

FRICTION EFFECT IN A PLANE PROBLEM OF PUNCH ACTING ON A HALF-SPACE WEAKENED BY CRACKS

Adam TOMCZYK*

*Białystok University of Technology, Faculty of Mechanical Engineering, ul. Wiejska 45C, 15-351 Białystok, Poland

a.tomczyk@pb.edu.pl

Abstract: This work makes use of singular integral equations method to solve plane contact problem for a half-space containing cracks. This method is based on complex variables. Relationships are presented which help find the contact pressure under the punch and the value of stress intensity factors at the crack tips. A detailed solution is presented for the problem of a punch pressed to the surface of a half-space weakened by a single straight crack. It includes both a situation where the punch has a flat and parabolic base. The influence of friction and the position of the crack on the distribution of the values of stress intensity factors K_I i K_{II} in the functions of crack length and distance from the contact zone was analyzed in detail.

Key words: Contact Problem, Elastic Half-Space, Crack, Stress Intensity Factor, Friction

1. INTRODUCTION

In the mechanical engineering or in the technology of abrasive machining, we often meet the problem connected with friction between cooperating elements. Many classic examples of such problems can be found in various friction brakes, gear units, in grinding processes, or in the problems concerning wheels rolling and braking on rails. In the analysis of such cases it is usually assumed that both elements have no defects, such as edge or internal cracks. It is obvious that the appearance of such defects leads to a redistribution and most of all strong concentration of stresses connected with the effect of a crack.

This work concentrates on the analysis of effects connected with the "sliding" of a punch pressed against an elastic half-space weakened by edge or internal cracks. The analysis makes use of the singular integral equations method including Muskhelishvili's complex potentials (Muskhelishvili, 1962). It covers a plane problem assuming that the punch is perfectly rigid and neglecting the problems of heat generation. In many papers, such problems are modeled by applying load in the form of an appropriate contact pressure on the surface of a half-space in the place where the punch is applied (Hills et al., 1993; Hills and Nowell, 1994). It can be also found for example in the series of works by Goshima and co-authors, which concentrate the problem of rolling (including sliding) of a rigid cylindrical punch on a damaged elastic half-space. This analysis included both fatigue and temperature effects. The analyzed defects included a single edge crack extending to the surface of the elastic half-space (Goshima and Keer, 1990), two edge cracks of varying orientation (Goshima and Kamishima, 1996), a set of periodic cracks (Goshima and Kamishima, 1994; Goshima, 2003). We can also find the solution of a plane problem with a defect in the form of a single internal crack (Goshima and Soda, 1997) and problems with a three-dimensional edge crack (Goshima, 2003; Goshima et al., 1990). Bryant et al. (1984) examined the interaction effects between cylindrical indenter and single-cracked half-space but only for the case when surface-braking crack was presented in contact zone. Other paper by Bryant (Keer and Bryant, 1983) presents a fatigue

problem where cylindrical damaged indenter were considered or problem where a half-space is weakened by two cracks: vertical edge crack and horizontal subsurface crack (Keer et al., 1982). Similar subjects are covered in works by Hasebe et al., (Hasebe, 1981; Hasebe et al., 1989; Okumura et al., 1990; Hasebe and Qian, 1995, 1997, 1998; Qian and Hasebe, 1997), where for example a punch is applied with one rounded and one sharp end (Hasebe and Qian, 1998). The problem of contact between the punch and a half-space weakened by cracks was also analyzed by Panasyuk and co-authors (Panasyuk et al., 1995 2000; Datsyshyn et al., 2001), where trajectories are found for edge cracks in the context of fatigue (Datsyshyn et al., 2001). A detailed analysis has also been provided for the problem of a flat-based punch on the surface of a half-space weakened by a internal (subsurface) crack, where the faces of the crack are in contact (Panasyuk et al., 1995). For last years a problem of interaction between crack and punch has been applied to FGM materials. It was analyzed parabolic or cylindrical stamp acting on graded coatings (Guler and Erdogan, 2007) or the case when the base of the punch was assumed to be flat (Dag and Erdogan, 2002).

2. FORMULATION OF THE PROBLEM

Let us assume that into the $y = 0$ surface of an elastic half-space $-\infty < x < \infty$, $y \leq 0$ with internal cracks a punch of any shape (Fig. 1) is pressed with normal force P and slide under the influence of tangential force ρP , ρ - friction coefficient. Let us also assume that under the influence of such loading the punch is in a limit equilibrium condition. Size of the contact zone between punch and surface of half-space is $2a$. It should be noted that moment affects the punch preventing it from turning.

The boundary conditions on the $y = 0$ plane can be noted as:

$$\begin{aligned} \sigma_{xy} + \rho\sigma_y = 0, \quad v = f(x) + C \\ x \in L_0 = \{x: -a < x < a\} \end{aligned} \quad (1)$$

$$\sigma_y - i\sigma_{xy} = p_0(x) = 0, \quad x \notin L_0 \quad (2)$$

where σ_y , σ_{xy} are the normal and tangent component of the stress tensor, v is a component of the displacement vector in the y direction, $f(x)$ – a function describing the shape of the punch base, and C – constant (vertical displacement of the punch).

