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SPECTRUM OF DISCRETE $2n$-TH ORDER DIFFERENCE OPERATOR WITH PERIODIC BOUNDARY CONDITIONS AND ITS APPLICATIONS

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Abstract. Let $n \in \mathbb{N}^*$, and $N \geq n$ be an integer. We study the spectrum of discrete linear $2n$-th order eigenvalue problems

$$\begin{cases} \sum_{k=0}^{n}(-1)^k \Delta^{2k} u(t-k) = \lambda u(t), & t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)), & i \in [0,2n-1]_{\mathbb{Z}}, \end{cases}$$

where λ is a parameter. As an application of this spectrum result, we show the existence of a solution of discrete nonlinear $2n$-th order problems by applying the variational methods and critical point theory.

Keywords: discrete boundary value problems, $2n$-th order, variational methods, critical point theory.

Mathematics Subject Classification: 39A10, 34B08, 34B15, 58E30.

1. INTRODUCTION

Let $n \geq 1$ be a positive integer. We consider the following nonlinear $2n$-th order boundary value problems:

$$\begin{cases} \sum_{k=0}^{n}(-1)^k \Delta^{2k} u(t-k) = f(t,u(t)), & t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)), & i \in [0,2n-1]_{\mathbb{Z}}, \end{cases} \tag{1.1}$$

where $N \geq n$ is an integer, $[1,N]_{\mathbb{Z}}$ denotes the discrete interval $\{1,2,\dots,N\}$,

Δ is the forward difference operator defined by

$$\begin{aligned}\Delta u(t) &= u(t+1) - u(t),\\ \Delta^0 u(t) &= u(t),\\ \Delta^i u(t) &= \Delta^{i-1}(\Delta u(t)) \quad \text{for } i = 1, 2, 3, \ldots, 2n\end{aligned}$$

and $f \in C([1,N]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$.

As usual, a solution of (1.1) is a function $u : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (1.1).

Let us consider the spectrum of the linear boundary value problem corresponding to the problem (1.1):

$$\begin{cases}\sum_{k=0}^{n}(-1)^k \Delta^{2k} u(t-k) = \lambda u(t), & t \in [1,N]_{\mathbb{Z}},\\ \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)), & i \in [0, 2n-1]_{\mathbb{Z}}.\end{cases} \tag{1.2}$$

In [1], Agarwal studied the second-order linear eigenvalue problem

$$\begin{cases}-\Delta^2 u(t-1) = \mu u(t), & t \in [1,N]_{\mathbb{Z}},\\ u(0) = u(N+1) = 0.\end{cases} \tag{1.3}$$

He obtained $\mu_r = 4\sin^2(\frac{r\pi}{2(N+1)})$ for $r \in [1,N]_{\mathbb{Z}}$, where μ_r is the eigenvalue of (1.3) and $\xi_r = (\xi_r(1), \xi_r(2), \ldots, \xi_r(N))^T$ is an eigenvector corresponding to the eigenvalue μ_r, where $\xi_r(j) = \sin(\frac{rj\pi}{N+1})$ for $j \in [1,N]_{\mathbb{Z}}$.

In [12], Kelly and Peterson studied the following eigenvalue problems:

$$\begin{cases}\Delta(p(t-1)\Delta u(t-1)) + q(t)u(t) + \mu m(t)u(t) = 0, & t \in [1,N]_{\mathbb{Z}},\\ u(0) = u(N+1) = 0,\end{cases} \tag{1.4}$$

where $p, m \in C([1,N]_{\mathbb{Z}},]0,\infty[)$ and $q \in C([1,N]_{\mathbb{Z}}, \mathbb{R})$. They proved that the problem (1.4) has exactly N real and simple eigenvalues μ_t, $t \in [1,N]_{\mathbb{Z}}$ satisfying $\mu_1 < \mu_2 < \ldots < \mu_N$ and the eigenfunction corresponding to μ_t has exactly $t-1$ simple generalized zeros.

Moreover, when $m(t) = 1$, Agarwal *et al.* [2] generalized the results of the problem (1.4) to the dynamic equations on time scales with Sturm–Liouville boundary condition.

It is well known that in different fields of research, such as computer science, economics, neural networks, biological systems, population dynamics, mechanical engineering, the mathematical modeling of important questions leads naturally to the consideration of nonlinear difference equations. As a result, in recent years, many existence results of nontrivial solutions for differential equations have been obtained due to the relatively fast development of studying the boundary value problems for differential equations, where various methods and techniques have been used, for example, fixed point theorems methods, coincidence degree theory, topological degree theory, we refer to [3–6, 11, 15]. Critical point theory as well as variational methods are powerful tools to investigate the existence of solutions of various problems on differential equations [7–10, 13, 14, 16].

In this paper, we study the spectrum of the problem (1.2), via matrix theory. And at last, as an application of this spectrum result, we show the existence of a solution of discrete nonlinear $2n$-th order problems (1.1) by variational methods and critical point theory. The main results in this paper are the following theorems:

Theorem 1.1. *If $N \geqslant 2n+1$, then the problem* (1.2) *has exactly N real eigenvalues λ_j, $j \in [0, N-1]_{\mathbb{Z}}$, which satisfies*

$$\begin{cases} \lambda_j = a_0 + 2\sum_{l=1}^{n} a_l \cos(\frac{2\pi lj}{N}), & j \in [0, N-1]_{\mathbb{Z}}, \\ \lambda_j = \lambda_{N-j}, & j \in [1, N-1]_{\mathbb{Z}}, \end{cases}$$

with $a_l = (-1)^l \sum_{j=l}^{n} C_{2j}^{j+l}$ for any $l \in [0, n]_{\mathbb{Z}}$. Moreover, the eigenspace $E(\lambda_j)$ corresponding to λ_j, $j \in [0, N-1]_{\mathbb{Z}}$, is given as follows:

$$\begin{cases} E(\lambda_0) = \mathrm{span}(\phi_0), \\ E(\lambda_j) = \mathrm{span}(\phi_j, \psi_j), \quad j \in [1, N-1]_{\mathbb{Z}}, \end{cases}$$

where

$$\phi_j = (\phi_j(0), \phi_j(1), \phi_j(2), \dots, \phi_j(N-1))^T,$$
$$\psi_j = (\psi_j(0), \psi_j(1), \psi_j(2), \dots, \psi_j(N-1))^T$$

for $j \in [0, N-1]_{\mathbb{Z}}$ with $\phi_j(r) = \cos(\frac{2\pi rj}{N})$ and $\psi_j(r) = \sin(\frac{2\pi rj}{N})$ for $r \in [0, N-1]_{\mathbb{Z}}$.

