

# Impossibility of Perfect Recovering Cosinusoidal Signal of Any Phase Sampled with Nyquist Rate

A. Borys

*Gdynia Maritime University, Gdynia, Poland*

P. Korohoda

*AGH University of Science and Technology, Kraków, Poland*

**ABSTRACT:** In this paper, a problem of a perfect recovering cosinusoidal signal of any phase being sampled critically is considered. It is shown that there is no general solution to this problem. Its detailed analysis presented here shows that recovering both the original cosinusoidal signal amplitude and its phase is not possible at all. Only one of this quantities can be recovered under the assumption that the second one is known. And even then, performing some additional calculations is needed. As a byproduct, it is shown here that a transfer function of the recovering filter that must be used in the case of the critical sampling differs from the one which is used when a cosinusoidal signal is sampled with the use of a sampling frequency greater than the Nyquist rate. All the results achieved in this paper are soundly justified by thorough derivations.

## 1 INTRODUCTION

We are talking about a critical sampling of an analog signal when the sampling frequency used for performing this operation equals exactly twice the value of a maximal frequency occurring in the spectrum of this signal. Note that equivalently we can talk in this case about a critical sampling frequency (rate).

It is well known (Marks II R. J. 1991), (Korohoda P., Borgosz J. 1999), (Osgood B. 2014), (Borys A., Korohoda P. 2017) that signal sampling performed at the critical sampling rate can provide some unwanted problems when carrying out the inverse operation. That is in reconstructing (restoring) the analog signal from its samples.

In this paper, we illustrate the problems mentioned above on an example of a perfect recovering a cosinusoidal signal of any phase being sampled critically. We do this through performing very detailed analysis of that what really happens

when the cosinusoidal signal is sampled with the Nyquist rate and afterwards reconstructed from its samples.

We show in this paper that recovering both the original cosinusoidal signal amplitude and its phase is not possible. But, what is possible then? It is possible to recover one of these quantities under the assumption that the second one is known. However, carrying out some additional calculations is also then needed.

In this paper, we show also that transfer functions of the recovering filters used in the cases of critical and non-critical sampling are not identical. Their form is derived here.

The remainder of the paper is organized as follows. Section 2 introduces an example of a cosinusoidal signal of any phase, which is discussed throughout this paper. Also, in this section, a thorough analysis of the effects appearing during recovery of the cosinusoidal signal sampled critically

is presented. In Section 3, we consider the reconstruction formula and the form of a transfer function occurring in it for the case of occurrence of Dirac deltas in the signal spectrum together with its critical sampling. In the next section, complementary results are presented for the case of non-critical sampling. Finally, Section 5 concludes the paper.

## 2 A SIMPLE EXAMPLE ILLUSTRATING THE PROBLEM

Let us start consideration of the problem we are discussing in this paper with an example. And, to this end, let us take into account the sampling of a sinusoidal signal of the form

$$x(t) = \cos(2\pi f_m t + \varphi), \quad (1)$$

where  $f_m$  and  $\varphi$  are its frequency and phase, respectively. For simplicity, the amplitude of this signal is assumed here to be equal to one, and  $t$  in (1) means a continuous time.

So, sampling of (1) will be critical, when the sampling rate,  $f_s$ , being the inverse of the distance between samples,  $T$ , is equal to

$$f_s = 1/T = 2f_m. \quad (2)$$

Moreover, see that the Fourier transform of the sinusoidal signal given by (1) has the following form:

$$X(f) = \frac{1}{2} [\delta(f + f_m) + \delta(f - f_m)] \exp(j\varphi f / f_m), \quad (3)$$

where  $\delta(\cdot)$  means the so-called Dirac delta impulse (Dirac P. A. M. 1947), (Marks II R. J. 1991), (Osgood B. 2014), which is also called the Dirac delta function (improperly) or the Dirac distribution (properly) in the literature.

Next, using the sifting property of the Dirac delta impulse in (3), we get

$$X(f) = \frac{1}{2} [\delta(f + f_m) \exp(-j\varphi) + \delta(f - f_m) \exp(j\varphi)]. \quad (4)$$

Note further that by applying the Euler formula to  $\exp(-j\varphi)$  and  $\exp(j\varphi)$  in (4) we arrive at an equivalent form of the latter, i.e.

$$X(f) = \frac{1}{2} \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] - \frac{1}{2} j \sin(\varphi) [\delta(f + f_m) - \delta(f - f_m)]. \quad (5)$$

In the literature, the operation of signal sampling is modeled as a modulation of the so-called Dirac comb (Marks II R. J. 1991), (Osgood B. 2014), i.e.

$$\text{comb}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad (6)$$

by a given analog signal to be sampled. In other words, the above operation can be expressed as a multiplication of the Dirac comb by this signal. That is by

$$x_s(t) = x(t) \cdot \text{comb}_T(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT), \quad (7)$$

where  $x_s(t)$  means a continuous-time sampled version of the signal  $x(t)$ . So, we get from (7)

$$x_s(t) = \sum_{n=-\infty}^{\infty} \cos(2\pi f_m nT + \varphi) \delta(t - nT). \quad (8)$$

in the case of (1).

