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Examples of series-“m out of k” systems and their limit reliability functions

Keywords

reliability, large system, asymptotic approach, limit reliability function

Abstract

The paper is concerned with mathematical methods in asymptotic approach to systems reliability analysis. The complexity of the reliability investigation of large-scale systems is proposed to be approximately solved by assuming that the number of system components tends to infinity and finding the limit reliability function of the system. Some general results in the form of auxiliary theorems and examples of limit reliability functions of homogeneous and regular series-“*m* out of *k*” systems with exponential and Weibull reliability functions of system components are presented.

1. Introduction - Asymptotic approach to system reliability

In the reliability investigation of large-scale systems, the problem of the complexity of their reliability functions arises. This problem may be approximately solved by assuming that the number of system components tends to infinity and finding the limit reliability function of the system. This approach is well recognized for basic systems. Gnedenko [1] has solved it for series and parallel systems, whereas Smirnov [7] for “*k* out of *n*” systems. They both have found the classes of possible limit reliability functions of these systems. All current results on asymptotic approach to reliability of large systems with typical structures are partly given in [2] and [4]-[6] and completely presented in the monograph [3].

In the paper these two areas considered by Gnedenko and Smirnov are brought together and some new results of investigations are showed.

We assume that the lifetime distributions do not necessarily have to be concentrated on the interval $<0,\infty)$. Then, a reliability function does not have to satisfy the usually demanded condition

$$R(t)=1 \text{ for } t\in(-\infty,0).$$

It is a generalization of the usually used concept of a reliability function. This generalization is convenient in the theoretical considerations. At the same time, the achieved results for the generalized reliability functions, also hold for the usually used reliability function. From these agreement it follows that between a reliability function $R(t)$ and a distribution function $F(t)$ there exists an explicit correspondence given by

$$R(t)=1-F(t),\ t\in(-\infty,\infty).$$

According to the properties of a distribution function, a reliability function $R(t)$ is non-increasing, right-continuous, $R(-\infty)=1$ and $R(+\infty)=0$.

We will deal with a reliability functions of the forms:

$$\mathfrak{R}^{(m)}(t)=1-\sum_{i=0}^{m-1}\exp[-V(t)]\frac{[V(t)]^i}{i!}, \tag{1}$$

$$\tilde{\mathfrak{R}}^{(\lambda)}(t)=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-L(t)}\exp[-\frac{x^2}{2}]dx, \tag{2}$$

and

$$\overline{\Re}^{(\overline{m})}(t) = \sum_{i=0}^{\overline{m}} \exp[-\overline{V}(t)] \frac{[\overline{V}(t)]^i}{i!}, \qquad (3)$$

where $m \in N$, $\lambda \in (0,1)$ and $\overline{m} \in N$.

The Asymptotic approach to systems reliability is based on investigating limit distributions of a standardized random variable

$$(T - b_n)/a_n,$$

where T is the lifetime of the system and $a_n > 0$, $b_n \in (-\infty, \infty)$ are some suitably chosen numbers. And since

$$P(\frac{T - b_n}{a_n} > t) = P(T > a_n t + b_n) = \boldsymbol{R}_n(a_n t + b_n),$$

where $\boldsymbol{R}n(t)$ is a reliability function of the system, then we assume the following definition.

Definition 1. A reliability function $\Re$(t) is called the limit reliability function of the system if there exist normalising constants $a_n > 0$, $b_n \in (-\infty, \infty)$ such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \Re(t) \quad \text{for} \quad t \in C_{\Re},$$

where $C_{\Re}$ is the set of continuity points of $\Re$.

Hence, for sufficiently large *n*, we get the following approximate formula approximate formula

$$\boldsymbol{R}_n(t) \cong \Re(t - b_n / a_n), \; t \in (-\infty, \infty).$$

2. Reliability of series-“m_n out of k_n” systems

Suppose that

$$E_{ij}, \; i = 1,2,...,k_n, \; j = 1,2,...,l_n, \; k_n, \; l_n \in N$$

are components of a system having reliability functions

$$R_{ij}(t) = P(T_{ij} > t), \; t \in (-\infty, \infty),$$

where T_{ij} are independent random variables representing the lifetimes of E_{ij}, having distribution functions

$$F_{ij}(t) = P(T_{ij} \le t), \; t \in (-\infty, \infty).$$

Definition 2. A system is called regular series-“m_n out of k_n” if its lifetime is given by

$$T = T_{(k_n - m_n + 1)}, \; 0 < m_n \le k_n,$$

where $T_{(k_n - m_n + 1)}$ is the m_n-th maximal order statistics in a sample of random variables

$$T_i = \min_{1 \le j \le l_i} \{T_{ij}\}, \quad i = 1,2,...,k_n,$$

representing the lifetimes of series subsystems of the system.