Theorem 1.2. *Assume that there exist $\alpha, \beta \in]0, \infty[$ and $l \in [0, N-2]_{\mathbb{Z}}$ such that*

$$\lambda_l s^2 < \alpha s^2 \leq f(t,s)s \leq \beta s^2 < \lambda_{l+1} s^2 \text{ for } |s| \geq r > 0 \text{ and } t \in [1, N]_{\mathbb{Z}}. \tag{1.5}$$

Then the problem (1.1) *has at least one solution.*

The paper is arranged as follows. Section 2 contains some preliminary lemmas. The main results are proved in Sections 3 and 4.

2. PRELIMINARY LEMMAS

In the present paper, we define a vector space E_N by

$$E_N = \{u : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)),$$
$$i = 0, 1, 2, 3, \dots, 2n-1\},$$

E_N can be equipped with inner product $\langle \cdot, \cdot \rangle_{E_N}$ and norm $\| \cdot \|_{E_N}$ as follows:

$$\langle u, v \rangle_{E_N} = \sum_{t=1}^{N} u(t)v(t), \quad \|u\|_{E_N} = \left(\sum_{t=1}^{N} |u(t)|^2 \right)^{1/2} \quad \text{for all } u, v \in E_N.$$

Remark 2.1. It is easy to see that, for any $u \in E_N$, we have

$$\begin{aligned} u(-(n-1)) &= u(N-(n-1)), \\ u(-(n-1)+1) &= u(N-(n-1)+1), \\ u(-(n-1)+2) &= u(N-(n-1)+2), \\ &\vdots \\ u(0) &= u(N), \\ u(1) &= u(N+1), \\ &\vdots \\ u(n) &= u(N+n). \end{aligned} \tag{2.1}$$

Clearly, $(E_N, \|\cdot\|_{E_N})$ is an N-dimensional reflexive Banach space, since it is isomorphic to the finite dimensional space $\mathbb{R}^N$. When we say that the vector $u = (u(1), \ldots, u(N)) \in \mathbb{R}^N$, we understand that u can be extended to a vector in E_N so that (2.1) holds, that is, u can be extended to the vector

$$(u(N-(n-1)), u(N-(n-1)+1), \ldots, u(N), u(1), u(2), \ldots, u(N), u(1), \ldots, u(n)) \in E_N$$

and when we write $E_N = \mathbb{R}^N$, we mean the elements in $\mathbb{R}^N$ which have been extended in the above sense.

Lemma 2.2 ([1]). *Let $u(t)$ be defined on $\mathbb{Z}$. Then, for all $k \in \mathbb{N}^*$ we have*

$$\Delta^k u(t) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i u(t+i), \quad t \in \mathbb{Z}.$$

Lemma 2.3. *Let $n \in \mathbb{N}^*$. For all $u, v \in E_N$ we have*

$$\sum_{t=1}^{N} \Delta^k u(t-k) \Delta^k v(t-k) = (-1)^k \sum_{t=1}^{N} \Delta^{2k} u(t-k) v(t), \quad k \in [0, n]_{\mathbb{Z}}. \tag{2.2}$$

Proof. For $k = 0$, it is easy to check the conclusion is true. We suppose that (2.2) is true for $k \in [0, n-1]_{\mathbb{Z}}$ and we prove that it is true for $k+1$, i.e.,

$$\sum_{t=1}^{N} \Delta^{k+1} u(t-(k+1)) \Delta^{k+1} v(t-(k+1)) = (-1)^{k+1} \sum_{t=1}^{N} \Delta^{2k+2} u(t-(k+1)) v(t).$$

By the summation by parts formula and the fact that $v(N+1) = v(1)$ and $\Delta^{2k+1}u(N-k) = \Delta^{2k+1}u(-k)$, it follows that

$$\begin{aligned}\sum_{t=1}^{N} \Delta^{2k+2}u(t-(k+1))v(t) &= \Delta^{2k+1}u(N-k)v(N+1)\\ &\quad - \Delta^{2k+1}u(-k)v(1)\\ &\quad - \sum_{t=1}^{N} \Delta^{2k+1}u(t-k)\Delta v(t)\\ &= -\sum_{t=1}^{N} \Delta^{2k+1}u(t-k)\Delta v(t).\end{aligned}$$

So it follows from $\Delta^{2k}u(N-(k-1)) = \Delta^{2k}u(-(k-1))$, $\Delta v(N+1) = \Delta v(1)$ and by the summation by parts formula, we get

$$\begin{aligned}\sum_{t=1}^{N} \Delta^{2k+1}u(t-k)\Delta v(t) &= \Delta^{2k}u(N+1-k)\Delta v(N+1)\\ &\quad - \Delta^{2k}u(1-k)\Delta v(1)\\ &\quad - \sum_{t=1}^{N} \Delta^{2k}u(t-(k-1))\Delta^2 v(t)\\ &= -\sum_{t=1}^{N} \Delta^{2k}u(t-k)\Delta^2 v(t-1)\\ &= (-1)^{k+1}\sum_{t=1}^{N} \Delta^{k}u(t-k)\Delta^{k}\left[\Delta^2 v(t-k-1)\right]\\ &= (-1)^{k+1}\sum_{t=1}^{N} \Delta^{k}u(t-k)\Delta^{k+2} v(t-k-1).\end{aligned}$$