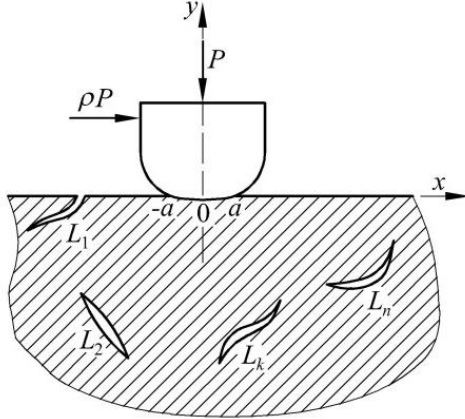


Fig. 1. Scheme of considered problem

The edges of the crack are assumed to be under a self-balanced load in the following form:

$$N^\pm(t) + iT^\pm(t) = p(t), \quad t \in L = \bigcup_{k=1}^n L_k \quad (3)$$

where N and T are a normal and tangential component of the surface stress vector on the faces of the crack, $L_k (k = \overline{1, n})$ – the contour of k -th crack, “+” and “-” denote left and right edges of the L_k contour respectively.

General complex stress potentials for a system of curved cracks in a half-space affected by external load, are as follows (Muskhelishvili, 1962; Savruk, 1981):

$$\Phi_1(z) = \Phi_0(z) + \Phi(z), \quad \Psi_1(z) = \Psi_0(z) + \Psi(z), \quad (4)$$

$$\Phi_0(z) = -\frac{1}{2\pi i} \int_{L_0} \frac{p_0(t) dt}{t-z} \quad (5)$$

$$\Psi_0(z) = -\frac{1}{2\pi i} \int_{L_0} \left[\frac{p_0(t)}{t-z} - \frac{tp_0(t)}{(t-z)^2} \right] dt$$

$$\Phi(z) = \frac{1}{2\pi} \int_L \left[\left(\frac{1}{t-z} - \frac{1}{\bar{t}-z} \right) g'(t) dt + \frac{t-\bar{t}}{(\bar{t}-z)^2} \overline{g'(t)} d\bar{t} \right]$$

$$\Psi(z) = \frac{1}{2\pi} \int_L \left[\left(\frac{1}{t-z} - \frac{1}{\bar{t}-z} - \frac{(t-\bar{t})(\bar{t}+z)}{(\bar{t}-z)^3} \right) \overline{g'(t)} d\bar{t} - \left(\frac{\bar{t}}{(t-z)^2} - \frac{\bar{t}}{(\bar{t}-z)^2} \right) g'(t) dt \right]$$

where p_0 denotes loading of the edge of the half-space and $g'(t)$ is an unknown function on the contours of the cracks.

The derivative of vertical displacement of the edge of the half-space can be given as follows (Gallin, 1980):

$$2G \frac{\partial v(x,0)}{\partial x} = -\frac{\kappa-1}{2} \tau(x) + \frac{\kappa+1}{2\pi} \int_{L_0} \frac{\sigma(t)}{t-z} dt + \text{Im} \left[\kappa \Phi(x) - \overline{\Phi(x)} - x \overline{\Phi'(x)} - \overline{\Psi(x)} \right] \quad (6)$$

where G is shear modulus, ν – Poisson's ratio, $\kappa = 3 - 4\nu$, σ , τ – respectively: the normal and tangential component of the load on the surface of the half-space.

Satisfying the boundary conditions (1) with the Eq. (6) leads to the following integral equation:

$$\varepsilon \rho \sigma(x) + \frac{1}{\pi} \int_{L_0} \frac{\sigma(t) dt}{t-x} + 2(\kappa+1)^{-1} \text{Im} \left[\kappa \Phi(x) - \overline{\Phi(x)} - x \overline{\Phi'(x)} - \overline{\Psi(x)} \right] = 4G(\kappa+1)^{-1} f'(x), \quad x \in L_0 \quad (7)$$

where $\varepsilon = \frac{\kappa-1}{\kappa+1}$.

To Eq. (7) we have to add a punch equilibrium condition:

$$\int_{L_0} \sigma(t) dt = -P \quad (8)$$

which guarantees uniqueness solution of the Eq. (7)

In the case of a homogeneous half-space, with no cracks, $\Phi(x) = \Psi(x)$, and Eqs. (7) and (8) are known (see Muskhelishvili, 1962).

Integral equations on the contours of the cracks can be written in condensed form (Savruk, 1981):

$$\Phi_1(t') + \overline{\Phi_1(t')} + \frac{d\bar{t}'}{dt'} \left[t' \overline{\Phi_1'(t')} + \overline{\Psi_1(t')} \right] = p(t'), \quad t' \in L, \quad (9)$$

where $\Phi_1(t')$ i $\Psi_1(t')$ – direct values of complex potentials (4).

To Eq. (9) we also have to add displacement uniqueness conditions while considering each internal crack:

$$\int_{L_k} g'(t) dt = 0, \quad k = \overline{2, n} \quad (10)$$

Eqs. (7) and (9) form a system of singular integral equations of the first and second kind, which allows for the finding of normal contact pressure $\sigma(x)$ and a derivative of the displacement jumps vector $g'(t)$ on the contours of cracks. Including conditions (8) and (10) this system has a single solution for any right part in the function class, which have integrable singularities at the ends of the ranges of integration (Muskhelishvili, 1962).

