{\exp(-3\int \frac{dx}{x})}{x^{-2}}dx = \frac{1}{x}\int \frac{\exp[-3\ln(x)]}{x^{-2}}dx = \tag{113}$$

$$= \frac{1}{x}\int \frac{\exp[\ln(x^{-3})]}{x^{-2}}dx = \frac{1}{x}\int \frac{x^{-3}}{x^{-2}}dx = \frac{1}{x}\int \frac{dx}{x} = \frac{1}{x}\ln(x),$$

$$y_2(x) = \frac{\ln(x)}{x}. \tag{114}$$

The general integral of the homogeneous equation (108) finally has the following form:

$$y(x) \equiv \tilde{y}(x) = C_1 \cdot \frac{1}{x} + C_2 \cdot \frac{\ln(x)}{x} \tag{115}$$

where C_1 and C_2 are any real constants.

The particular solution of the non-homogeneous equation (107) is found by the constant variation method. Therefore:

$$y(x) = C_1(x)\cdot\frac{1}{x} + C_2(x)\cdot\frac{\ln(x)}{x}. \tag{116}$$

Unknown functions $C_1(x)$ and $C_2(x)$ in (116) are determined from the following system:

$$\begin{cases} \frac{dC_1}{dx}\cdot\frac{1}{x} + \frac{dC_2}{dx}\cdot\frac{\ln(x)}{x} = 0 \\ \frac{dC_1}{dx}\cdot\left(\frac{-1}{x^2}\right) + \frac{dC_2}{dx}\cdot\frac{1-\ln(x)}{x^2} = \frac{1}{x^3}. \end{cases} \tag{117}$$

We solve the system (117) using Cramer method. The main determinant has the form:

$$W(x) \equiv \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1-\ln(x)}{x^2} \end{vmatrix} = \frac{1}{x^3} - \frac{\ln(x)}{x^3} + \frac{\ln(x)}{x^3} = \frac{1}{x^3}, \quad x \neq 0. \tag{118}$$

Thus, the determinants for the functions $\frac{dC_1}{dx}$ and $\frac{dC_2}{dx}$ have the form:

$$W_{C_1^{'}(x)}(x) \equiv \begin{vmatrix} 0 & \frac{\ln(x)}{x} \\ \frac{1}{x^3} & \frac{1-\ln(x)}{x^2} \end{vmatrix} = -\frac{\ln(x)}{x^4}, \tag{119}$$

$$W_{C_1^{'}(x)}(x) \equiv \begin{vmatrix} \frac{1}{x} & 0 \\ -\frac{1}{x^2} & \frac{1}{x^3} \end{vmatrix} = \frac{1}{x^4}. \tag{120}$$

Using Cramer formulae, we get:

$$\frac{dC_1}{dx} = -\frac{\ln(x)}{x^4} \cdot \frac{x^3}{1} = -\frac{\ln(x)}{x}, \tag{121}$$

$$\frac{dC_2}{dx} = \frac{1}{x^4} \cdot \frac{x^3}{1} = \frac{1}{x}. \tag{122}$$

Both sides of above equations we integrate relative to x variable:

$$C_1(x) = -\int \frac{\ln(x)}{x} dx = \left\langle \begin{matrix} \ln(x) = t \\ \frac{dx}{x} = dt \end{matrix} \right\rangle = -\int t\,dt = -\frac{t^2}{2} = -\frac{1}{2}\ln^2(x) \tag{123}$$

$$C_2(x) = \int \frac{dx}{x} = \ln(x). \tag{124}$$

After integration in (123) and (124) we have:

$$C_1(x) = -\frac{1}{2}\ln^2(x), \tag{125}$$

$$C_2(x) = \ln(x). \tag{126}$$

The particular solution of the non-homogeneous equation (107) has the following form:

$$y(x) \equiv \hat{y}(x) = -\frac{1}{2}\ln^2(x) \cdot \left(\frac{1}{x}\right) + \ln(x) \cdot \left(\frac{\ln(x)}{x}\right) = -\frac{1}{2}\frac{\ln^2(x)}{x} + \frac{\ln^2(x)}{x} = \frac{1}{2}\frac{\ln^2(x)}{x}. \tag{127}$$

$$y(x) \equiv \hat{y}(x) = \frac{\ln^2(x)}{2x}. \tag{128}$$

Finally, the general solution of the non-homogeneous equation (107) has the formula:

$$y(x) \equiv \tilde{y}(x) + \hat{y}(x) = C_1\frac{1}{x} + C_2\left(\frac{\ln(x)}{x}\right) + \frac{\ln^2(x)}{2x}, \tag{129}$$

$$y(x) = \frac{C_1}{x} + \frac{C_2 \ln(x)}{x} + \frac{\ln^2(x)}{2x} \tag{130}$$

where C_1 and C_2 are the real constants.