Thus, we obtain

$$\sum_{t=1}^{N} \Delta^{2k+2}u(t-(k+1))v(t) = (-1)^{k}\sum_{t=1}^{N} \Delta^{k}u(t-k)\Delta^{k+2} v(t-k-1).$$

Similarly, using the summation by parts formula, $\Delta^{k+1}v(N-k) = \Delta^{k+1}v(-k)$ and $\Delta^k u(N-(k-1)) = \Delta^k u(-(k-1))$, and we have

$$\begin{aligned}
\sum_{t=1}^{N} \Delta^k u(t-k)\Delta^{k+2}v(t-k-1) &= \Delta^k u(N-(k-1))\Delta^{k+1}v(N-k) \\
&\quad - \Delta^k u(-(k-1))\Delta^{k+1}v(-k) \\
&\quad - \sum_{t=1}^{N} \Delta^{k+1}u(t-k)\Delta^{k+1}v(t-k) \\
&= - \sum_{t=1}^{N} \Delta^{k+1}u(t-k)\Delta^{k+1}v(t-k).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\sum_{t=1}^{N} \Delta^{2k+2}u(t-(k+1))v(t) &= (-1)^{k+1}\sum_{t=1}^{N} \Delta^{k+1}u(t-k)\Delta^{k+1}v(t-k) \\
&= (-1)^{k+1}\sum_{t=1}^{N} \Delta^{k+1}u(t-(k+1))\Delta^{k+1}v(t-(k+1))
\end{aligned}$$

which means that

$$\sum_{t=1}^{N} \Delta^{k+1}u(t-(k+1))\Delta^{k+1}v(t-(k+1) = (-1)^{k+1}\sum_{t=1}^{N} \Delta^{2k+2}u(t-(k+1))v(t).$$

The proof is complete. □

For $u \in E_N$, let Φ be the functional denoted by

$$\Phi(u) = \frac{1}{2}\sum_{t=1}^{N}\sum_{k=0}^{n} |\Delta^k u(t-k)|^2 - \sum_{t=1}^{N} F(t,u(t)),$$

where $F(t,x) = \int_0^x f(t,s)ds$ for $(t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$. Then, it is easy to see that $\Phi \in C^1(E_N,\mathbb{R})$ and its derivative $\Phi'(u)$ at $u \in E_N$ is given by

$$\Phi'(u).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n} \Delta^k u(t-k)\Delta^k v(t-k) - f(t,u(t))v(t)\right] \quad \text{for any } v \in E_N.$$

By Lemma 2.3, Φ' can be written as

$$\Phi'(u).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n} (-1)^k \Delta^{2k}u(t-k) - f(t,u(t))\right] v(t) \quad \text{for any } v \in E_N.$$

Thus, finding solutions of (1.1) is equivalent to finding critical point of the functional Φ.

Finally, we introduce the saddle point theorem, which will be used later in Section 4.

Definition 2.4. Let E be a real Banach space, and $\Phi \in C^1(E,\mathbb{R})$ is a continuously Fréchet differentiable functional defined on E. Recall that Φ is said to satisfy the Palais–Smale (PS) condition if every sequence $(u_m) \subset E$, such that $(\Phi(u_m))$ is bounded and $\Phi'(u_m) \to 0$ as $m \to \infty$, has a convergent subsequence. Here, the sequence (u_m) is called a (PS) sequence.

Let B_ρ denote the open ball in E about 0 of raduis ρ and let ∂B_ρ denote its boundary.

Theorem 2.5 ([14], the saddle point theorem)**.** *Let $E = V \oplus W$ be a Banach space with $V \neq \{0\}$, $\dim V < \infty$, $\Phi \in C^1(E,\mathbb{R})$ and $Q = \overline{B_\rho} \cap V$ with $\rho > 0$. If*

(1) $a = \max\limits_{\partial Q} \Phi < b = \inf\limits_{W} \Phi$,
(2) Φ *satisfies the* $(PS)_c$ *condition, where* $c = \inf\limits_{\gamma\in\Gamma}\sup\limits_{u\in Q} \Phi(\gamma(u))$, *and*

$$\Gamma = \{\gamma \in C(Q,E) \mid \gamma(u) = u \ \ on \ \ \partial Q\},$$

then c is a critical value of Φ such that $c \geq b$.

3. SPECTRUM OF (1.2)

We consider the linear eigenvalue problem (1.2) corresponding to the problem (1.1).

Definition 3.1. λ is called eigenvalue of (1.2) if there exists $u \in E_N \setminus \{0\}$ such that

$$\sum_{t=1}^{N}\sum_{k=0}^{n}(-1)^k \Delta^{2k} u(t-k)v(t) = \lambda \sum_{t=1}^{N} u(t)v(t) \quad \text{for every } v \in E_N.$$

For proving Theorem 1.1, we start with three auxiliary results.

Lemma 3.2. *Let $n \in \mathbb{N}^*$. The eigenvalues of* (1.2) *are exactly the eigenvalues of the matrix $\sum_{k=0}^{n} A_k$, where A_k, $k \in [0,n]_{\mathbb{Z}}$, is a symmetric matrix and its general form for $N \geqslant 2k+1$ is $A_k = [a_{ij}]_{1\leq i,j\leq N}$, with*

$$\begin{aligned}
a_{ii} &= C_{2k}^{k}, && i \in [1,N]_{\mathbb{Z}},\\
a_{ii+j} &= (-1)^j C_{2k}^{k+j}, && j \in [1,k]_{\mathbb{Z}}, i \in [1,N-j]_{\mathbb{Z}},\\
a_{ii+j} &= 0, && j \in [k+1, N-(k+1)]_{\mathbb{Z}}, i \in [1,N-j]_{\mathbb{Z}},\\
a_{ii+j} &= (-1)^{N-j} C_{2k}^{k+N-j}, && j \in [N-k, N-1]_{\mathbb{Z}}, i \in [1,N-j]_{\mathbb{Z}},
\end{aligned}$$

that is,

$$A_k = \begin{pmatrix} C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^k C_{2k}^{2k} & \cdots & (-1)^1 C_{2k}^{k+1} \\ (-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} & (-1)^2 C_{2k}^{k+2} \\ \vdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} \\ 0 & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\ \vdots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^k C_{2k}^{k} & \vdots & 0 & \vdots & \ddots & \ddots & (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} \\ \vdots & (-1)^3 C_{2k}^{k+3} & \vdots & \ddots & \cdots & \ddots & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} \\ (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} & (-1)^3 C_{2k}^{k+3} & \cdots & \cdots & \cdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k} \end{pmatrix}.$$

