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THE CAUCHY AND LAGRANGE INTEGRAL OF THE EULER EQUATION OF MOTION

Streszczenie

Autorzy artykułu dokonują analizy metody zastosowanej przez Cauchy'ego i Lagrange'a dla uzyskania całki równania ruchu Eulera. Na tej podstawie stawiają hipotezę, że całka Cauchy'ego i Lagrange'a nie jest jedyną całką równania ruchu Eulera. Autorzy artykułu przedstawiają krótką procedurę wykorzystującą twierdzenie Schwarza, której zastosowanie doprowadziło do uzyskania rozwiązania równania ruchu Eulera składającego się z dwóch całek. Przedstawione przez autorów rozwiązanie problemu całkowania równania ruchu Eulera stanowi w istocie przypadek jakościowo inny, bo o większym stopniu ogólności.

1. INTRODUCTION

The first pages of this article present an analysis of the method used by Cauchy and Lagrange to obtain the integral of the Euler equation of motion. This analysis is based on a step by step reconstruction of what the authors thought was likely course of this integral. For this reason, the article reproduces all of the steps that led to the well-known result. As a result of this analysis, we can assume that the integral of Cauchy and Lagrange may not be the only integral of the Euler equation of motion.

In Chapters 2 and 3, the authors present the derivation of the Cauchy and Lagrange integral of the Euler equation of motion in commonly known form and the derivation of the equation of potential. The authors considered it expedient to place these patterns in the article. This is partially because of the possibility of spectacular comparing them with the results presented in the rest of the article without consulting the appropriate manual of fluid mechanics.

Chapter 4 shows that the Cauchy and Lagrange integral is not the only integral of the Euler equation of motion. In this section, the authors propose a method using the Schwarz theorem. This application resulted in a solution containing two integrals.

One of these integrals is the Cauchy and Lagrange integral, and the other is the integral obtained by the authors through the use of the described method. It appears to be important complement to the Cauchy and Lagrange solution. Therefore, it appears that both integrals constitute a set representing the most general solution to the problem of the integration of the Euler equation of motion.

Chapter 4 also presents the equation of potential, which was derived using the set of two integrals of the Euler equation of motion mentioned above. The presented equation is significantly different from the well-known equation of the same name.

2. DERIVATION OF THE CAUCHY AND LAGRANGE INTEGRAL OF THE EULER EQUATION OF MOTION

Let us repeat here the Cauchy and Lagrange integral derivation of the Euler equation of motion. In the case of one-dimensional unsteady flow, the Euler equation boils down to a very simple form:

$$\frac{\partial p}{\partial x} = -\rho \cdot \frac{du}{dt} \tag{2.1}$$

In this embodiment, the following are true:

1. The fluid is not in the field of mass forces unit, in a potential field, or in any other field.
2. There is no concern that the movement could be a whirl, because it is a one-dimensional flow.

Therefore, in order to effectively obtain the Cauchy and Lagrange integral, it must be assumed that the fluid is barotropic.

Before proceeding with further transformations, we should divide the parties of equation (2.1) by ρ_0, i.e. the density of the fluid at rest. This is a formality which does not prevent the transformation of this equation and will not affect the final result. We do this to avoid the use of values appointed in the final result.

We have:

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right) = -\frac{\rho}{\rho_0} \cdot \frac{du}{dt} \tag{2.2}$$

Then, we divide both sides of the equation (2.2) by $\frac{\rho}{\rho_0}$, while developing the right side:

$$\frac{1}{\left(\frac{\rho}{\rho_0}\right)} \cdot \frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right) = -u \cdot \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \tag{2.3}$$