It has to be noted that the system of integral Eqs. (7)–(9) was found under conditions (3). This means that there can be given loads on crack faces or, in a particular case, no load. Such boundary conditions can be realized only when the cracks are within the tensile stress fields. However, as a result of pressing the punch in, some or even all cracks can be found in the field of compression stresses, which is produced in the half-space under the punch. In that situation the faces of the cracks will be in contact. If this is so along the whole length of the crack, than Eqs (7)–(10) can be easily modified, including the continuity of normal displacement on the contours of the crack and friction according to Amonton's law. In the general case we receive a mixed problem on the faces of the crack, which significantly increases the complexity of the solution. However, most works in this field, including this one, analyze these questions without including contact between crack faces.

Complex potential (5) can be presented in the form:

$$\begin{aligned} \Phi_0(z) &= \frac{1}{\pi} \int_{L_0} f_{11}(t, z) \sigma(t) dt \\ \Psi_0(z) &= \frac{1}{\pi} \int_{L_0} h_{11}(t, z) \sigma(t) dt \\ \Phi_0'(z) &= \frac{1}{\pi} \int_{L_0} g_{11}(t, z) \sigma(t) dt \\ \Phi(z) &= \frac{1}{\pi} \int_L [f_{21}(t, z) g'(t) dt + f_{22}(t, z) \overline{g'(t)} d\bar{t}] \\ \Psi(z) &= \frac{1}{\pi} \int_L [h_{21}(t, z) g'(t) dt + h_{22}(t, z) \overline{g'(t)} d\bar{t}] \\ \Phi'(z) &= \frac{1}{\pi} \int_L [g_{21}(t, z) g'(t) dt + g_{22}(t, z) \overline{g'(t)} d\bar{t}] \end{aligned} \quad (11)$$

where:

$$\begin{aligned} f_{11}(t, z) &= -\frac{1+i\rho}{2i} \frac{1}{t-z}, & f_{21}(t, z) &= \frac{1}{2} \left(\frac{1}{t-z} - \frac{1}{\bar{t}-z} \right) \\ f_{22}(t, z) &= \frac{t-\bar{t}}{2(\bar{t}-z)^2}, & g_{11}(t, z) &= -\frac{1+i\rho}{2i} \frac{1}{(t-z)^2} \\ g_{21}(t, z) &= \frac{1}{2} \left(\frac{1}{(t-z)^2} - \frac{1}{(\bar{t}-z)^2} \right), & g_{22}(t, z) &= \frac{t-\bar{t}}{(\bar{t}-z)^3} \\ h_{11}(t, z) &= -\frac{1}{2i} \left[\frac{1-i\rho}{t-z} - \frac{t(1+i)}{(t-z)^2} \right] \\ h_{21}(t, z) &= -\frac{\bar{t}}{2} \left(\frac{1}{(t-z)^2} - \frac{1}{(\bar{t}-z)^2} \right) \\ h_{22}(t, z) &= \frac{1}{2} \left(\frac{1}{t-z} - \frac{1}{\bar{t}-z} - \frac{(t-\bar{t})(\bar{t}+z)}{(\bar{t}-z)^3} \right) \end{aligned} \quad (12)$$

Taking into account Eqs. (11) and (12), the system of integral Eqs. (7) and (9) can be noted as:

$$\begin{aligned} \varepsilon \rho \sigma(x) + \frac{1}{\pi} \int_{L_0} \frac{\sigma(t) dt}{t-x} + \frac{1}{\pi} \operatorname{Im} \int_L [K_1(t, x) g'(t) dt + \\ + L_1(t, x) \overline{g'(t)} d\bar{t}] = \frac{4G}{\kappa+1} f'(x), \quad x \in L_0 \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{1}{\pi} \int_L [K_2(t, t') g'(t) dt + L_2(t, t') \overline{g'(t)} d\bar{t}] + \\ + \frac{1}{\pi} \int_{L_0} M(t, t') \sigma(t) dt = p(t'), \quad t' \in L \end{aligned} \quad (14)$$

Kernels of the system of Eqs. (13) and (14) are given by functions f_{ij} , g_{ij} and h_{ij} ($i, j = 1, 2$) described by relations (12), can be represent as follows:

$$\begin{aligned} K_1(t, x) &= \frac{2}{\kappa+1} \left[\kappa f_{21}(t, x) - \overline{f_{22}(t, x)} - x g_{22}(t, x) - \overline{h_{22}(t, x)} \right] \\ L_1(t, x) &= \frac{2}{\kappa+1} \left[\kappa f_{22}(t, x) - \overline{f_{21}(t, x)} - x g_{21}(t, x) - \overline{h_{21}(t, x)} \right] \\ K_2(t, t') &= f_{21}(t, t') + \overline{f_{22}(t, t')} + \frac{dt'}{dt} \left[t' g_{22}(t, t') + \overline{h_{22}(t, t')} \right] \\ L_2(t, t') &= f_{22}(t, t') + \overline{f_{21}(t, t')} + \frac{dt'}{dt} \left[t' g_{21}(t, t') + \overline{h_{21}(t, t')} \right] \\ M(t, t') &= f_{11}(t, t') + \overline{f_{11}(t, t')} + \frac{dt'}{dt} \left[t' g_{11}(t, t') + \overline{h_{11}(t, t')} \right] \end{aligned} \quad (15)$$

Note, that function $K_2(t, t')$ includes Cauchy's singular kernel. This part will not be specially singled out, unless numerical integration formulas will be used to carry out the numerical calculations, which are correct for both singular and regular integrals.