Proof. Let $n \in \mathbb{N}^*$, $k \in [0,n]_{\mathbb{Z}}$ and $u, v \in E_N$. It is clear to see that the application

$$L_k : (u,v) \longrightarrow \sum_{t=1}^{N} (-1)^k \Delta^{2k} u(t-k) v(t),$$

is bilinear and symmetric.

From the Riesz theorem, there exists a unique symmetric matrix A_k such that

$$L_k(u,v) = \langle A_k u, v\rangle_{E_N} \quad \text{for all } u, v \in E_N.$$

Thus the eigenvalues of (1.2) are exactly the eigenvalues of the matrix $\sum_{k=0}^{n} A_k$. Now we will determine the matrix A_k. Using Lemma 2.2, we have

$$\begin{aligned} \langle A_k u, u\rangle_{E_N} &= \sum_{t=1}^{N} (-1)^k \Delta^{2k} u(t-k) u(t) \\ &= \sum_{t=1}^{N} (-1)^k \left[\sum_{i=0}^{2k} (-1)^{2k-i} C_{2k}^{i} u(t-k+i) \right] u(t) \\ &= \sum_{t=1}^{N} (-1)^k u(t-k) u(t) + (-1)^{k-1} C_{2k}^{1} u(t-(k-1)) u(t) + \ldots \\ &\quad + (-1)^1 C_{2k}^{k-1} u(t-1) u(t) + C_{2k}^{k} u^2(t) + (-1)^1 C_{2k}^{k+1} u(t+1) u(t) \\ &\quad + \ldots + (-1)^k C_{2k}^{2k} u(t+k) u(t) \\ &= \sum_{t=1}^{N} C_{2k}^{k} u^2(t) + 2 \times (-1)^1 C_{2k}^{k+1} u(t) u(t+1) + \ldots \\ &\quad + 2 \times (-1)^k C_{2k}^{2k} u(t) u(t+k). \end{aligned}$$

So, we deduce that

$$A_k = \begin{pmatrix} C_{2k}^k & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^k C_{2k}^{2k} & \cdots & (-1)^1 C_{2k}^{k+1} \\ (-1)^1 C_{2k}^{k+1} & C_{2k}^k & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} & (-1)^2 C_{2k}^{k+2} \\ \vdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^k & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} \\ 0 & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\ \vdots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^k C_{2k}^k & \vdots & 0 & \vdots & \ddots & \ddots & (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} \\ \vdots & (-1)^3 C_{2k}^{k+3} & \vdots & \ddots & \cdots & \ddots & C_{2k}^k & (-1)^1 C_{2k}^{k+1} \\ (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} & (-1)^3 C_{2k}^{k+3} & \cdots & \cdots & \cdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^k \end{pmatrix}.$$

The proof of Lemma 3.2 is complete. □

Remark 3.3. If v is replaced by u in (2.2), we get

$$\sum_{t=1}^{N} |\Delta^k u(t-k)|^2 = \sum_{t=1}^{N} (-1)^k \Delta^{2k} u(t-k) u(t) = \langle A_k u, u \rangle, \quad k \in [0,n]_{\mathbb{Z}}. \tag{3.1}$$

Thus $A_0 = I_N$ is positive definite and A_k, $k \in [1,n]_{\mathbb{Z}}$, are positive semidefinite.

Put

$$\begin{aligned} a_l &= (-1)^l \sum_{j=l}^{n} C_{2j}^{j+l}, \quad l \in [0,n]_{\mathbb{Z}}, \\ a_l &= 0, \quad l \in [n+1, N-(n+1)]_{\mathbb{Z}}, \\ a_l &= (-1)^{N-l} \sum_{j=N-l}^{n} C_{2j}^{j+N-l}, \quad l \in [N-n, N-1]_{\mathbb{Z}}. \end{aligned}$$

We can write the matrix $\sum_{k=0}^{n} A_k$ for $N \geq 2n+1$ in the following form:

$$\sum_{k=0}^{n} A_k = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{N-(n+1)} & a_{N-n} & a_{N-(n-1)} & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & a_{n+1} & \cdots & \vdots & \vdots & a_{N-n} & \cdots & a_{N-3} & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & \vdots & \vdots & \vdots & \cdots & a_{N-4} & a_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_{N-1} & a_0 & a_1 \\ a_1 & a_2 & a_3 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_{N-2} & a_{N-1} & a_0 \end{pmatrix}.$$

Let J be the following matrix:

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By some calculations, it is easy to check that

$$\sum_{k=0}^{n} A_k = a_0 I_N + a_1 J + a_2 J^2 + \ldots + a_{N-1} J^{N-1} = R(J), \tag{3.2}$$

where $R(x) = \sum_{l=0}^{N-1} a_l x^l$.