In equation (2.3), we see that the operations of the partial derivatives with respect to *x* refer to the word on the left and the first term on the right side of the equation; the second term on the right side shows the operation of the partial derivative with respect to *t*. Therefore, the direct integration of the Euler equation of motion in the form of (2.3) is impossible. An exception is the first term on the right hand side of equation (2.3), which could be presented in the form of:

$$u \cdot \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \tag{2.4}$$

Now, lets check the word on the left side of equation (2.3). If the fluid is barotropic, the pressure is a function only of fluid density. Therefore, the function is:

$$\frac{p}{\rho_0} = f\left(\frac{\rho}{\rho_0}\right) \tag{2.5}$$

It will be used to transform a suitable expression on the left side of equation (2.3). Thus, we find the partial derivative of the expression with respect to *x* (2.5):

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right)=f'\left(\frac{\rho}{\rho_0}\right)\cdot\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\right) \tag{2.6}$$

Now, we multiply (2.6) the parties by $\frac{1}{\left(\frac{\rho}{\rho_0}\right)}$:

$$\frac{1}{\left(\frac{\rho}{\rho_0}\right)}\cdot\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right)=\frac{1}{\left(\frac{\rho}{\rho_0}\right)}\cdot f'\left(\frac{\rho}{\rho_0}\right)\cdot\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\right) \tag{2.7}$$

Then, we assume that:

$$\frac{1}{\left(\frac{\rho}{\rho_0}\right)}\cdot f'\left(\frac{\rho}{\rho_0}\right)=F'\left(\frac{\rho}{\rho_0}\right) \tag{2.8}$$

where:

$$F'\left(\frac{\rho}{\rho_0}\right)=\frac{d}{d\left(\frac{\rho}{\rho_0}\right)}\cdot\left[F\left(\frac{\rho}{\rho_0}\right)\right] \tag{2.9}$$

and $F\left(\frac{\rho}{\rho_0}\right)$ is an arbitrary function of $\frac{\rho}{\rho_0}$.

After substituting (2.8) for (2.7), we obtain:

$$\frac{1}{\left(\frac{\rho}{\rho_0}\right)}\cdot\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right)=F'\left(\frac{\rho}{\rho_0}\right)\cdot\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\right) \tag{2.10}$$

And finally, we have:

$$\frac{1}{\left(\frac{\rho}{\rho_0}\right)}\cdot\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right)=\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right] \tag{2.11}$$

Thus, the expression on the left side of equation (2.3) is presented in the form of a partial derivative function $F\left(\frac{\rho}{\rho_0}\right)$ with respect to *x*.

The second term on the right side of equation (2.3) remains there. There is no problem with it, because the assumption that the fluid is barotropic is sufficient for the existence of the space-time movement velocity potential φ:

$$\varphi=\varphi(x,t) \tag{2.12}$$

In other words, making the fluid barotropic makes the Euler equation of motion possible to integrate. So, we have the following relationship [1, page 99]:

$$u=\frac{\partial\varphi}{\partial x} \tag{2.13}$$

We now processes the partial derivative with respect to time (2.13):

$$\frac{\partial u}{\partial t}=\frac{\partial^2\varphi}{\partial x\partial t} \tag{2.14}$$

Hence we obtain:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t}\right) \tag{2.15}$$

by substituting (2.4), (2.11) and (2.15) into equation (2.3):

$$\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=-\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t}\right) \tag{2.16}$$

After elevating the operator $\frac{\partial}{\partial x}$ before the bracket, after collecting the words on the left side of the equal sign, and after ordering, we get:

$$\frac{\partial}{\partial x}\left[\frac{\partial\varphi}{\partial t}+\frac{u^2}{2}+F\left(\frac{\rho}{\rho_0}\right)\right]=0 \tag{2.17}$$

After performing the integral (2.17) (indefinite integral along the line *t* = *const*), we obtain:

$$\boxed{\frac{\partial\varphi}{\partial t}+\frac{u^2}{2}+F\left(\frac{\rho}{\rho_0}\right)=g(t)} \tag{2.18}$$

where *g(t)* is an arbitrary function of time.

This is the Cauchy and Lagrange integral of the Euler equation of motion for the one-dimensional unsteady flow.