3. PARTICULAR CASE OF A SINGLE EDGE CRACK

Let us consider a case, where a half-space is weakened by only one straight crack (Fig. 2) Let us introduce a parametric notation of the contour of crack L and the zone of contact between the punch and the surface of half-space L_0 :

$$t = a\xi, \quad x = a\eta, \quad -1 < \xi, \eta < 1, \quad (t, x \in L_0)$$

$$t = a\omega(\xi), \quad t' = a\omega(\eta), \quad -1 < \xi, \eta < 1, \quad (t, t' \in L) \quad (16)$$

$$\omega(\xi) = -b^* - \frac{l^*}{2} (\xi+1) e^{i\varphi}, \quad b^* = \frac{b}{a}, \quad l^* = \frac{l}{a}$$

where φ is angle of crack orientation defined according to Fig. 2.

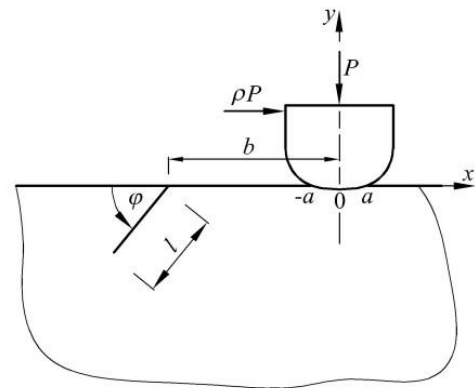


Fig. 2. Scheme of a pressed punch „sliding” on a surface of a single-cracked half-space

Moving to non-dimensional variables and functions the system of integral Eqs. (13) and (14) and the condition of solvability (8) has the following form:

$$\begin{aligned} \varepsilon \rho \sigma(\eta) + \frac{1}{\pi} \int_{-1}^1 \frac{\sigma(\xi) d\xi}{\xi-\eta} + \frac{1}{\pi} \operatorname{Im} \int_{-1}^1 [K_1(\xi, \eta) g'(\xi) + \\ + L_1(\xi, \eta) \overline{g'(\xi)}] d\xi = F(\eta), \quad |\eta| < 1 \end{aligned} \quad (17)$$

$$\frac{1}{\pi} \int_{-1}^1 \left[K_2(\xi, \eta) g'(\xi) + L_2(\xi, \eta) \overline{g'(\xi)} \right] d\xi + \frac{1}{\pi} \int_{-1}^1 M(\xi, \eta) \sigma(\xi) d\xi = p(\eta), \quad |\eta| < 1 \quad (18)$$

$$\int_{-1}^1 \sigma(\xi) d\xi = A = \frac{-P}{a}, \quad (19)$$

where:

$$\begin{aligned} \sigma(a\eta) &= \sigma(\eta), & g'(a\omega(\xi)) \omega'(\xi) &= g'(\xi) \\ p(a\omega(\xi)) &= p(\eta), & 4G(\kappa+1)^{-1} f'(a\eta) &= F(\eta) \\ aK_1(a\omega(\xi), a\eta) &= K_1(\xi, \eta) \\ aL_1(a\omega(\xi), a\eta) &= L_1(\xi, \eta) \\ aK_2(a\omega(\xi), a\omega(\eta)) &= K_2(\xi, \eta) \\ aL_2(a\omega(\xi), a\omega(\eta)) &= L_2(\xi, \eta) \\ aM(a\xi, a\omega(\eta)) &= M(\xi, \eta) \end{aligned} \quad (20)$$

A solution for the system of integral Eqs. (17)-(18), which satisfies condition (19) will be sought in the class of functions, which have integrable singularities:

$$\begin{aligned} \sigma(\xi) &= w(\xi) q(\xi) \\ w(\xi) &= (1-\xi)^\alpha (1+\xi)^\beta, \quad -1 < \alpha, \beta < 0 \\ g'(\xi) &= (1-\xi^2)^{-1/2} u(\xi) \end{aligned} \quad (21)$$

where $q(\xi)$ and $u(\xi)$ are continuous functions in a closed interval $[-1, 1]$, and parameters α and β are roots $\alpha = -0.5 + \mu$, $\beta = -0.5 - \mu$, $\mu = \arctan \varepsilon \rho$ of characteristic equations: $\cot \pi \alpha = -\varepsilon \rho$, $\cot \pi \beta = \varepsilon \rho$.

Let us remind that only for internal cracks condition (10) must be satisfied. However for the case of the edge crack singularity is produced only at the one crack tip (i.e. for $\xi = +1$). At the second tip ($\xi = -1$), which extends to the surface, singularity disappears and also condition (10) can not be satisfied. So the solution of Eqs. (17)-(18) will be sought in the same class of functions (21) but condition (10) should be replaced by:

$$u(-1) = 0 \quad (22)$$

In the case of a homogeneous half-space, with no cracks, there is $g'(\xi) = 0$ and a system of integral Eqs. (17) and (18) is reduced to a single Eq. (18) when condition (19) is satisfy.

4. NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS SYSTEM

For a numerical solution of the system of Eqs. (17)-(19) we will use the quadrature method (or the mechanical quadrature method) (Savruk, 1981; Savruk et al., 1999, Savruk and Tomczyk, 2010) using Gauss-Chebyshev nodes. As a result we receive the following system of linear algebraic equations:

$$\sum_{k=1}^n a_{mk} q(\xi_k) + \text{Im} \sum_{k=1}^{\hat{n}} \left[b_{1mk} u(\hat{\xi}_k) + c_{1mk} \overline{u(\hat{\xi}_k)} \right] = F(\eta_m), \quad m = 1, \dots, n-1 \quad (23)$$

$$\sum_{k=1}^{\hat{n}} \left[b_{2mk} u(\hat{\xi}_k) + c_{2mk} \overline{u(\hat{\xi}_k)} \right] + \sum_{k=1}^n d_{mk} q(\xi_k) = p(\hat{\eta}_m), \quad m = 1, \dots, \hat{n}-1$$

$$\sum_{k=1}^n a_{nk} q(t_k) = A, \quad \sum_{k=1}^{\hat{n}} (-1)^k \sqrt{\frac{1-\hat{\xi}_k}{1+\hat{\xi}_k}} u(\hat{\xi}_k) = 0 \quad (24)$$

$$\begin{aligned} \xi_k &= \cos \frac{2k-1}{2n} \pi, & \hat{\xi}_k &= \cos \frac{2k-1}{2\hat{n}} \pi \\ \eta_m &= \cos \frac{\pi m}{n}, & \hat{\eta}_m &= \cos \frac{\pi m}{\hat{n}} \end{aligned} \quad (25)$$

The unknown values here are functions $q(\xi)$ and $u(\xi)$ respectively in nodes ξ_k and $\hat{\xi}_k$. The coefficients of the unknowns in the system of Eqs. (23) are given by the following formulas:

$$\begin{aligned} a_{mk} &= -\frac{1}{n} (-1)^k \sqrt{1-\xi_k^2} \left\{ \frac{X_0}{1-\eta_m} \left[\frac{1}{1-\xi_k} - \frac{T_n(\eta_m)}{\eta_m - \xi_k} \right] + \right. \\ &\quad \left. + \frac{R_0}{1+\eta_m} \left[\frac{(-1)^n}{1+\xi_k} + \frac{T_n(\eta_m)}{\eta_m - \xi_k} \right] + X_3(\eta_m) + X_4(\eta_m) \frac{T_n(\eta_m)}{\eta_m - \xi_k} \right\} \\ a_{nk} &= \frac{\pi}{n} \sqrt{1-\xi_k^2} \left[w(\xi_k) - (-1)^k \frac{X_0}{1-\xi_k} + (-1)^{k+n} \frac{R_0}{1+\xi_k} \right] \\ b_{1mk} &= \frac{1}{\hat{n}} K_1(\hat{\xi}_k, \eta_m), & b_{2mk} &= \frac{1}{\hat{n}} K_2(\hat{\xi}_k, \hat{\eta}_m) \\ c_{1mk} &= \frac{1}{\hat{n}} L_1(\hat{\xi}_k, \eta_m), & c_{2mk} &= \frac{1}{\hat{n}} L_2(\hat{\xi}_k, \hat{\eta}_m) \\ d_{mk} &= \frac{1}{n} \sqrt{1-\xi_k^2} \left[M(\xi_k, \hat{\eta}_m) w(\xi_k) - \right. \\ &\quad \left. - (-1)^k \frac{X_0 M(1, \hat{\eta}_m)}{1-\xi_k} + (-1)^{n+k} \frac{R_0 M(-1, \hat{\eta}_m)}{1+\xi_k} \right] \\ X_0 &= \frac{1}{2} \left[\frac{2\alpha}{\sin \pi \alpha} - \frac{1}{n} \sum_{k=1}^n X(\xi_k) \sqrt{1-\xi_k^2} \right] \\ R_0 &= \frac{1}{2} \left[\frac{2\beta}{\sin \pi \beta} - \frac{1}{n} \sum_{k=1}^n R(\xi_k) \sqrt{1-\xi_k^2} \right] \end{aligned} \quad (26)$$

$$X_1(x) = \frac{1}{2} \left[X(x) \cot \pi \alpha - \frac{1}{\sin \pi \alpha} - R(x) \cot \pi \beta + \frac{1}{\sin \pi \beta} \right]$$

$$X_3(x) = \frac{1}{n} \sum_{r=1}^n w(\xi_r) \sqrt{1-\xi_r^2} F(x, \xi_r, \xi_k)$$

$$X_4(x) = \varepsilon \rho w(x) + X_1(x)$$

$$X(x) = \left(\frac{1-x}{1+x} \right)^\alpha, \quad R(x) = \left(\frac{1+x}{1-x} \right)^\beta$$

$$F(x, \xi_r, \xi_k) = \begin{cases} \frac{T_n(x)}{(x-\xi_k)(x-\xi_r)}, & x \neq \xi_k, x \neq \xi_r, \xi_k \neq \xi_r \\ -\frac{n(-1)^r}{\sqrt{1-\xi_r^2}(\xi_r-x)} + \frac{T_n(x)}{(\xi_r-x)^2}, & x \neq \xi_k = \xi_r \end{cases}$$

where $Tn(x) = \cos(n \arccos x)$ is a first kind Chebyshev polynomial of n degree.