• Numerical solution

In numerical analysis we take into account the solution (130) where C[1]=C[2]=0,25; C[1]=C[2]=0,5; C[1]=C[2]=0,75; C[1]=[2]=1; [1], [3], [5], [7], [14]. Hence:

$$y(x)=\frac{0,25}{x}+\frac{0,25\cdot\ln(x)}{x}+\frac{\ln^2(x)}{2x}, \tag{131}$$

$$y(x)=\frac{0,5}{x}+\frac{0,5\cdot\ln(x)}{x}+\frac{\ln^2(x)}{2x}, \tag{132}$$

$$y(x)=\frac{0,75}{x}+\frac{0,75\cdot\ln(x)}{x}+\frac{\ln^2(x)}{2x}, \tag{133}$$

$$y(x)=\frac{1}{x}+\frac{\ln(x)}{x}+\frac{\ln^2(x)}{2x}. \tag{134}$$

Program 4. (Mathematica 7.0)

```
In[1]:= DSolve[y''[x] + (3/x)*y'[x] + (1/x^2)*y[x] == 1/x^3 y[x], x] /.
{{C[1]→0.25,C[2]→0.25},{C[1]→0.5,C[2]→0.5},{C[1]→0.75,C[2]→0.75},{C[1]→1,C[2]→1}}
Plot[Evaluate[y[x] /. %], {x, 0.1, 0.6}, Background → RGBColor[0.95,0,1],
PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
{{RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
{RGBColor[1,0,1], Thickness[0.009]}},PlotRange → {0, 5}, AxesOrigin → {0.15, 0},
 AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
 GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]
```

Out[1] = $\left\{\left\{\left\{y[x]\to\frac{0.25}{x}+\frac{0.25\mathrm{Log}[x]}{x}+\frac{\mathrm{Log}[x]^2}{2x}\right\},\left\{y[x]\to\frac{0.5}{x}+\frac{0.5\mathrm{Log}[x]}{x}+\frac{\mathrm{Log}[x]^2}{2x}\right\},\right.\right.$
$\left.\left.\left\{y[x]\to\frac{0.75}{x}+\frac{0.75\mathrm{Log}[x]}{x}+\frac{\mathrm{Log}[x]^2}{2x}\right\},\left\{y[x]\to\frac{1}{x}+\frac{\mathrm{Log}[x]}{x}+\frac{\mathrm{Log}[x]^2}{2x}\right\}\right\}\right\}$

Out[2] = = Graphics =

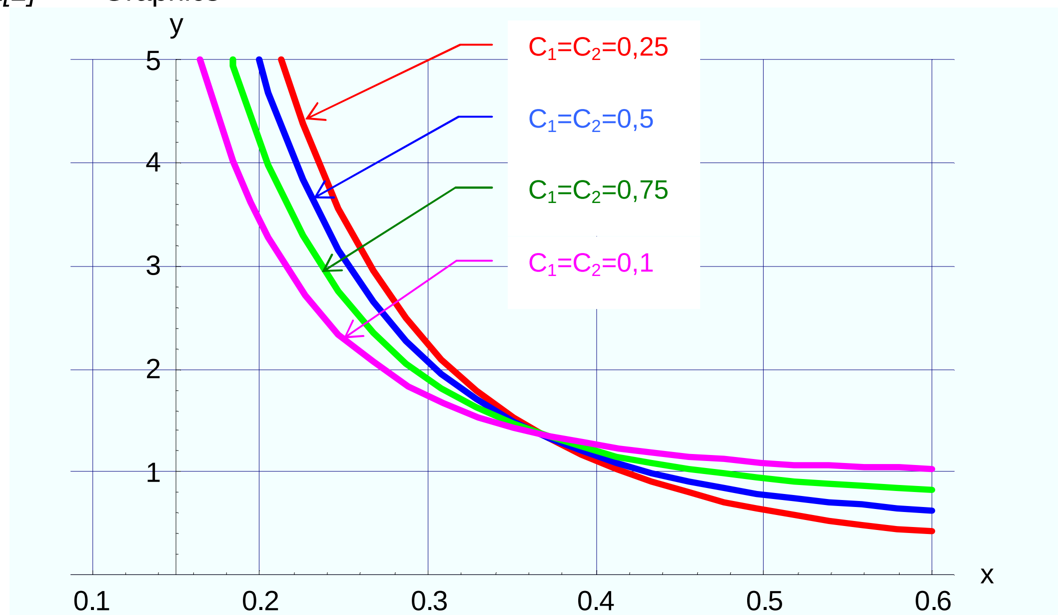

Fig. 4. Graphs of the functions (131)-(134), some solutions of the second order linear non-homogeneous differential equation (107) with changeable coefficients and for constants: $C_1=C_2=1$, $C_1=C_2=0{,}75$, $C_1=C_2=0{,}5$, $C_1=C_2=0{,}25$

Source: Program and graphs in Mathematica elaborated by the Authors

3. Conclusions

- Taking into account Liouville-Ostrogradzki formula and the constant variation method it is possible to solve the linear non-homogeneous differential equations of the second order with changeable coefficients.
- Using the *Mathematica* program for numerical solution, of the linear non-homogeneous differential equations of the second order with changeable coefficients, you can quickly get a solution and create its graphical interpretation.

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