3. DERIVATION OF THE EQUATION OF POTENTIAL

Finding the partial derivative of equation (2.18) with respect to *t*, we obtain:

$$\frac{\partial^2\varphi}{\partial t^2}+u\cdot\frac{\partial u}{\partial t}+\frac{\partial}{\partial t}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=\frac{dg(t)}{dt} \tag{3.1}$$

Then, we substitute into it the equation (2.14):

$$\frac{\partial^2\varphi}{\partial t^2}+u\cdot\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial}{\partial t}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=\frac{dg(t)}{dt} \tag{3.2}$$

Subsequently, taking into account that according to (2.13), the following holds:

$$\frac{\partial u}{\partial x}=\frac{\partial^2\varphi}{\partial x^2} \tag{3.3}$$

We find the partial derivative of the equation (2.18) with respect to *x*:

$$\frac{\partial^2\varphi}{\partial t\partial x}+u\cdot\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=0 \tag{3.4}$$

Then, we multiply the parties of (3.4) by *u*:

$$u\cdot\frac{\partial^2\varphi}{\partial t\partial x}+u^2\cdot\frac{\partial^2\varphi}{\partial x^2}+u\cdot\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=0 \tag{3.5}$$

Thus, we take the prepared equations 3.2 and 3.5 and add pages, which yields:

$$\begin{gathered}u^2\cdot\frac{\partial^2\varphi}{\partial x^2}+2\mathrm{u}\cdot\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial^2\varphi}{\partial t^2}+\\+u\cdot\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right]+\frac{\partial}{\partial t}\left[F\left(\frac{\rho}{\rho_0}\right)\right]=\frac{dg(t)}{dt}\end{gathered} \tag{3.6}$$

Given that the fourth and fifth term on the left side of equation (3.6) represent a substantial derivative of the function $F\left(\frac{\rho}{\rho_0}\right)$, we can make the equation significantly shorter:

$$u^2 \cdot \frac{\partial^2\varphi}{\partial x^2} + 2\mathrm{u} \cdot \frac{\partial^2\varphi}{\partial x \partial t} + \frac{\partial^2\varphi}{\partial t^2} + \frac{d}{dt}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = \frac{dg(t)}{dt} \tag{3.7}$$

Let us now put the fourth word in (3.7) in a more desired form. We will start from the action:

$$\frac{d}{dt}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = F'\left(\frac{\rho}{\rho_0}\right) \cdot \frac{d}{dt}\left(\frac{\rho}{\rho_0}\right) \tag{3.8}$$

If in (3.8) we have to deal with substantial derivatives, we can use the equation of continuity:

$$\frac{d}{dt}\left(\frac{\rho}{\rho_0}\right) = -\left(\frac{\rho}{\rho_0}\right) \cdot \frac{\partial u}{\partial x} \tag{3.9}$$

Substituting them according to equation (3.8):

$$\frac{d}{dt}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = -F'\left(\frac{\rho}{\rho_0}\right) \cdot \left(\frac{\rho}{\rho_0}\right) \cdot \frac{\partial u}{\partial x} \tag{3.10}$$

Subsequently, in equation (3.10), we substitute equation (3.3):

$$\frac{d}{dt}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = -F'\left(\frac{\rho}{\rho_0}\right) \cdot \left(\frac{\rho}{\rho_0}\right) \cdot \frac{\partial^2\varphi}{\partial x^2} \tag{3.11}$$

But according to equation (2.8), we have:

$$F'\left(\frac{\rho}{\rho_0}\right) \cdot \left(\frac{\rho}{\rho_0}\right) = f'\left(\frac{\rho}{\rho_0}\right) \tag{3.12}$$

Substituting equation (3.12) for (3.11), we obtain:

$$\frac{d}{dt}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = -f'\left(\frac{\rho}{\rho_0}\right) \cdot \frac{\partial^2\varphi}{\partial x^2} \tag{3.13}$$