Lemma 3.4. *The matrix J satisfies the following proprieties:*

(1) *the eigenvalues of J are $\omega_k = e^{i\frac{2k\pi}{N}}$, $k \in [0, N-1]_{\mathbb{Z}}$,*
(2) *J is diagonalizable on $\mathbb{C}$,*
(3) *$E(\omega_k) = \operatorname{span}(X_k)$, $k \in [0, N-1]_{\mathbb{Z}}$, where $E(\omega_k)$ is the ω_k-eigenspace and $X_k = (1, \omega_k, \omega_k^2, \ldots, \omega_k^{(N-1)})^T$.*

Proof. (1) Let $P_J(x)$ the characteristic polynomial of J:

$$P_J(x) = \det(J - xI_N) = \begin{vmatrix} -x & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -x \end{vmatrix}.$$

Developing with respect to the first column, we get

$$P_J(x) = -x \begin{vmatrix} -x & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -x \end{vmatrix} + (-1)^{N+1} \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -x & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -x & 1 \end{vmatrix}$$
$$= (-x)^N + (-1)^{N+1} = (-1)^N (x^N - 1).$$

However, the set of eigenvalues of J is the following:

$$\mathbf{U}_N = \left\{ \omega_k = e^{i\frac{2k\pi}{N}} : k \in [0, N-1]_{\mathbb{Z}} \right\}.$$

(2) Since the eigenvalues of J are simple, then J is diagonalizable on $\mathbb{C}$.
(3) Let $X = (x_1, x_2, \dots, x_N)^T \in \mathbb{C}^N$. Since $JX = (x_2, x_3, x_4, \dots, x_N, x_1)^T$, we get

$$X \in E(\omega_k) = Ker(J - \omega_k I_N) \iff \begin{cases} x_2 &= \omega_k x_1, \\ x_3 &= \omega_k x_2, \\ & \vdots \\ x_N &= \omega_k x_{N-1}, \\ x_1 &= \omega_k x_N \end{cases}$$
$$\iff X \in \operatorname{span}(X_k), \quad k \in [0, N-1]_{\mathbb{Z}}.$$

Therefore, Lemma 3.4 is proved. □

Remark 3.5.
(1) $B = (X_0, X_1, \dots, X_{N-1})$ is a basis formed by the eigenvectors of J.
(2) The matrix J can be written as

$$J = PDP^{-1}, \tag{3.3}$$

with

$$D = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \omega_{N-1} \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_1 & \omega_2 & \cdots & \omega_{N-1} \\ 1 & \omega_1^2 & \omega_2^2 & \cdots & \omega_{N-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_1^{N-1} & \omega_2^{N-1} & \cdots & \omega_{N-1}^{(N-1)} \end{pmatrix},$$

where P is the invertible matrix from B to B_1, $B_1 = (e_1, e_2, \dots, e_N)$ and $e_j, j \in [1, N]_{\mathbb{Z}}$, is a column vector, where all terms are equal to 0 except the j-th term which is equal to 1.

Lemma 3.6. *The matrix $\sum_{k=0}^{n} A_k$ is diagonalizable and*

$$\operatorname{Sp}\Big(\sum_{k=0}^{n} A_k\Big) = \{R(\lambda) : \lambda \in \operatorname{Sp}(J)\},$$

where $\operatorname{Sp}(\sum_{k=0}^{n} A_k)$ and $\operatorname{Sp}(J)$ are the spectrum of the matrices $\sum_{k=0}^{n} A_k$ and J, respectively.

Proof. It is clear that the matrix $\sum_{k=0}^{n} A_k$ is diagonalizable. From (3.3) we easily deduce that

$$J^k = PD^kP^{-1} \quad \text{for any } k \in [0, N-1]_{\mathbb{Z}}. \tag{3.4}$$

Combining (3.2) and (3.4), we obtain

$$\sum_{k=0}^{n} A_k = R(J) = PR(D)P^{-1}, \tag{3.5}$$

where

$$R(D) = \begin{pmatrix} R(1) & 0 & \cdots & \cdots & 0 \\ 0 & R(\omega_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & R(\omega_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & R(\omega_{N-1}) \end{pmatrix}.$$

Thus, one has

$$\mathrm{Sp}\Big(\sum_{k=0}^{n} A_k\Big) = \{R(\lambda) : \lambda \in \mathrm{Sp}(J)\}.$$

The proof is complete. $\square$

Proof of Theorem 1.1. This proof is divided into two steps.
Step 1. Let λ_j, $j \in [0, N-1]_{\mathbb{Z}}$, be the eigenvalue of $\sum_{k=0}^{n} A_k$. From Lemma 3.6 we have

$$\lambda_j = R(\omega_j),$$

where $\omega_j = e^{i\frac{2\pi j}{N}}$ and $R(x) = \sum_{l=0}^{N-1} a_l x^l$. Therefore,

$$\lambda_j = \sum_{l=0}^{N-1} a_l \omega_j^l = \sum_{l=0}^{N-1} a_l \omega_j^l = a_0 + \sum_{l=1}^{n} a_l \omega_j^l + \sum_{l=N-n}^{N-1} a_l \omega_j^l.$$

Since $\omega_j^{N-l} = \overline{\omega_j^l}$ and $a_{N-l} = a_l$ for any $l \in [1, N-1]_{\mathbb{Z}}$, we get

$$\begin{aligned} \lambda_j &= a_0 + \sum_{l=1}^{n} a_l \omega_j^l + \sum_{l=1}^{n} a_l \overline{\omega_j^l} \\ &= a_0 + \sum_{l=1}^{n} a_l \left[\omega_j^l + \overline{\omega_j^l}\right] = a_0 + 2\sum_{l=1}^{n} a_l \cos\Big(\frac{2\pi l j}{N}\Big). \end{aligned} \tag{3.6}$$

Using again (3.6), we deduce that for any $j \in [1, N-1]_{\mathbb{Z}}$

$$\lambda_{N-j} = a_0 + 2\sum_{l=1}^{n} a_l \cos\Big(\frac{2\pi l}{N}(N-j)\Big) = a_0 + 2\sum_{l=1}^{n} a_l \cos\Big(2\pi l - \frac{2\pi l j}{N}\Big) = \lambda_j.$$