Using Lagrange's interpolating polynomial for nodes ξ_k we can find the values of function $q(\xi)$ at any point:

$$q(\xi) = -\frac{1}{n} \sum_{k=1}^n (-1)^k q(\xi_k) \sqrt{1-\xi^2} \frac{T_n(x)}{\xi - \xi_k} \quad (27)$$

Particularly, at the ends of the interval $[-1, 1]$ we receive:

$$q(\pm 1) = \mp \frac{(\pm 1)^n}{n} \sum_{k=1}^n (-1)^k \sqrt{\frac{1 \pm \xi_k}{1 \mp \xi_k}} q(\xi_k) \quad (28)$$

Similarly, we search for the values of function $u(\xi_k)$ in nodes ξ_k :

$$u(\xi) = -\frac{1}{n} \sum_{k=1}^n (-1)^k u(\xi_k) \sqrt{1-\xi^2} \frac{T_n(x)}{\xi - \xi_k}, \quad -1 \leq \xi \leq 1 \quad (29)$$

$$u(\pm 1) = \mp \frac{(\pm 1)^n}{n} \sum_{k=1}^n (-1)^k \sqrt{\frac{1 \pm \xi_k}{1 \mp \xi_k}} u(\xi_k) \quad (30)$$

Stress intensity factors at the crack tip $\xi = 1 (K_{I,II}^+)$ are found with these formulas (Savruk, 1981):

$$K_I^+ - iK_{II}^+ = -\sqrt{\pi a} \left| \omega'(1) \right| \frac{u(1)}{\omega'(1)} \quad (31)$$

It has to be noted that in the case of a flat-based punch contact zone L_0 is known. However, if the punch base is rounded at one or both ends, the dimensions of the contact zone are not known. In these cases the stresses at both ends of the contact zone are finite and for the purpose of finding the location of this zone we have one or two additional conditions. Particularly, for the parabolic punch, which also was the subject of our research, we have additional conditions:

$$q(-1) = q(1) = 0 \quad (32)$$

thus we determine the half width of contact zone a and the eccentricity e (distance of the center of these zone from the punch axis).

5. ANALYSIS OF RESULTS

The theoretical relations presented in the previous paragraph were used to create numerical algorithms which allow to determine the distribution of contact pressure and the values of stress intensity factors at the crack tip. As it was already mentioned, two different shapes of the base of the punch were considered – flat and parabolic. It was analyzed in detail the influence of the shape of the punch base on the distribution of dimensionless stress intensity factors $F_I = \frac{K_{I}\sqrt{a}}{P}$, $F_{II} = \frac{K_{II}\sqrt{a}}{P}$. It was assumed in the numerical calculations that the edges of the crack are free of load ($N = T = 0$ in Eq. (3)).

Results of numerical analysis prove that existence of crack (or set of cracks) has a significant influence on contact pressure distribution (see Fujimoto et al, 1992, Tomczyk 2011; Savruk and Tomczyk, 2010). It corresponds with both flat and parabolic punch particularly for crack orientation angles from the range

(90°, 180°). For these angles crack tip can be found in material under contact zone directly and produce a “rapidly” perturbation of typical contact pressure distribution. Assuming regular contact pressure distribution in the place where punch acting can produce considerable errors in many contact problems. Only taking into consideration a mutual interaction between crack (or set of cracks) and punch gives a detailed information about character of pressure distribution. This interaction can be neglected for the case of homogenous half-space or when the crack is far away from the punch. As it was mentioned for the case of flat punch contact zone size a is known. For parabolic punch this size should be obtained by using condition of pressure reducing to zero in the ends of contact zone (see Eq. (32)).

In present analysis the dependence of stress intensity factors values on distance between crack and punch, crack length and its orientation and friction coefficient is discussed. It is also analysed the effect of punch shape (parabolic or flat) on stress intensity factors.

In the case of no friction or for small values of friction coefficient crack located close to the punch can be present in fields of compression stresses. Situation like this can take place for both: when $0^\circ < \varphi < 90^\circ$ (Fig. 3a), and when $90^\circ < \varphi < 180^\circ$ (Fig. 3b).

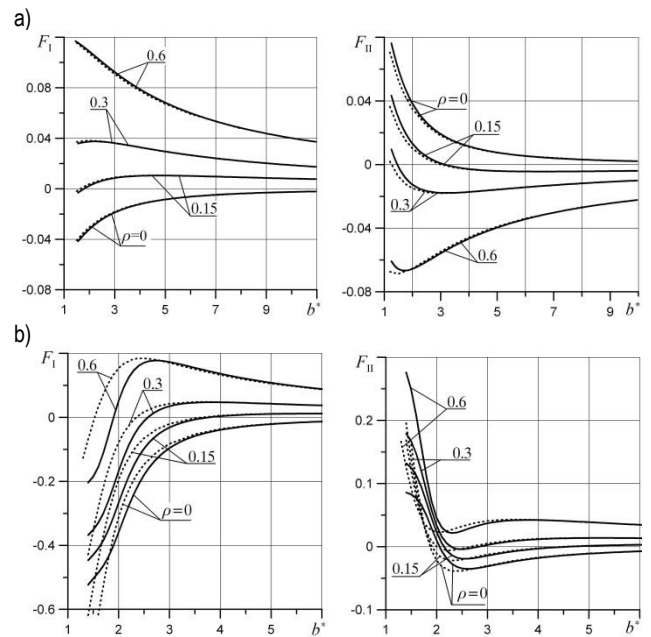


Fig 3. The effect of distance between crack and punch on dimensionless stress intensity factors F_I and F_{II} for different values of friction coefficient ($f = 1, \nu = 0.3$): a) $\varphi = 45^\circ$, b) $\varphi = 135^\circ$ (dashed lines – parabolic punch, continuous lines – flat punch)

The effect of punch shape is distinctly seen for $\varphi \in (90^\circ, 180^\circ)$ particular for close cracks when $b^* < 2.5$ (Fig. 3b, Fig. 4). For cracks defined by $0^\circ < \varphi < 90^\circ$ the effect of punch shape can be neglected even for small values of distance between crack and punch. When the suitable conditions exist the crack can close when the flat punch is acting while for the case of parabolic punch the same crack is opened. The effect like this is more probable when the angle φ is close to 180° and values of b^* are very small (Fig. 5).

