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BETTER VERSUS LONGER SERIES OF HEADS AND TAILS

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Abstract. This article considers a part of games theory by Penney. We intuitionally believe that a shorter series is always a better one. We will prove that this is not always so, and a longer series may happen to be better (see also [1]). In the case of Penney's game, in which players can choose their series, the proved theorems can be a part of a player's game strategy.

Definition 1. Let $k \in \mathbb{N}$ and $k \geq 1$. Each result of the k-fold variation of the $\{H, T\}$ set, which is a result of the k-fold coin toss, is called *a series of heads and tails.* We mark its length as $|\alpha|$.

Definition 2. Let α and β be series of heads and tails. We can say that *the series α is not included in the series β* if it is not the subsequence of the successive elements of the series β.

Definition 3. Let α and β be series of heads and tails. Let the series α be k long and the series β be l long. Let us also assume that the series α is not included in the β one. We repeat a coin toss so long that we get k last results forming the series α or l last results forming the series β. We call this experiment *waiting for one of the two stated series of results* and mark it as $\delta_{\alpha-\beta}$ (see [3], pp. 406–415).

Let us consider a game of two players, G_α and G_β. In the game they conduct the experiment $\delta_{\alpha-\beta}$. If the waiting finishes with the series α – the player G_α wins, and if it finishes with the series β – the player G_β wins. We shall call this game *the Penney game*[1] and mark it as $g_{\alpha-\beta}$.

Let us consider the waiting of $\delta_{\alpha-\beta}$. The sequence ω having its elements from the set $\{H,T\}$ is a result of the experiment $\delta_{\alpha-\beta}$ if it fulfills the following conditions:

- the subsequence of k last results forms the series α or the subsequence of l last results forms the series β, and
- no subsequence of k or l successive results forms the series α or β.

We mark the set of all such sequences (results of the experiment $\delta_{\alpha-\beta}$) as $\Omega_{\alpha-\beta}$.

If the result ω of the experiment $\delta_{\alpha-\beta}$ is an n-element sequence, it is a specific result of an n-fold coin toss. Its probability equals $\left(\frac{1}{2}\right)^n$.
Let $p_{\alpha-\beta}$ be a function of ω,

$$p_{\alpha-\beta}(\omega) = \left(\frac{1}{2}\right)^{|\omega|} \text{ for } \omega \in \Omega_{\alpha-\beta},$$

and $|\omega|$ be the ω sequence length (the number of elements). This function is the distribution of probability in the set $\Omega_{\alpha-\beta}$, and the pair $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$ is a probabilistic model of the waiting $\delta_{\alpha-\beta}$.

Let us state two opposite events in the space $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$:

$$A = \{\text{the waiting } \delta_{\alpha-\beta} \text{ gives the series } \alpha \text{ at the end}\},$$
$$B = \{\text{the waiting } \delta_{\alpha-\beta} \text{ gives the series } \beta \text{ at the end}\}.$$

Definition 4. If $P(A) = P(B)$, we say that *the series α and β are equally good* and mark them as $\alpha \approx \beta$.

Definition 5. If $P(A) > P(B)$, we say that *the series α is better than the series β* and mark them as $\alpha \gg \beta$.

In the game $g_{\alpha-\beta}$ we conduct the experiment $\delta_{\alpha-\beta}$. If the event A occurs, the player G_α wins. If the experiment ends with the event B, the game winner is the player G_β. Stating the probability of the events A and B, we can also determine the fairness of the Penney game. If the series α and β are equally good, the players have equal chance to win. The game $g_{\alpha-\beta}$ is fair. If one of the series is better than the other, the players chances to win are not equal and the game is not fair.

[1]Proposed by Walter Penney, see [2].

Let $\delta_{\alpha-\beta}$ be waiting for one of two series of heads and tails and k and l be the lengths of series α and β. Let $m \in \{1, 2, 3, \ldots, \min\{k, l\}\}$, $\alpha^{(m)}$, $\beta^{(m)}$ mean sequences of first m elements of series α and β, respectively, and $\alpha_{(m)}$, $\beta_{(m)}$ mean last m elements of the series α and β, respectively. Let us define the sets

$$A_\alpha = \{m : \alpha_{(m)} = \alpha^{(m)}\}, \qquad A_\beta = \{m : \alpha_{(m)} = \beta^{(m)}\},$$

$$B_\beta = \{m : \beta_{(m)} = \beta^{(m)}\}, \qquad B_\alpha = \{m : \beta_{(m)} = \alpha^{(m)}\},$$

and the following sums

$$\alpha : \alpha = \sum_{j \in A_\alpha} 2^j, \qquad \alpha : \beta = \sum_{j \in A_\beta} 2^j,$$

$$\beta : \beta = \sum_{j \in B_\beta} 2^j, \qquad \beta : \alpha = \sum_{j \in B_\alpha} 2^j.$$

Theorem 1. *In the probabilistic space of $\delta_{\alpha-\beta}$ the equation*

$$\frac{P(B)}{P(A)} = \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

called the Conway equation, is true[2].