Step 2. It is easy to see that $E(\lambda_0) = \text{span}(\phi_0)$. Let $Y = (y_1, y_2, \dots, y_N)^T \in E(\lambda_j)$, $j \in [1, N-1]_{\mathbb{Z}}$. Denote $Z = P^{-1}Y = (z_1, z_2, \dots, z_N)^T$. From (3.5), obviously for any $j \in [1, N-1]_{\mathbb{Z}}$,

$$R(D)Z = \lambda_j Z.$$

Since $R(D)Z = (\lambda_0 z_1, \lambda_1 z_2, \dots, \lambda_{N-1} z_N)^T$, we have

$$\begin{cases} \lambda_0 z_1 = \lambda_j z_1, \\ \lambda_1 z_2 = \lambda_j z_2, \\ \vdots \\ \lambda_{N-j} z_{N-j+1} = \lambda_j z_{N-j+1}, \\ \vdots \\ \lambda_{N-1} z_N = \lambda_j z_N. \end{cases}$$

This implies that

$$\begin{cases} Z = z_{j+1} e_{j+1} + z_{N-j+1} e_{N-j+1}, \\ Y = PZ = z_{j+1}(1, \omega_j, \omega_j^2, \dots, \omega_j^{(N-1)})^T + z_{N-j+1}(1, \omega_{N-j}, \omega_{N-j}^2, \dots, \omega_{N-j}^{(N-1)})^T. \end{cases}$$

Since $z_j \in \mathbb{C}$ for any $j \in [1, N]_{\mathbb{Z}}$, we can write z_j as $z_j = z'_j + iz''_j$. Then we get

$$Y = \begin{pmatrix} (z'_{j+1} + iz''_{j+1}) \times 1 \\ (z'_{j+1} + iz''_{j+1}) \times e^{i\frac{2\pi j}{N}} \\ (z'_{j+1} + iz''_{j+1}) \times e^{i\frac{4\pi j}{N}} \\ \vdots \\ (z'_{j+1} + iz''_{j+1}) \times e^{i\frac{2(N-1)\pi j}{N}} \end{pmatrix} + \begin{pmatrix} z'_{N-j+1} + iz''_{N-j+1} \\ (z'_{N-j+1} + iz''_{N-j+1}) \times e^{i\frac{2\pi(N-j)}{N}} \\ (z'_{N-j+1} + iz''_{N-j+1}) \times e^{i\frac{4\pi(N-j)}{N}} \\ \vdots \\ (z'_{N-j+1} + iz''_{N-j+1}) \times e^{i\frac{2(N-1)\pi(N-j)}{N}} \end{pmatrix}.$$

As $Y \in E_N$, we deduce that

$$Y = (z'_{j+1} + z'_{N-j+1})\phi_j + (z''_{N-j+1} - z''_{j+1})\psi_j$$

with

$$\phi_j = \begin{pmatrix} 1 \\ \cos(\frac{2\pi j}{N}) \\ \cos(\frac{4\pi j}{N}) \\ \vdots \\ \cos(\frac{2(N-1)\pi j}{N}) \end{pmatrix} \quad \text{and} \quad \psi_j = \begin{pmatrix} 0 \\ \sin(\frac{2\pi j}{N}) \\ \sin(\frac{4\pi j}{N}) \\ \vdots \\ \sin(\frac{2(N-1)\pi j}{N}) \end{pmatrix}.$$

Consequently, $E(\lambda_j) = \text{span}(\phi_j, \psi_j)$, $j \in [1, N-1]_{\mathbb{Z}}$. The proof of Theorem 1.1 is complete. □

Remark 3.7. We denote $r = \frac{N-1}{2}$ when N is odd, or $r = \frac{N}{2}$ when N is even. Since $\lambda_j = \lambda_{N-j}$ for every $j \in [1, N-1]_{\mathbb{Z}}$, the matrix $\sum_{k=0}^{n} A_k$ has $r+1$ different eigenvalues. Therefore, these numbers can be written in the following way:

$$0 < \lambda_0 < \lambda_1 < \lambda_2 \dots < \lambda_r.$$

By (3.1), Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\Big\langle \sum_{k=0}^{n} A_k u, u \Big\rangle_{E_N} - \sum_{t=1}^{N} F(t, u(t)).$$

4. PROOF OF THEOREM 1.2

To apply Theorem 2.5 we shall do separate studies of the "geometry" of Φ and its "compactness". We decompose $E_N = V \bigoplus W$, where $V = \bigoplus_{i=0}^{l} E(\lambda_i)$ and $W = \bigoplus_{i=l+1}^{N-1} E(\lambda_i)$.

Lemma 4.1. *Under assumption* (1.5)*, the functional* Φ *has the following properties:*

(1) $\Phi(u) \longrightarrow -\infty$ *as* $\|v\|_{E_N} \to \infty$, $v \in V$,
(2) $\Phi(u) \longrightarrow \infty$ *as* $\|w\|_{E_N} \to \infty$, $w \in W$.

Proof. (1) Assume by contradiction that there exist a constant A and a sequence $(v_m) \subset V$ with $\|v_m\|_{E_N} \to \infty$ such that

$$A \le \Phi(v_m). \tag{4.1}$$

According to (1.5), there exists $r > 0$ such that

$$\frac{1}{2}\lambda_l x^2 < \frac{1}{2}\alpha x^2 \le F(t,x) \le \frac{1}{2}\beta x^2 < \frac{1}{2}\lambda_{l+1} x^2, \quad (t, |x|) \in [1, N]_{\mathbb{Z}} \times]r, \infty[\,. \tag{4.2}$$

Therefore,

$$\frac{1}{2}\lambda_l x^2 - F(t,x) \le 0, \quad (t, |x|) \in [1, N]_{\mathbb{Z}} \times]r, \infty[\,.$$