The above definition means that a series-“m_n out of k_n” system is not failed if and only if at least m_n of its k_n series subsystems are not failed.

Definition 3. A regular series-“m_n out of k_n” system is called homogeneous if component lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t), \; t \in (-\infty, \infty),$$

where $i = 1,2,...,k_n$, $j = ,2,...,l_n$,

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t), \; t \in (-\infty, \infty).$$

The reliability function of the homogeneous regular series-“ m_n out of k_n” system is given by

$$\boldsymbol{R}_{k_n, l_n}^{(m_n)}(t) = 1 - \sum_{i=0}^{m_n - 1} \binom{k_n}{i} [R^{l_n}(t)]^i [1 - R^{l_n}(t)]^{k_n - i}, \qquad (4)$$

or by formula

$$\overline{\boldsymbol{R}}_{k_n, l_n}^{(\overline{m}_n)}(t) = \sum_{i=0}^{\overline{m}_n} \binom{k_n}{i} [1 - R^{l_n}(t)]^i [R^{l_n}(t)]^{k_n - i}, \qquad (5)$$

where $t \in (-\infty, \infty)$, $\overline{m}_n = k_n - m_n$, k_n is the number of series subsystems of a system and l_n is the number of components in series subsystems.

3. Examples of series-“m_n out of k_n” systems and their limit reliability functions

It is important to notice, that the form of limit reliability function of homogeneous regular series–“m_n out of k_n” system depends not only on the reliability function of system components, but also on relation between m_n and the number k_n of series subsystems of the system and moreover between k_n and the number l_n of components in series subsystems of our system. The paper presents some spectacular solutions for the problem of possible limit reliability functions for homogeneous and regular series–“m_n out of k_n” systems with

Weibull or exponential reliability function of the system components. The proofs of presented lemmas can be found in [2] and [5].

Agreement 1. We assume the following notation for any positive functions $x(n)$ and $y(n)$:

$$y(n) = o(x(n)) \text{ means that } \lim_{n\to\infty} \frac{y(n)}{x(n)} = 0.$$

If $x(n) = o(1)$ and numbers a, α are such that $a \neq 0, \alpha \neq 0$ and $\alpha \neq 1$ then we may use following equations:

$$e^{\pm x(n)} = 1 \pm x(n) + o(x(n)), \tag{6}$$

$$(1 + x(n))^{\alpha} = 1 + \alpha x(n) + o(x(n)), \tag{7}$$

$$(a + x(n))^{\alpha} = a^{\alpha} + a^{\alpha-1}\alpha x(n) + o(x(n)) \tag{8}$$

Lemma1. If

(i) $\boldsymbol{R}^{(m_n)}_{k_n,l_n}(t)$ is a reliability function of the regular homogeneous series–„m_n out of k_n" system given by (4),

(ii) $\Re^{(m)}(t)$ a non-degenerate reliability function given by (1),

(iii) $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty} m_n = m = \text{const}$,

$$\left(\frac{m_n}{k_n} \to 0 \text{ przy } k_n \to \infty\right),$$

(iv) $a_n > 0$, $b_n \in (-\infty, \infty)$ are some functions.