A very interesting situation we can observe in the Fig. 6. It presents the effect of translation of crack tip into material under

contact zone directly produced by increasing in crack length. It is clearly seen that crack starts to close for certain values of l^* . This effect appears for small values of friction coefficient earlier than for higher values. Increasing in friction delays crack closure process. Let us note that there is no any influence of punch shape on stress intensity factors in Fig. 6.

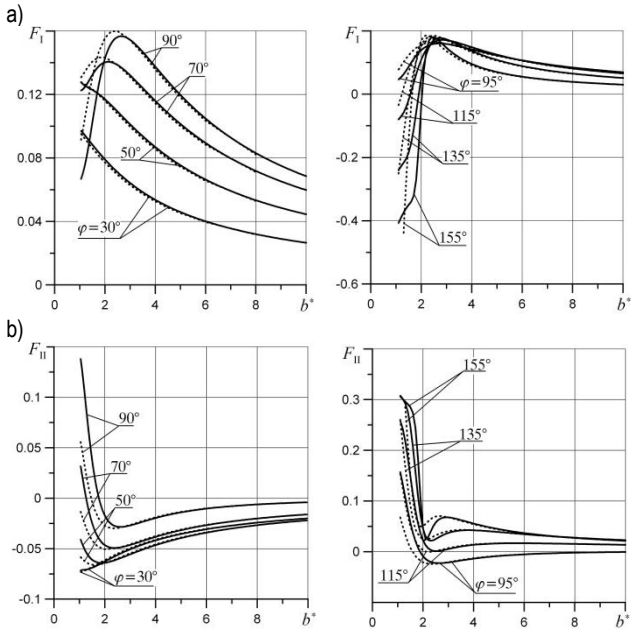


Fig. 4. The effect of distance between crack and punch on dimensionless stress intensity factors F_I (a) and F_{II} (b) for various crack orientations ($l^* = 1, \nu = 0.3, \rho = 0.6$) (dashed lines – parabolic punch, continuous lines – flat punch)

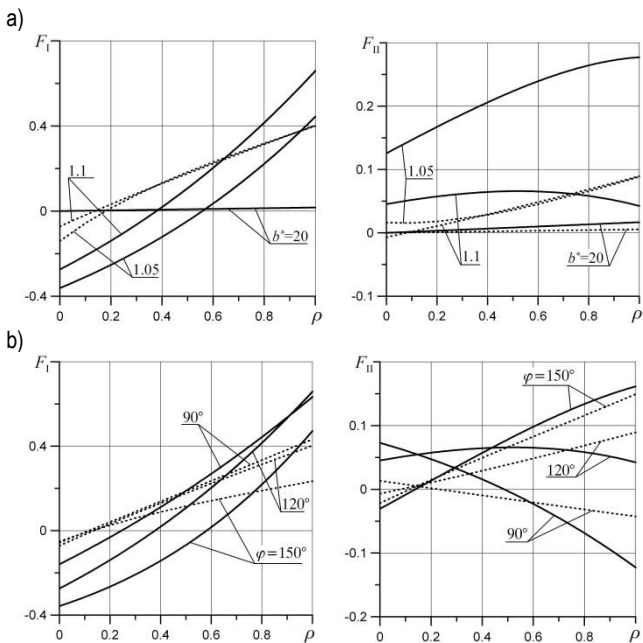


Fig. 5. Influence of friction on dimensionless stress intensity factors F_I and F_{II} for flat (continuous lines) and parabolic (dashed lines) punch: a) the effect of distance between crack and punch ($l^* = 0.1, \nu = 0.3, \varphi = 120^\circ$), b) the effect of crack orientation ($l^* = 0.1, b^* = 1.1, \nu = 0.3$)

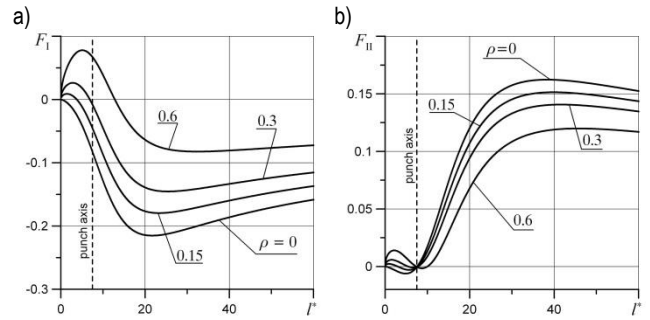


Fig. 6. The effect of crack length on dimensionless stress intensity factors F_I (a) i F_{II} (b) for various values of friction coefficient ($b^* = 15, \nu = 0.3, \varphi = 120^\circ$)