Then, for any $(t,x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$, we have

$$\frac{1}{2}\lambda_l x^2 - F(t,x) \le \max_{|x| \le r} \Big| \frac{1}{2}\lambda_l x^2 - F(t,x) \Big| = \Psi(t). \tag{4.3}$$

Let $x_m = \frac{v_m}{\|v_m\|_{E_N}}$, then $\|x_m\|_{E_N} = 1$. Since $\dim V < \infty$, there exists some $x \in V$ such that

$$\|x_m - x\|_{E_N} \underset{m\to\infty}{\longrightarrow} 0, \quad \|x\|_{E_N} = 1.$$

In particular, $x \ne 0$. We put $H_1 = \{t \in [1, N]_{\mathbb{Z}} : x(t) \ne 0\}$. For $t \in H_1$, $|v_m(t)| \longrightarrow \infty$, and by (4.2) we get

$$\sum_{t \in H_1} \frac{1}{2}\lambda_l |v_m(t)|^2 - F(t, v_m(t)) \le \frac{1}{2}(\lambda_l - \alpha) \sum_{t \in H_1} |v_m(t)|^2 \longrightarrow -\infty, \tag{4.4}$$

as $m \to \infty$.

So that using (4.3), (4.4) and the fact that $(v_m) \subset V$, we obtain

$$\begin{aligned}
\Phi(v_m) &= \frac{1}{2}\Big\langle \sum_{k=0}^{n} A_k v_m, v_m \Big\rangle_{E_N} - \sum_{t=1}^{N} F(t, v_m(t)) \\
&\leq \frac{1}{2}\lambda_l \|v_m\|_{E_N}^2 - \sum_{t=1}^{N} F(t, v_m(t)) \\
&\leq \sum_{t=1}^{N} \frac{1}{2}\lambda_l |v_m(t)|^2 - F(t, v_m(t)) \\
&= \sum_{t \in H_1} \frac{1}{2}\lambda_l |v_m(t)|^2 - F(t, v_m(t)) \\
&\quad + \sum_{t \in [1,N]_{\mathbb{Z}} \setminus H_1} \frac{1}{2}\lambda_l |v_m(t)|^2 - F(t, v_m(t)) \\
&\leq \sum_{t \in H_1} \frac{1}{2}\lambda_l |v_m(t)|^2 - F(t, v_m(t)) + \sum_{t \in [1,N]_{\mathbb{Z}} \setminus H_1} \Psi(t) \underset{m\to\infty}{\longrightarrow} -\infty.
\end{aligned}$$

This is contradiction with (4.1).

(2) Suppose on the contrary that Φ is not coercive in W. Thus, there is some constant B and some sequence $(w_m) \subset W$, with $\|w_m\|_{E_N} \to \infty$, such that

$$\Phi(w_n) \leq B. \tag{4.5}$$

Since $x \longrightarrow \frac{1}{2}\lambda_{l+1} x^2 - F(t,x)$ is continuous and by (4.2), we have

$$\frac{1}{2}\lambda_{l+1} x^2 - F(t,x) \geqslant \xi_t, \quad (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}, \tag{4.6}$$

where

$$\xi_t = \max_{t \in [1,N]_{\mathbb{Z}}} \left\{ \min_{|x| \leq r} \left[\frac{1}{2}\lambda_{l+1} x^2 - F(t,x) \right], 0 \right\}.$$

Let $y_m = \frac{w_m}{\|w_m\|_{E_N}}$, then $\|y_m\|_{E_N} = 1$. Since $\dim W < \infty$, there exists some $y \in W$ such that

$$\|y_m - y\|_{E_N} \underset{m\to\infty}{\longrightarrow} 0, \quad \|y\|_{E_N} = 1.$$

In particular, $y \neq 0$. We put $H_2 = \{t \in [1,N]_{\mathbb{Z}} / y(t) \neq 0\}$. For $t \in H_2$, $|w_m(t)| \underset{m\to\infty}{\longrightarrow} \infty$ and again by (4.2), we obtain

$$\sum_{t \in H_2} \frac{1}{2}\lambda_{l+1} |w_m(t)|^2 - F(t, w_m(t)) \geq \frac{1}{2}(\lambda_{l+1} - \beta) \sum_{t \in H_2} |w_m(t)|^2 \longrightarrow \infty, \tag{4.7}$$

as $m \to \infty$.

Using again (4.6), (4.7) and $(w_m) \subset W$, we have

$$\begin{aligned}
\Phi(w_m) &= \frac{1}{2}\Big\langle \sum_{k=0}^{n} A_k w_m, w_m \Big\rangle_{E_N} - \sum_{t=1}^{N} F(t, w_m(t)) \\
&\geq \frac{1}{2}\lambda_{l+1}\|w_m\|_{E_N}^2 - \sum_{t=1}^{N} F(t, w_m(t)) \\
&\geq \sum_{t=1}^{N} \frac{1}{2}\lambda_{l+1}|w_m(t)|^2 - F(t, w_m(t)) \\
&= \sum_{t\in H_2} \frac{1}{2}\lambda_{l+1}|w_m(t)|^2 - F(t, w_m(t)) \\
&\quad + \sum_{t\in[1,N]_{\mathbb{Z}}\setminus H_2} \frac{1}{2}\lambda_{l+1}|w_m(t)|^2 - F(t, w_m(t)) \\
&\geq \sum_{t\in H_2} \frac{1}{2}\lambda_{l+1}|w_m(t)|^2 - F(t, w_m(t)) + \sum_{t\in[1,N]_{\mathbb{Z}}\setminus H_2} \xi_t \underset{m\to\infty}{\longrightarrow} \infty.
\end{aligned}$$

This contradicts to (4.5). The proof of Lemma 4.1 is complete. □

Now, we show that Φ satisfies the (PS) condition.

Lemma 4.2. *Under the assumption* (1.5), Φ *satisfies the* (PS) *condition on* E_N.