Then the assertion

$$\lim_{n\to\infty} \boldsymbol{R}^{(m_n)}_{k_n,l_n}(a_n t + b_n) = \Re^{(m)}(t), t \in C_{\Re^{(m)}},$$

is equivalent to the assertion

$$\lim_{n\to\infty} k_n R^{l_n}(a_n t + b_n) = V(t), t \in C_V.$$

Proof. The proof may be found in [2] and [5]. □

Proposition 1. If components of the regular homogeneous regular series-"m_n out of k_n" system have exponential reliability functions

$$R(t) = \begin{cases} 1 & \text{for } t \le 0 \\ (t+1)^{-c} & \text{for } t > 0, c > 0, \end{cases}$$

and moreover if

(i) $\lim_{n\to\infty} m_n = \text{const}$, $\lim_{n\to\infty} k_n = \infty$, $l_n = c$,

(ii) $a_n = k_n^{\frac{1}{c^2}}$, $b_n = -1$,

then limit reliability functions of the system is

$$\Re_1^{(m)}(t) = \begin{cases} 1 & \text{for } t < 0, \\ 1 - \sum_{i=0}^{m-1} \exp(-t^{-c^2}) \dfrac{t^{-ic^2}}{i!} & \text{for } t \ge 0. \end{cases}$$

Justification: According to *Lemma 1* it is enough to show that

$$\lim_{n\to\infty} k_n R^{l_n}(a_n t + b_n) = V_1(t), \ t \in C_{V_1}, \tag{9}$$

where

$$V_1(t) = \begin{cases} \infty, & t < 0 \\ t^{-c^2}, & t > 0, \alpha > 0. \end{cases}$$

Since from (i) and (ii)

$$a_n t + b_n = k_n^{\frac{1}{c^2}} t - 1,$$

we get for n large enough

$$a_n t + b_n < 0 \text{ for } t \le 0$$

and

$$a_n t + b_n > 0 \text{ for } t > 0.$$

Therefore

$$R(a_n t + b_n) = \begin{cases} 1 & \text{for } t \le 0, \\ (a_n t + b_n + 1)^{-c} & \text{for } t > 0, c > 0. \end{cases}$$

According (i) and (ii) we get

$$k_n R^{l_n}(a_n t + b_n) = \begin{cases} k_n & \text{for } t \le 0, \\ t^{-c^2} & \text{for } t > 0, c > 0, \end{cases}$$

what according (i) means, that (9) holds. □

Proposition 2. If components of the regular homogeneous regular series-"m_n out of k_n" system have Weibull reliability functions

$$R(t) = \begin{cases} 1, & \text{for } t < 0, \\ \exp[-\beta t^{\alpha}], & \text{for } t \ge 0, \alpha > 0, \beta > 0, \end{cases}$$

and moreover if

(i) $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty} m_n = m = \text{const}$,

(ii) $a_n = \dfrac{b_n}{\alpha \log k_n}$, $b_n = \left(\dfrac{\log k_n}{\beta l_n}\right)^{\frac{1}{\alpha}}$,

then limit reliability functions of the system is

$$\mathfrak{R}_3^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \exp[-e^{-t}] \frac{e^{-it}}{i!} \text{ for } t \in (-\infty, \infty).$$

Justification: By *Lemma 1* it is sufficient to show that

$$\lim_{n\to\infty} k_n R^{l_n}(a_n t + b_n) = e^{-t} \quad \text{for} \quad t \in (-\infty, \infty).$$

According to (i) and (ii), for n large enough, we get

$$a_n t + b_n = b_n \left(\frac{t}{\alpha \log k_n} + 1\right) > 0 \quad \text{for } t \in (-\infty, \infty)$$

Hence for any $t \in (-\infty, \infty)$

$$R(a_n t + b_n) =$$

$$= \exp[-\beta (a_n t + b_n)^{\alpha}] =$$

$$= \exp[-\beta \left(b_n \left(\frac{t}{\alpha \log k_n} + 1\right)\right)^{\alpha}] =$$

$$= \exp[-\frac{\log k_n}{l_n} \left(\frac{t}{\alpha \log k_n} + 1\right)^{\alpha}] .$$

Using (7) for $\left(\dfrac{t}{\alpha \log k_n} + 1\right)^{\alpha}$ we get for $t \in (-\infty, \infty)$

$$R(a_n t + b_n) = \exp[-\frac{\log k_n}{l_n} - \frac{t}{l_n} - o(\frac{1}{l_n})].$$

Thus, for $t \in (-\infty, \infty)$

$$\lim_{n\to\infty} k_n R^{l_n}(a_n t + b_n) =$$

$$= \lim_{n\to\infty} k_n \exp[-\log k_n - t - o(1)] =$$

$$= \lim_{n\to\infty} \exp[-t - o(1)] = e^{-t}. \qquad \square$$

Directly, from the above *Proposition*, it follows next result.