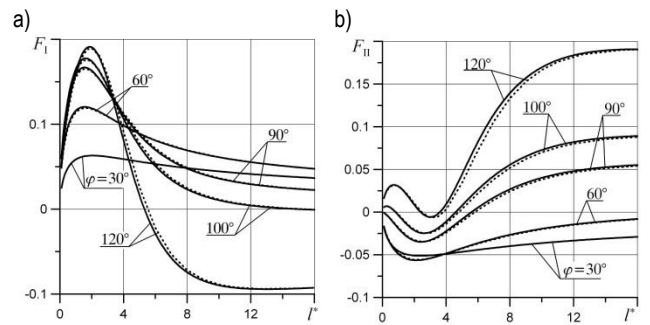


Fig. 7. The effect of crack length on dimensionless stress intensity factors F_I (a) i F_{II} (b) for various crack orientations ($b^* = 5, \nu = 0.3, \rho = 0.75$) (dashed lines – parabolic punch, continuous lines – flat punch)

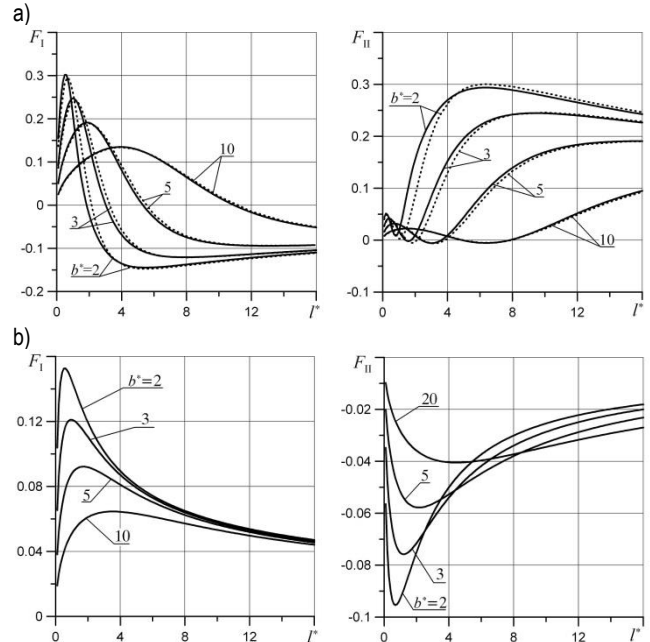


Fig. 8. The effect of crack length on dimensionless stress intensity factors F_I and F_{II} for various values of distance between crack and punch ($\rho = 0.75, \nu = 0.3$): a) $\varphi = 120^\circ$, b) $\varphi = 45^\circ$ (dashed lines – parabolic punch, continuous lines – flat punch)

It is very characteristic for $90^\circ < \varphi < 180^\circ$ that factors F_I and F_{II} decrease to zero considerably slower (Fig. 8a) than for $0^\circ < \varphi < 90^\circ$ (Fig. 8b). The effect of crack closure can be achieved much more earlier for higher values of φ (Fig. 7a). This

effect appears more distinctly for $90^\circ < \varphi < 180^\circ$. For the case of $0^\circ < \varphi < 90^\circ$ it can be neglected particularly for small values of φ when it is not visible.

The effect of crack length is very strong in the region of short cracks where $l < 6$ (Fig. 8). The most dangerous situations appear not for shortest cracks but for cracks where stress intensity factors reaches maximal values. This characteristic length depends on distance between crack and punch mainly and increases as b^* increases. The effect of punch shape is as clearly as crack is located closer to the punch and for $90^\circ < \varphi < 180^\circ$ only.

6. CONCLUSIONS

This work presents a solution of the problem of a rigid punch acting on a half-space weakened by a set of cracks. In the analysis the method of singular integral equations has been used. It has been analyzed in detail the case of a half-space weakened by a single edge crack while the base of the punch is parabolic or flat. Therefore it has been observed the influence of shape of the base of the punch on the the values of K_I and K_{II} . This influence can be neglected for the crack orientation angle $\varphi \in (0^\circ, 90^\circ)$ as opposed to $\varphi \in (90^\circ, 180^\circ)$. For a numerical solution of the system of integral equations the quadrature method has been used, which allowed to reduce this system to a system of linear algebraic equations.

Presented method gives possibility for simultaneously calculating the contact pressure and the values of stress intensity factors. This means that it is possible to take into account the mutual interaction between crack and punch. While analyzing the influence of the punch shape, it has to be noted that it has a significant influence not only on the characteristics of contact pressure distribution, but also on the values of stress intensity factors. This influence is particularly notable for cracks close to the contact zone and decreases as the crack "moves away". So it shows that for far cracks it doesn't matter a character of applied loading: parabolic, asymptotic or even in the form of concentrate force. On the other hand for close cracks it should be noted that even not accounting for friction the shape of the punch is not without importance. Furthermore, it was made a detailed analysis of the influence of the orientation and length of the crack on the values of stress intensity factors K_I and K_{II} . For vertically oriented cracks the shape of the punch has no influence on the values of stress intensity factors even for close cracks. Proposed method also allows for description of the changes in the contact zone size and the eccentricity of its center when the parabolic punch is applied.

The method presented here can be used for the analysis of problems concerning a half-space weakened by a set of edge or internal cracks. An attempt could also be made to use this solution to analyze problems of single cracks or sets of cracks under the surface of a half-space or internal curvilinear macroinclusions or holes. Furthermore, the singular integral equations method may be used in future to solve problems of bilateral contact, with the assumption that both the half-space and the punch are not rigid.

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