Substituting equation (3.13) into the equation (3.7), we obtain:

$$u^2 \cdot \frac{\partial^2\varphi}{\partial x^2} + 2\mathrm{u} \cdot \frac{\partial^2\varphi}{\partial x \partial t} + \frac{\partial^2\varphi}{\partial t^2} - f'\left(\frac{\rho}{\rho_0}\right) \cdot \frac{\partial^2\varphi}{\partial x^2} = \frac{dg(t)}{dt} \tag{3.14}$$

After arranging the equation, (3.14) finally becomes its final form:

$$\boxed{\left[u^2 - f'\left(\frac{\rho}{\rho_0}\right)\right] \cdot \frac{\partial^2\varphi}{\partial x^2} + 2\mathrm{u} \cdot \frac{\partial^2\varphi}{\partial x \partial t} + \frac{\partial^2\varphi}{\partial t^2} = \frac{dg(t)}{dt}} \tag{3.15}$$

This is a partial differential equation of second quasi-linear order, which is satisfied by the velocity potential φ. This equation is called the equation of potential. Note that by finding the substantial derivative of both sides of (2.5), we get:

$$\frac{d}{dt}\left(\frac{p}{\rho_0}\right) = f'\left(\frac{\rho}{\rho_0}\right) \cdot \frac{d}{dt}\left(\frac{\rho}{\rho_0}\right) \tag{3.16}$$

Then do we get:

$$\frac{dp}{d\rho} = f'\left(\frac{\rho}{\rho_0}\right) \tag{3.17}$$

And because:

$$\frac{dp}{d\rho} = c^2 \tag{3.18}$$

where c is the local speed of sound; Thus, it appears that:

$$f'\left(\frac{\rho}{\rho_0}\right) = c^2 \tag{3.19}$$

Using this in equation (3.15), we obtain:

$$\boxed{\left(u^2 - c^2\right) \cdot \frac{\partial^2\varphi}{\partial x^2} + 2\mathrm{u} \cdot \frac{\partial^2\varphi}{\partial x \partial t} + \frac{\partial^2\varphi}{\partial t^2} = \frac{dg(t)}{dt}} \tag{3.20}$$

Assuming that the arbitrary function (2.5) expresses the pressure dependence of the fluid's density, the equation (3.15) is the most general and formally mathematically correct. According the actions in (3.16), (3.17), (3.18) and (3.19) leading to the equation (3.20) clearly show that this case concerns isentropic pressure depending on the density of the fluid:

$$\frac{p}{\rho_0} = \frac{p_0}{\rho_0} \cdot \left(\frac{\rho}{\rho_0}\right)^k \tag{3.21}$$

This means that equation (3.20) is not only formal mathematically correct, but it also describes physically possible movement.

4. OBTAINING THE INTEGRAL OF THE EULER EQUATION OF MOTION USING THE SCHWARZ THEOREM

Cited in Chapter 2 of the Cauchy and Lagrange integral derivation of the Euler equation of motion is mostly a repetition of the output contained in many textbooks of fluid mechanics (in example in [1]). It seems that contemporary authors thought they could repeat the historical course of the authors of the integral of the Euler equation of motion, i.e. Augustus Cauchy and Joseph Lagrange, by presenting reasoning leading to the integral in their textbooks.

With full respect for both eminent mathematicians and the recognition accuracy, we believe there is a different, but somewhat simpler way to integrate Euler equation of motion, using the Schwarz theorem[1]. This claim relates to (in the case considered here) equal mixed second partial derivatives of the function (two independent variables), expressing the physical size of the flow of fluid in space-time of flow.