Proposition 3. If components of the homogeneous regular series-"m_n out of k_n" have exponential reliability functios:

$$R(t) = \begin{cases} 1 & \text{for} \quad t < 0, \\ \exp[-\beta t] & \text{for} \quad t \ge 0,\ \beta > 0, \end{cases}$$

and moreover if

(i) $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty} m_n = m = \text{const}$,

(ii) $a_n = \dfrac{1}{\beta l_n}$, $b_n = \dfrac{1}{\beta l_n} \log k_n$,

then limit reliability functions of the system is

$$\mathfrak{R}_3^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \exp[-\exp(-t)] \frac{e^{-it}}{i!}$$

for $t \in (-\infty, \infty)$.

Lemma 2. If

(i) $\boldsymbol{R}_{k_n, l_n}^{(m_n)}(t)$ is given by (4) a reliability function of the homogeneous regular series–„m_n out of k_n" system,

(ii) $\mathfrak{R}(t)$ is non-degenerate reliability function,

(iii) $\lim_{n\to\infty} k_n = k,\ k > 0,\ \ \lim_{n\to\infty} m_n = m,\ \ 0 < m \le k,$

$\lim_{n\to\infty} l_n = \infty$,

(iv) $a_n > 0,\ \ b_n \in (-\infty, \infty)$ are some sequences of constants,

then the assertion

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n, l_n}^{(m_n)}(a_n t + b_n) = \mathfrak{R}(t), \quad t \in C_{\mathfrak{R}} \ ,$$

is equivalent to the assertion

$$\lim_{n\to\infty} R^{l_n}(a_n t + b_n) = \mathfrak{R}_0(t), \quad t \in C_{\mathfrak{R}_0},$$

where $\mathfrak{R}_0$ in non-degenerate reliability function. Moreover for $t \in (-\infty, \infty)$.

$$\mathfrak{R}(t) = 1 - \sum_{i=0}^{m-1} \binom{k}{i} [\mathfrak{R}_0(t)]^i [1 - \mathfrak{R}_0(t)]^{k-i}.$$

Proof: The proof may be found in [5].

Proposition 4. If components of the regular homogeneous regular series-"m_n out of k_n" system have Weibull reliability functions

$$R(t) = \begin{cases} 1, & t < 0, \\ \exp[-\beta t^{\alpha}], & t \ge 0, \alpha > 0, \beta > 0, \end{cases}$$

moreover if $\lim_{n\to\infty} m_n = m$ and pairs (k_n, l_n) and (a_n, b_n) fulfill conditions:

(i) $\lim_{n\to\infty} k_n = k,\ k > 0,\ \lim_{n\to\infty} l_n = \infty_n$,

(ii) $a_n = (\beta l_n)^{\frac{-1}{\alpha}},\ b_n = 0$,

then limit reliability functions of the system is

$$\mathfrak{R}_9^{(m)}(t) = 1 \text{ for } t < 0$$

and

$$\mathfrak{R}_9^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \binom{k}{i} [\exp(-t^{\alpha})]^i [1 - \exp(-t^{\alpha})]^{k-i}$$

for $t \ge 0, \alpha > 0$.

Justification: According to *Lemma 2* it is sufficient to show that

$$\lim_{n\to\infty} R^{l_n}(a_n t + b_n) = \overline{\mathfrak{R}}_2(t),\ t \in C_{\overline{\mathfrak{R}}_2}, \tag{10}$$

where

$$\overline{\mathfrak{R}}_2(t) = 1 \text{ for } t < 0$$

and

$$\overline{\mathfrak{R}}_2(t) = \exp[-t^{\alpha}] \text{ for } t \ge 0, \alpha > 0.$$

According to (ii) we have for $t \ge 0$,

$$a_n t + b_n < 0 \text{ for } t < 0 \text{ and } a_n t + b_n \ge 0$$

and next

$$R(a_n t + b_n) = \begin{cases} 1 & \text{for } t < 0, \\ \exp[-t^{\alpha} l_n^{-1}] & \text{for } t \ge 0, \alpha > 0. \end{cases}$$

Thus, we have

$$R^{l_n}(a_n t + b_n) = 1 \text{ for } t < 0$$

and for $t \ge 0$ we get

$$R^{l_n}(a_n t + b_n) = \left(\exp[-t^{\alpha} l_n^{-1}]\right)^{l_n} = \exp[-t^{\alpha}].$$

From the above we conclude that (10) holds. □

From *Proposition 4* the next result follows immediately.