Remark 1. *From the preceding equation, we can tell that if*

$$\mu := \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

then

$$\mu > 1 \Leftrightarrow \beta \gg \alpha,$$

$$\mu = 1 \Leftrightarrow \alpha \approx \beta,$$

$$\mu < 1 \Leftrightarrow \alpha \gg \beta.$$

Example 1. Let $\alpha = HTHTHT$ and $\beta = HHTHTH$. Let us notice that $\alpha_{(1)} = T \neq H = \alpha^{(1)}$, so $1 \notin A_\alpha$. Analogously

$$\left.\begin{matrix}\mathbf{HT}HTHT \\ HTHT\mathbf{HT}\end{matrix}\right\} \Rightarrow 2 \in A_\alpha, \qquad \left.\begin{matrix}\mathbf{HTH}THT \\ HTH\mathbf{THT}\end{matrix}\right\} \Rightarrow 3 \notin A_\alpha,$$

$$\left.\begin{matrix}\mathbf{HTHT}HT \\ HT\mathbf{HTHT}\end{matrix}\right\} \Rightarrow 4 \in A_\alpha, \qquad \left.\begin{matrix}\mathbf{HTHTH}T \\ H\mathbf{THTHT}\end{matrix}\right\} \Rightarrow 5 \notin A_\alpha,$$

[2]Discovered by John Horton Conway; the proof of its correctness is shown in [4].

$$\left.\begin{array}{l}\mathbf{HTHTHT}\\ \mathbf{HTHTHT}\end{array}\right\} \Rightarrow 6 \in A_\alpha.$$

Therefore

$$A_\alpha = \{2, 4, 6\},$$

so

$$\alpha : \alpha = 2^2 + 2^4 + 2^6 = 84.$$

In the same way, we come to the following:

$$\alpha : \beta = 0, \qquad \beta : \beta = 66, \qquad \beta : \alpha = 42,$$

so

$$\frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha} = \frac{84 - 0}{66 - 42} = \frac{21}{6} > 1.$$

Therefore $HHTHTH \gg HTHTHT$, and this means that $g_{HTHTHT-HHTHTH}$ is not a fair one.

Let $\delta_{\alpha-\beta}$ be waiting for one of the series α or β. Let us assume that $|\alpha| > |\beta|$. Intuitionally we can presume that the series β, being shorter than the series α, is a better one.

Let us consider two series: $\alpha = HHTT...TT$ and $\beta = TT...TT$. The series are such that $|\alpha| = |\beta| + 1 = k + 1$, where $k \geq 2$. In this case

$$\alpha : \alpha = 2^{k+1}, \qquad \alpha : \beta = \sum_{j=1}^{k-1} 2^j,$$

$$\beta : \beta = \sum_{j=1}^{k} 2^j, \qquad \beta : \alpha = 0.$$

From the Conway equation we know that

$$\frac{P(A)}{P(B)} = \frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} = \frac{\sum_{j=1}^{k} 2^j}{2^{k+1} - \sum_{j=1}^{k-1} 2^j}.$$

Let us notice that $\sum_{j=1}^{n} 2^j$ is a sum of first n elements of the geometrical sequence which has the first element 2 and the quotient 2, so

$$\sum_{j=1}^{n} 2^j = 2\frac{1-2^n}{1-2} = 2^{n+1} - 2. \tag{1}$$

Therefore

$$\frac{\sum_{j=1}^{k} 2^j}{2^{k+1} - \sum_{j=1}^{k-1} 2^j} = \frac{2 \cdot 2^k - 2}{2 \cdot 2^k - 2^k + 2} = \frac{1 - \frac{1}{2^k}}{\frac{1}{2} + \frac{1}{2^k}} > \frac{1}{\frac{1}{2}},$$

and

$$\frac{P(A)}{P(B)} > 2,$$

so $\alpha \gg \beta$ even if the series α is longer than the β one.

If we narrow our consideration to pairs of series that differ by more than one element in length, we can easily see that the shorter series is a better one.

Theorem 2. *Let $\delta_{\alpha-\beta}$ be waiting for one of the α or β series of heads and tails which lengths fulfill the condition $|\alpha| \geq |\beta|+2$. Then the series β is better than the series α.*

Proof. Let α and β be series of heads and tails and $|\alpha| = k$, $|\beta| = l$. Let $m \geq 2$ be such a number that $k = l + m$. As the series cannot include each other, we have

$$\{k\} \subset A_\alpha \subset \{1,2,3,...,k\}, \qquad A_\beta \subset \{1,2,3,...,l-1\},$$
$$\{k\} \subset B_\beta \subset \{1,2,3,...,l\}, \qquad B_\alpha \subset \{1,2,3,...,l-1\},$$

which leads us to the following approximations:

$$2^k \leq \alpha : \alpha \leq \sum_{j=1}^{k} 2^j, \qquad 0 \leq \alpha : \beta \leq \sum_{j=1}^{l-1} 2^j$$

$$2^l \leq \beta : \beta \leq \sum_{j=1}^{l} 2^j, \qquad 0 \leq \alpha : \beta \leq \sum_{j=1}^{l-1} 2^j.$$

Then

$$\frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} \leq \frac{\sum_{j=1}^{l} 2^j - 0}{2^k - \sum_{j=1}^{l-1} 2^j}.$$

From (1) we get

$$\frac{\sum_{j=1}^{l} 2^j}{2^k - \sum_{j=1}^{l-1} 2^j} = \frac{2 \cdot 2^l - 2}{2^{m+l} - (2^l - 2)} < \frac{2 \cdot 2^l}{2^m \cdot 2^l - (2^l - 2)} = \frac{2}{2^m - (1 - \frac{2}{2^l})} < \frac{2}{4-1},$$

therefore

$$\frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} < 1.$$

Considering the remark 1, we get $\beta \gg \alpha$.

References

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