Proof. Let $(u_m) \subset E_N$ be a (PS) sequence, i.e.,

$$|\Phi(u_m)| \leq M \quad \text{and} \quad \Phi'(u_m) \longrightarrow 0, \text{ as } m \to \infty,$$

where M is a constant. It clearly suffices to show that (u_m) remains bounded in (E_N). We argue by contradiction. Defining $z_m = \frac{u_m}{\|u_m\|_{E_N}}$, we have $\|z_m\|_{E_N} = 1$. There is a convergent subsequence of (z_m), call it (z_m) again, such that $z_m \longrightarrow z \in E_N$ as $m \to \infty$, $\|z\|_{E_N} = 1$. For every $y \in E_N$, we have

$$\frac{\langle \Phi'(u_m), y\rangle_{E_N}}{\|u_m\|_{E_N}} \longrightarrow 0, \quad \text{as } m \to \infty,$$

which means that

$$\Big\langle \sum_{k=0}^{n} A_k z_m, y \Big\rangle_{E_N} - \sum_{t=1}^{N} \frac{f(t, u_m(t))}{\|u_m\|_{E_N}} y(t) \longrightarrow 0, \quad \text{as } m \to \infty. \tag{4.8}$$

Set $H_3 = \{t \in [1, N]_{\mathbb{Z}} : z(t) \neq 0\}$. From (1.5) it is clear that

$$\lambda_l < \alpha \leq \frac{f(t, u_m(t))}{u_m(t)} \leq \beta < \lambda_{l+1}, \quad t \in H_3,$$

which implies that there exists a subsequence of (u_m), still called (u_m), and $\gamma_t \in [\alpha, \beta]$ such that

$$\lim_{m\to\infty} \frac{f(t, u_m(t))}{u_m(t)} = \gamma_t \quad \text{for } t \in H_3.$$

If $t \in [1, N]_{\mathbb{Z}} \setminus H_3$, then $\frac{f(t,u_m(t))}{\|u_m\|_{E_N}} \longrightarrow 0$ as $m \to \infty$. Thus we can rewrite (4.8) as

$$\Big\langle \sum_{k=0}^{n} A_k z_m, y \Big\rangle_{E_N} - \sum_{t\in H_3} \frac{f(t, u_m(t))}{u_m(t)} z_m(t)y(t) \longrightarrow 0, \quad \text{as } m \to \infty. \tag{4.9}$$

On the other hand, it easy to see that

$$\Big\langle \sum_{k=0}^{n} A_k z_m, y \Big\rangle_{E_N} - \sum_{t\in H_3} \frac{f(t, u_m(t))}{u_m(t)} z_m(t)y(t) \longrightarrow \Big\langle \sum_{k=0}^{n} A_k z, y \Big\rangle_{E_N} - \sum_{t\in H_3} \gamma_t z(t)y(t), \tag{4.10}$$

as $m \to \infty$. Combining (4.9) and (4.10), we get

$$\Big\langle \sum_{k=0}^{n} A_k z, y \Big\rangle_{E_N} = \sum_{t\in H_3} \gamma_t z(t)y(t) \quad \text{for } y \in E_N.$$

We put

$$\widehat{\gamma}_t = \begin{cases} \gamma_t, & t \in H_3, \\ \frac{\alpha+\beta}{2}, & t \in [1, N]_{\mathbb{Z}} \setminus H_3. \end{cases}$$

Since $z(t) = 0$ for any $t \in [1, N]_{\mathbb{Z}} \setminus H_3$, we have

$$\Big\langle \sum_{k=0}^{n} A_k z, y \Big\rangle_{E_N} = \sum_{t=1}^{N} \widehat{\gamma}_t z(t)y(t) \quad \text{for every } y \in E_N. \tag{4.11}$$

Let $z = z^- + z^+$, where $z^- \in V = \bigoplus_{i=0}^{l} E(\lambda_i)$ and $z^+ \in W = \bigoplus_{i=l+1}^{N-1} E(\lambda_i)$. Since $z \neq 0$, then $z^+ \neq 0$ or $z^- \neq 0$. Assume that $z^+ \neq 0$. Setting $y = z^+ - z^-$ in (4.11), we obtain

$$\Big\langle \sum_{k=0}^{n} A_k z^+, z^+ \Big\rangle_{E_N} - \sum_{t=1}^{N} \widehat{\gamma}_t z^+(t)^2 = \Big\langle \sum_{k=0}^{n} A_k z^-, z^- \Big\rangle_{E_N} - \sum_{t=1}^{N} \widehat{\gamma}_t z^-(t)^2. \tag{4.12}$$

In other words, we have

$$\Big\langle \sum_{k=0}^{n} A_k z^+, z^+ \Big\rangle_{E_N} - \sum_{t=1}^{N} \widehat{\gamma}_t z^+(t)^2 \geq (\lambda_{l+1} - \beta)\|z^+\|_{E_N}^2 > 0$$

and

$$\Big\langle \sum_{k=0}^{n} A_k z^-, z^- \Big\rangle_{E_N} - \sum_{t=1}^{N} \widehat{\gamma}_t z^-(t)^2 \leq (\lambda_l - \alpha)\|z^-\|_{E_N}^2 \leq 0.$$

But this gives us once more a contradiction from (4.12). The case where $z^- \neq 0$ can be proved similarly. This completes the proof. $\square$

Proof of Theorem 1.2. In view of Lemma 4.1 and Lemma 4.2 we may apply the saddle point theorem. We set

$$W = \bigoplus_{i=l+1}^{N-1} E(\lambda_i) \quad \text{and} \quad Q = \left\{ v \in V = \bigoplus_{i=0}^{l} E(\lambda_i) | \|u\|_{E_N} \leq R \right\}$$

with $R > 0$ being such that

$$a = \max_{\partial Q} \Phi < b = \inf_{W} \Phi.$$

It follows that the functional Φ has a critical value $c \geq b$ and hence the problem (1.1) has at least one solution. The proof of Theorem 1.2 is complete. $\square$

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