Proposition 5. If components of the regular homogeneous regular series-"m_n out of k_n" have an exponential reliability functions

$$R(t) = \begin{cases} 1 & \text{for } t < 0, \\ \exp[-\beta t] & \text{for } t \ge 0, \beta > 0, \end{cases}$$

and moreover

$$\lim_{n\to\infty} m_n = m$$

and the pairs (k_n, l_n) and (a_n, b_n) fulfill the conditions:

(i) $\lim_{n\to\infty} k_n = k,\ k > 0,\ \lim_{n\to\infty} l_n = \infty$,

(ii) $a_n = \dfrac{1}{\beta l_n},\ b_n = 0$,

then limit reliability functions of the system is

$$\mathfrak{R}_9^{(m)}(t) =$$

$$\begin{cases} 1 & \text{for } t < 0, \\ 1 - \sum_{i=0}^{m-1} \binom{k}{i} [\exp(-t)]^i [1 - \exp(-t)]^{k-i} & \text{for } t \ge 0. \end{cases}$$

Lemma 3. If

(i) $\overline{\boldsymbol{R}}_{k_n, l_n}^{(\overline{m}_n)}(t)$ is a reliability function of the homogeneous regular series–„m_n out of k_n" system given by (5),

(ii) $\overline{\mathfrak{R}}^{(\overline{m})}(t)$ is a non-degenerate reliability function given by (3),

(iii) $n \to \infty,\ k_n \to \infty,\ \dfrac{m_n}{k_n} \to 1$,

$$\overline{m}_n = k_n - m_n,\ \overline{m}_n \to \overline{m} = \text{const},$$

(iv) $a_n > 0,\quad b_n \in (-\infty, \infty)$ are some functions,

then the assertion

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_{k_n, l_n}^{(\overline{m}_n)}(a_n t + b_n) = \overline{\mathfrak{R}}^{(\overline{m})}(t),\ t \in C_{\overline{\mathfrak{R}}^{(\overline{m})}},$$

is equivalent to the assertion

$$\lim_{n\to\infty} k_n l_n F(a_n t + b_n) = \overline{V}(t),\ t \in C_{\overline{V}}.$$

Proof. The proof may be found in [5]. □

Proposition 6. If components of the regular homogeneous series-"m_n out of k_n" system have Weibull reliability functions

$$R(t)=\begin{cases}1 & \text{for } t<0\\ \exp[-\beta t^{\alpha}] & \text{for } t\ge 0,\ \alpha>0,\ \beta>0,\end{cases}$$

and moreover if

(i) $\lim_{n\to\infty} k_n=\infty$, $\lim_{n\to\infty}(k_n-m_n)=\overline{m}=\text{const}$

(ii) $a_n=(\beta l_n k_n)^{\frac{-1}{\alpha}}$, $b_n=0$,

then the limit reliability functions of the system is

$$\overline{\mathfrak{R}}_2^{(\overline{m})}(t)=\begin{cases}1 & \text{for } t<0,\\ \sum_{i=0}^{\overline{m}}\exp[-t^{\alpha}]\dfrac{(t^{\alpha})^i}{i!} & \text{for } t\ge 0,\alpha>0.\end{cases}$$

Justification: According to *Lemma 3* it is enough to show that

$$\lim_{n\to\infty} k_n l_n F(a_n t+b_n)=\overline{V}_2(t),\ t\in C_{\overline{V}_2},$$

where

$$\overline{V}_2(t)=\begin{cases}0 & \text{for } t<0,\\ t^{\alpha} & \text{for } t>0,\alpha>0.\end{cases}$$