The equivalent of (7) in the frequency domain is given by

$$\begin{aligned} X_s(f) &= \mathcal{F}\{x_s(t)\} = \mathcal{F}\{x(t)\} \otimes \mathcal{F}\{\text{comb}_T(t)\} = \\ &= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s), \end{aligned} \quad (9)$$

where  $\mathcal{F}\{\cdot\}$  stands for performing the Fourier transformation of a signal occurring in braces of this symbol. Moreover, the symbols  $\otimes$  and  $f_s$  denote the convolution operation and sampling rate, respectively.

Looking at the result (9), we see that the Fourier transform of a sampled signal consists of an infinite sum of periodically shifted Fourier transforms of its un-sampled version multiplied by the sampling frequency  $f_s = 1/T$ . For details regarding derivation of (9), see, for example, (Marks II R. J. 1991), (Osgood B. 2014).

Further, applying (5) in (9) leads to

$$\begin{aligned} X_s(f) &= \frac{1}{2} f_s \sum_{n=-\infty}^{\infty} \{ \cos(\varphi) [\delta(f - nf_s + f_m) + \\ &+ \delta(f - nf_s - f_m)] - j \sin(\varphi) [\delta(f - nf_s + f_m) - \\ &- \delta(f - nf_s - f_m)] \}. \end{aligned} \quad (10)$$

In the next step, note that the general expression (10) can be highly simplified in the case of the critical sampling of the signal (1). That is with the use of  $f_s = 2f_m$ , as given by (2). Then, the following:

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \delta(f - 2nf_m + f_m) = \\
& = \sum_{n=-\infty}^{\infty} \delta(f - 2(n-1)f_m - 2f_m + f_m) = \\
& = \sum_{n'=-1=-\infty}^{\infty} \delta(f - 2n'f_m - 2f_m + f_m) = \\
& = \sum_{n' \rightarrow n=-\infty}^{\infty} \delta(f - 2nf_m - f_m)
\end{aligned} \tag{11}$$

holds. Further, application of (11) in (10) gives

$$\begin{aligned}
X_s(f) &= 2f_m \cos(\varphi) \sum_{n=-\infty}^{\infty} \delta(f - (2n-1)f_m) = \\
&= 2f_m \cos(\varphi) \sum_{n=-\infty}^{\infty} \delta(f - (2n+1)f_m) .
\end{aligned} \tag{12}$$

What does it mean the operation of recovering a signal in the frequency domain? See that it can be simply formulated as finding a filter possessing the transfer function, say  $H(f)$ , which fulfills the following:

$$H(f)X_s(f) = X(f) . \tag{13}$$

So, applying (9) in (13) gives

$$H(f) = \frac{X(f)}{X_s(f)} = \frac{X(f)}{f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)} . \tag{14}$$

Further, in the next step, one can check that the function

$$H(f) = \frac{1}{f_s} \text{rect}\left(\frac{f}{f_s}\right) , \tag{15}$$

with an auxiliary function  $\text{rect}(\cdot)$  defined by

$$\text{rect}(x) = 1 \text{ for } |x| \leq \frac{1}{2} \text{ and } 0 \text{ for } |x| > \frac{1}{2} , \tag{16}$$

is a solution in (14), when the function  $X(f)$  is well-defined (that is it is a function, not a distribution). Note also that such a filter as that one given by (15) is called an interpolation filter (Marks II R. J. 1991).

Let us now introduce (12) and (15) with  $f_s = 2f_m$  into (13). This leads to

$$\begin{aligned}
X_r(f) &= \frac{1}{2f_m} \text{rect}\left(\frac{f}{2f_m}\right) 2f_m \cos(\varphi) \cdot \\
& \cdot \sum_{n=-\infty}^{\infty} \delta(f - (2n+1)f_m) = \\
& = \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] .
\end{aligned} \tag{17}$$

Looking at (17), we see that the result obtained  $X_r(f)$  differs from the expected one, that is from  $X(f)$  given by (3) or equivalently by (4). And, for this reason, it is denoted differently, by  $X_r(f)$ . Further, note also that the inverse Fourier transform of  $X_r(f)$  is given by

$$x_r(t) = \mathcal{F}^{-1}\{X_r(f)\} = 2\cos(\varphi)\cos(2\pi f_m t) , \tag{18}$$

where  $\mathcal{F}^{-1}\{\cdot\}$  stands for performing the inverse Fourier transformation of a Fourier transform in braces of this symbol.