To demonstrate this, let us substitute compounds (2.4) and (2.11) into equation (2.3):

$$\frac{\partial}{\partial x}\left[F\left(\frac{\rho}{\rho_0}\right)\right] = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \frac{\partial u}{\partial t} \tag{4.1}$$

Then, we transform equation (4.1) as follows:

$$\boxed{\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{u^2}{2} + F\left(\frac{\rho}{\rho_0}\right)\right]} \tag{4.2}$$

Now, we write down the following relationship between the second derivatives of the mixed function φ (2.12), which expresses the Schwarz theorem:

$$\frac{\partial^2\varphi}{\partial x \partial t} = \frac{\partial^2\varphi}{\partial t \partial x} \tag{4.3}$$

Or, for a better effect, we may write equation (4.3) as follows:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t}\right)} \tag{4.4}$$

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[1] August Cauchy (1789-1857) and Joseph Lagrange (1736-1813) are not able to use the theorem of Hermann Schwarz (1843-1921), for obvious reasons.

If we now compare the left and right sides of equations (4.4) and (4.2) respectively, we obtain the following equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)=\frac{\partial u}{\partial t} \tag{4.5}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right)=-\frac{\partial}{\partial x}\left[\frac{u^2}{2}+F\left(\frac{\rho}{\rho_0}\right)\right] \tag{4.6}$$

They then integrate over, yielding:

$$\frac{\partial \varphi}{\partial x}=u+h(x) \tag{4.7}$$

$$\frac{\partial \varphi}{\partial t}=-\left[\frac{u^2}{2}+F\left(\frac{\rho}{\rho_0}\right)\right]+g(t) \tag{4.8}$$

In the above two integrals, the functions $h(x)$ and $g(t)$ are arbitrary functions.

It turns out that the integration of the Euler equation of motion using the Schwarz theorem yields two integrals, (4.7) and (4.8), expressing the partial derivatives of the velocity potential φ. The second integral (4.8) is, of course, identical to the integral of the Cauchy and Lagrange equation. However, the first integral (4.7), somehow obtained by the procedure, could replace the defining equation of velocity potential (2.13), which is a part of historical research. However, with its presence, the equations (4.7) and (4.8) represent a qualitatively different case because of the greater degree of generality.

Repeating the procedure mentioned in section 3, i.e. making the appropriate operations on equations (4.7) and (4.8) using (3.1)...(3.15) the equation of potential for the general case can be derived:

$$\boxed{\begin{aligned}&\left[u^2-f'\left(\frac{\rho}{\rho_0}\right)\right]\cdot\frac{\partial^2\varphi}{\partial x^2}+2u\cdot\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial^2\varphi}{\partial t^2}=\\&=\left[u^2-f'\left(\frac{\rho}{\rho_0}\right)\right]\cdot\frac{dh(x)}{dx}+\frac{dg(t)}{dt}\end{aligned}} \tag{4.9}$$

Transforming (4.9) using (3.16) ... (3.20) we get the equation of potential for the isentropic case:

$$\boxed{\begin{aligned}&\left(u^2-c^2\right)\cdot\frac{\partial^2\varphi}{\partial x^2}+2u\cdot\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial^2\varphi}{\partial t^2}=\\&=\left(u^2-c^2\right)\cdot\frac{dh(x)}{dx}+\frac{dg(t)}{dt}\end{aligned}} \tag{4.10}$$

These equations are significantly different from the formulas (3.15) and (3.20), which are a result of the classical solution to the Cauchy and Lagrange integral.

CONCLUSIONS

It follows that the solution of the integral of the Euler equation of motion created by Cauchy and Lagrange is a special case of the proposed solutions and occurs with the function h(x)≡0 in the integral (4.7).

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Abstract

The authors analyse the method used by Cauchy and Lagrange to obtain the integral of the Euler equation of motion. The authors hypothesize that the Cauchy and Lagrange integral is not the only integral of the Euler equation of motion. The authors present a brief procedure using the Schwarz theorem, which led to a solution of the Euler equation of motion consisting of two integrals. The solution presented by the authors is probably the most general and comprehensive solution to the problem of the integration of the Euler equation of motion.

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