According to (ii)

$$a_n t+b_n=(\beta l_n k_n)^{\frac{-1}{\alpha}}t.$$

Therefore

$$a_n t+b_n<0 \text{ for } t<0$$

and

$$a_n t+b_n\ge 0 \text{ for } t\ge 0\ .$$

Because for $t<0$

$$F(a_n t+b_n)=1-R(a_n t+b_n)=0,$$

thus, for $t<0$, we get

$$\lim_{n\to\infty} k_n l_n F(a_n t+b_n)=0.$$

However for any $t\ge 0$

$$F(a_n t+b_n)=1-R(a_n t+b_n)=$$

$$=1-\exp[-\beta(a_n t)^{\alpha}]$$

$$=1-\exp[-\beta((\beta l_n k_n)^{-\alpha^{-1}}t)^{\alpha}]=$$

$$=1-\exp[-(l_n k_n)^{-1}t^{\alpha}].$$

From the above for any $t\ge 0$, we get

$$\lim_{n\to\infty} k_n l_n F(a_n t+b_n)=\lim_{n\to\infty} k_n l_n\left(1-\exp[-\frac{t^{\alpha}}{k_n l_n}]\right).$$

Using (i) and (6), for $t\ge 0$, the above equation we can write as

$$\lim_{n\to\infty} k_n l_n F(a_n t+b_n)=$$

$$=\lim_{n\to\infty} k_n l_n\left(1-1+\frac{t^{\alpha}}{k_n l_n}-o(\frac{t^{\alpha}}{k_n l_n})\right)=t^{\alpha}.$$ □

From the above results the next proposition follows obviously.

Proposition 7. If components of the regular homogeneous regular series–"m_n out of k_n" have exponential reliability functions

$$R(t)=\begin{cases}1 & \text{for } t<0,\\ \exp[-\beta t] & \text{for } t\ge 0,\ \beta>0,\end{cases}$$

and moreover if

(i) $\lim_{n\to\infty} k_n=\infty$, $\lim_{n\to\infty}(k_n-m_n)=\overline{m}=\text{const}$,

(ii) $a_n=\dfrac{1}{\beta l_n k_n}$, $b_n=0$,

then limit reliability functions of the system is

$$\overline{\mathfrak{R}}_2^{(\overline{m})}(t)=\begin{cases}1 & \text{for } t<0,\\ \sum_{i=0}^{\overline{m}}\exp[-t]\dfrac{t^i}{i!} & \text{for } t\ge 0.\end{cases}$$

Lemma 4. If

(i) $\boldsymbol{R}_{k_n,l_n}^{(m_n)}(t)$, $t\in(-\infty,\infty)$, is a reliability function of the homogeneous regular series–„m_n out of k_n" given by (4),

(ii) $\tilde{\mathfrak{R}}^{(\lambda)}(t)$, is a non-degenerate reliability function given by (2),

(iii) $n\to\infty$, $k_n\to\infty$,

$$\frac{m_n}{k_n}=\lambda+o(\frac{1}{\sqrt{k_n}}),\ 0<\lambda<1,$$

(iv) $a_n > 0, b_n \in (-\infty, \infty)$ are some functions,

then the assertion

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n,l_n}^{(m_n)}(a_n t + b_n) = \tilde{\Re}^{(\lambda)}(t) \text{ for } t \in C_{\tilde{\Re}^{(\lambda)}},$$

is equivalent to the assertion

$$\lim_{n\to\infty} \frac{\sqrt{k_n+1}[R^{l_n}(a_n t + b_n) - \lambda]}{\sqrt{\lambda(1-\lambda)}} = L(t)$$

for $t \in C_L$.

Proof. The proof may be found in [5]. □

Proposition 8. If components of the regular homogeneous regular series−"m_n out of k_n" system have Weibull reliability functions:

$$R(t) = \begin{cases} 1, & t < 0, \\ \exp[-\beta t^{\alpha}], & t \ge 0, \alpha > 0, \beta > 0, \end{cases}$$

and moreover

(i) $\lim_{n\to\infty} k_n = \infty$, $m_n / k_n = \lambda + o(\frac{1}{\sqrt{k_n}})$,

$0 < \lambda < 1$,

(ii) $a_n = \dfrac{\sqrt{1-\lambda}}{\alpha(\beta l_n)^{\frac{1}{\alpha}} \sqrt{\lambda k_n} (-\log\lambda)^{\frac{\alpha-1}{\alpha}}}$,