Comparison of (18) with (1) shows that the reconstructed signal  $x_r(t)$  evidently differs from the original analog one,  $x(t)$ . In what follows, we will look for the cause of this. So, to this end, let us first check whether the transfer function given by (15), which was derived from (13) and (14), is a correct one in our case.

### 3 TRANSFER FUNCTION IN RECONSTRUCTION FORMULA IN CASE OF OCCURRENCE OF DIRAC DELTAS IN SIGNAL SPECTRUM AND CRITICAL SAMPLING

Consider now in more detail (13), which is the reconstruction formula expressed in the frequency domain, when both the signals  $X_s(f)$  and  $X(f)$  in it contain Dirac deltas.

Further, we continue in this section analysis of the example introduced in the previous section about a critical sampling of an analog cosinusoidal signal. And, note now that because of a critical character of the latter operation it is highly advisable to check correctness of calculations at edges of the characteristics of  $H(f)$ . That is for the left-hand side edge occurring at  $f = -f_s/2 = -f_m$  and for its right-hand side counterpart at  $f = f_s/2 = f_m$ . For this purpose, we modify slightly  $H(f)$  given by (15) by introducing there a coefficient  $c$ , which we assume to be unknown at first instance. So, we rewrite then (15) as

$$H_c(f) = \frac{c}{f_s} \text{rect}\left(\frac{f}{f_s}\right) , \tag{19}$$

where  $c$  means a real number and  $H_c(f)$  stands for a modified  $H(f)$ .

Substituting  $X_s(f)$  given by (12),  $H_c(f)$  given by (19), and  $X(f)$  given by (5) into (13), with  $f_s = 2f_m$ , leads to

$$\begin{aligned}
& \frac{c}{2f_m} \text{rect}\left(\frac{f}{2f_m}\right) 2f_m \cos(\varphi) \cdot \\
& \cdot \sum_{n=-\infty}^{\infty} \delta(f - (2n+1)f_m) = \\
& = \frac{1}{2} \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] - \\
& - \frac{1}{2} j \sin(\varphi) [\delta(f + f_m) - \delta(f - f_m)] .
\end{aligned} \tag{20}$$

In the next step, note that performing the operations indicated in the expression on the right-hand side of (20) results in

$$\begin{aligned}
& \frac{c}{2f_m} \text{rect}\left(\frac{f}{2f_m}\right) 2f_m \cos(\varphi) \cdot \\
& \cdot \sum_{n=-\infty}^{\infty} \delta(f - (2n+1)f_m) = \\
& = c \cdot \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] .
\end{aligned} \tag{21}$$

Next, introducing (21) into (20), we get finally

$$\begin{aligned}
& c \cdot \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] = \\
& = \frac{1}{2} \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] - \\
& - \frac{1}{2} j \sin(\varphi) [\delta(f + f_m) - \delta(f - f_m)] .
\end{aligned} \tag{22}$$

Observe now that there is no real number  $c$  that fulfills equation (22), in general, for all phases  $\varphi$  of the cosinusoidal signal given by (1). However, for a particular value of  $\varphi = 0$  radians, equation (22) simplifies to

$$\begin{aligned}
& c \cdot [\delta(f + f_m) + \delta(f - f_m)] = \\
& = \frac{1}{2} [\delta(f + f_m) + \delta(f - f_m)] ,
\end{aligned} \tag{23}$$

and the resulting equation (23) is identically satisfied for  $c = 1/2$ .

Observe also that our previous results (17) and (18) will be slightly modified if we take into account the coefficient  $c = 1/2$  in (19) instead of  $c = 1$ ; the latter follows from (14) and (15). Then, we will have

$$X_{rc}(f) = \cos(\varphi) \frac{1}{2} [\delta(f + f_m) + \delta(f - f_m)] \tag{24}$$

and

$$\begin{aligned}
x_{rc}(t) &= \mathcal{F}^{-1}\{X_{rc}(f)\} = \cos(\varphi) \cos(2\pi f_m t) = \\
& = A_\varphi \cos(2\pi f_m t)
\end{aligned} \tag{25}$$

respectively, where  $X_{rc}(f)$  means a modified version of  $X_r(f)$  and  $x_{rc}(t)$  is the inverse Fourier transform of the former one. Moreover,  $A_\varphi$  means an amplitude of the cosinusoidal signal in (25).

Now, see finally that for  $\varphi = 0$  radians the original signal  $x(t)$  given by (1) and its recovered version  $x_{rc}(t)$  given by (25) are identical. That is in this particular case we have to do with a perfect reconstruction. And, this is a very strong argument for the use of the transfer function (19) with  $c = 1/2$  instead of the one given by (15) in our further considerations.

By the way, note that using another method it has been shown in (Borys A., Korohoda P. 2017) that reasonable results of analysis of the example discussed in this paper are achieved only when the value of the coefficient  $c = 1/2$  in (19).

Let us now come back to consideration of (22) for phases  $\varphi \neq 0$  radians. It can be easily shown that there is no such real or complex-valued coefficient  $c$  that satisfies equation (22). Here, we propose to overcome this problem in a way presented below. And, afterwards, we check whether results achieved are reasonable.

We start with ignoring the imaginary component in (22). That is we ignore the existence of

$$-\frac{1}{2} j \sin(\varphi) [\delta(f + f_m) - \delta(f - f_m)] \tag{26}$$

in (22). Or, equivalently, we ignore the fact that the imaginary components on both side of (22) do not compensate each other. That is we ignore that the following:

$$0 \neq -\frac{1}{2} j \sin(\varphi) [\delta(f + f_m) - \delta(f - f_m)] \tag{27}$$

holds. And, what remains then? It remains consideration of the real parts on both sides of (22) exclusively. That is consideration of the following

$$\begin{aligned}
& c \cdot \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] = \\
& = \frac{1}{2} \cos(\varphi) [\delta(f + f_m) + \delta(f - f_m)] ,
\end{aligned} \tag{28}$$

where it is assumed that the coefficient  $c$  is real-valued.

Observe now that for  $\cos(\varphi) \neq 0$  equation (28) simplifies to (23). Therefore, its solution  $c = 1/2$  is identical with the previous one as well as are the related solutions (24) and (25) describing the form of the reconstructed signal.

Let us now take a closer look at the recovered signal given by (25). Obviously, it is not identical with the original one given by (1). So, the signal recovery process in the case of  $\varphi \neq 0$  and  $\cos(\varphi) \neq 0$  at the same time, as described above, is not perfect. However, not all is lost here. Namely, proceeding heuristically as shown below allows us to arrive at the original form of the signal (1) when we have the result

(25). Simply, considering the amplitude of the cosinusoidal signal given by (25), we can write

$$\varphi = \arccos(A_\varphi) . \quad (29)$$

With this in mind, we can modify the previous recovery algorithm for the case of  $\varphi \neq 0$  and  $\cos(\varphi) \neq 0$  at the same time - as follows.

- 1 Perform all the operations to get  $x_{rc}(t)$  as given by (25). Observe that frequency of the original signal is correctly recognized in  $x_{rc}(t)$ . However, signal amplitude there is equal to  $\cos(\varphi)$  instead of 1.
- 2 Calculate the phase  $\varphi$  from (29).
- 3 In (25), replace the amplitude  $\cos(\varphi)$  with 1 and add also there the calculated (in point II above) value of  $\varphi$  to the cosine function argument  $2\pi f_m t$ . It results finally in getting (1).

The algorithm sketched in points I, II, and III above can be viewed as an improved signal recovery algorithm for the case of recovery of cosinusoidal signals of any phase. As seen, its application leads then to a perfect signal reconstruction.

Note however that all this presented above works correctly only under the assumption of knowing the amplitude of a cosinusoidal signal before sampling. And, in this context, we recall that we assumed for simplicity in our example the signal given by (1) that amplitude is 1. Obviously, without this knowledge, the quantity  $A_\varphi$  in (25) cannot be fully attributed to  $\cos(\varphi)$ . Then, the recovered quantity  $A_\varphi$  is a product of an unknown amplitude of the un-sampled cosinusoidal signal and of a value of the cosine function of its phase. That is it is a product of two unknowns. Hence, in this case, nothing can be said about the amplitude and phase of the original cosinusoidal signal.

Furthermore, it follows from the detailed analysis performed in this section that there is no such a "recovery" transfer function  $H(f)$  in sense of equation (13), which allows a perfect recovery of a cosinusoidal signal of any phase sampled with Nyquist rate. As shown, this fact follows from the impossibility to satisfy equation (22).

#### 4 TRANSFER FUNCTION IN RECONSTRUCTION FORMULA IN CASE OF OCCURRENCE OF DIRAC DELTAS IN SIGNAL SPECTRUM AND NON-CRITICAL SAMPLING

It is worthy to complete our derivations presented in the previous section by considering also the case of a non-critical sampling of the cosinusoidal signal given by (1). That is we will sample now the signal given by (1) with a sampling frequency fulfilling the following inequality:

$$f_s = 1/T > 2f_m . \quad (30)$$

So, using the general formula (9) and (4), we arrive at

$$\begin{aligned} X_s(f) &= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) = \\ &= \frac{f_s}{2} \sum_{n=-\infty}^{\infty} \left[ \delta(f - nf_s + f_m) \exp(-j\varphi) + \right. \\ &\quad \left. + \delta(f - nf_s - f_m) \exp(j\varphi) \right] \end{aligned} \quad (31)$$

with, do not forget now,  $f_s > 2f_m$ .

Next, let us come to the operation of recovery of  $X(f)$  given by (4) from  $X_s(f)$  expressed by (31). To this end, we will take into account, as before, the reconstruction formula in the frequency domain presented in (13). Further, observe that a good candidate to play a role of  $H(f)$  in (13) is  $H_c(f)$  given by (19) because it filters out all the spectral components outside the range of frequencies  $f : |f| \leq f_s$ , as wanted. So, applying  $H_c(f)$  to (31) gives

$$\begin{aligned} H_c(f) X_s(f) &= \frac{c}{2} \left[ \delta(f + f_m) \exp(-j\varphi) + \right. \\ &\quad \left. + \delta(f - f_m) \exp(j\varphi) \right] . \end{aligned} \quad (32)$$

And, substituting (32) into (13) with  $X(f)$  given by (3) results in

$$\begin{aligned} \frac{c}{2} \left[ \delta(f + f_m) \exp(-j\varphi) + \delta(f - f_m) \exp(j\varphi) \right] &= \\ = \frac{1}{2} \left[ \delta(f + f_m) \exp(-j\varphi) + \delta(f - f_m) \exp(j\varphi) \right] . \end{aligned} \quad (33)$$

In the next step, grouping Dirac deltas with the same arguments in (33), we get

$$\begin{aligned} (c-1) \exp(-j\varphi) \delta(f + f_m) + \\ + (c-1) \exp(j\varphi) \delta(f - f_m) = 0 . \end{aligned} \quad (34)$$

Observe now that (34) will be satisfied if and only if the coefficients multiplying the Dirac deltas in (34) are equal zero. This follows from the theory of distributions (Hoskins R. F. 2009). That is from the fact that  $a \cdot \delta(\cdot) = 0$  if and only if  $a = 0$ . So, applying this in (34) leads to the following conclusion:  $c$  must be equal to 1 in the case considered in this section. Further, substituting  $c = 1$  in (19) results in

$$H_{c=1}(f) = \frac{1}{f_s} \text{rect}\left(\frac{f}{f_s}\right) = H(f) . \quad (35)$$

So, we can conclude finally that in the case of occurrence of Dirac deltas in signal spectrum and non-critical sampling the transfer function of a reconstruction filter is equal to  $H(f)$  given by (15).

At the end of this section, it is also worthy to draw attention to the fact that some algebraic operations on distributions are forbidden (Hoskins R. F. 2009). For example, the following notation:

$$\begin{aligned}
H(f) &= \frac{X(f)}{X_s(f)} = \\
&= \frac{\frac{1}{2}[\delta(f+f_m)\exp(-j\varphi) + \delta(f-f_m)\exp(j\varphi)]}{\frac{f_s}{2} \sum_{n=-\infty}^{\infty} [\delta(f-nf_s+f_m)\exp(-j\varphi) + \delta(f-nf_s-f_m)\exp(j\varphi)]} = \\
&= \frac{1}{f_s} \frac{\delta(f+f_m)\exp(-j\varphi) + \delta(f-f_m)\exp(j\varphi)}{\delta(f+f_m)\exp(-j\varphi) + \delta(f-f_m)\exp(j\varphi)} = \frac{1}{f_s} \cdot 1 = \frac{1}{f_s}
\end{aligned}
\tag{36}$$

is highly incorrect, mainly because of the occurrence of divisions of Dirac deltas. They are forbidden (or undefined) in the theory of distributions.

## 5 CONCLUSIONS

The reasons of impossibility to recover both the original cosinusoidal signal amplitude and its phase from samples of this signal sampled critically have been recognized in this paper. They follow from the very detailed analyses presented.

Furthermore, it has been shown that only when one of the aforementioned quantities is known in the process of signal reconstruction, the value of the second one can be recovered.

Finally, it has been also shown that a transfer function of the reconstruction filter that must be used in the case of a critical sampling differs from the one which is used when a cosinusoidal signal is not sampled critically.

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