$b_n = (\dfrac{-\log\lambda}{\beta l_n})^{\frac{1}{\alpha}}$,

then limit reliability functions of the system is

$$\tilde{\Re}_7^{(\lambda)}(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx \quad \text{for } t \in (-\infty, \infty).$$

Justification: According to *Lemma 4* it is enough to show that for $t \in (-\infty, \infty)$

$$\lim_{n\to\infty} \frac{\sqrt{k_n+1}[R^{l_n}(a_n t + b_n) - \lambda]}{\sqrt{\lambda(1-\lambda)}} = -t.$$

From (i) and (ii), because for n large enough and for $t \in (-\infty, \infty)$, we have

$$a_n t + b_n =$$

$$= \frac{1}{(\beta l_n)^{\frac{1}{\alpha}}} (\frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t + (-\log\lambda)^{\frac{1}{\alpha}}) > 0$$

it follows that

$$R(a_n t + b_n) =$$

$$\exp[-\beta(\frac{1}{(\beta l_n)^{\frac{1}{\alpha}}} (\frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t + (-\log\lambda)^{\frac{1}{\alpha}}))^{\alpha}]$$

and next for large n and $t \in (-\infty, \infty)$ we get

$$\lim_{n\to\infty} \frac{\sqrt{k_n+1}[R^{l_n}(a_n t + b_n) - \lambda]}{\sqrt{\lambda(1-\lambda)}} = \lim_{n\to\infty} \tag{11}$$

$$\frac{\sqrt{k_n+1}[\exp[-(\frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t + (-\log\lambda)^{\frac{1}{\alpha}})^{\alpha}] - \lambda]}{\sqrt{\lambda(1-\lambda)}}$$

Using (8) we get

$$(\frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t + (-\log\lambda)^{\frac{1}{\alpha}})^{\alpha} =$$

$$= ((-\log\lambda)^{\frac{1}{\alpha}})^{\alpha} +$$

$$+ \alpha(-\log\lambda)^{\frac{\alpha-1}{\alpha}} \frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t +$$

$$+ o(\frac{\sqrt{1-\lambda}}{\alpha\sqrt{\lambda k_n}(-\log\lambda)^{\frac{\alpha-1}{\alpha}}} t) =$$

$$= -\log\lambda + \frac{\sqrt{1-\lambda}}{\sqrt{\lambda k_n}} t + o(\frac{1}{\sqrt{k_n}}).$$

Continuing transformations (11) we get

$$\frac{\sqrt{k_n+1}[\exp[-(-\log\lambda + \frac{\sqrt{1-\lambda}}{\sqrt{\lambda k_n}} t + o(\frac{1}{\sqrt{k_n}}))] - \lambda]}{\sqrt{\lambda(1-\lambda)}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{k_n+1}[R^{l_n}(a_n t + b_n) - \lambda]}{\sqrt{\lambda(1-\lambda)}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{k_n+1}\left[\lambda(\exp[-\frac{\sqrt{1-\lambda}}{\sqrt{\lambda k_n}} t + o(\frac{1}{\sqrt{k_n}})] - 1)\right]}{\sqrt{\lambda(1-\lambda)}}.$$

From the above, using (6), we get for $t \in (-\infty, \infty)$

$$\lim_{n\to\infty}\frac{\sqrt{k_n+1}[R^{l_n}(a_n t+b_n)-\lambda]}{\sqrt{\lambda(1-\lambda)}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{k_n+1}\left[\lambda\left(1-\frac{\sqrt{1-\lambda}}{\sqrt{\lambda k_n}}t+o\left(\frac{1}{\sqrt{k_n}}\right)-1\right)\right]}{\sqrt{\lambda(1-\lambda)}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{k_n+1}\left[-\frac{\sqrt{\lambda(1-\lambda)}}{\sqrt{k_n}}t+o\left(\frac{1}{\sqrt{k_n}}\right)\right]}{\sqrt{\lambda(1-\lambda)}}=-t.$$

□

Next Proposition follows from the above immediately.

Proposition 9. If components of the regular homogeneous regular series−"m_n out of k_n" have exponential reliability function:

$$R(t)=\begin{cases}1 & \text{for} \quad t<0,\\ \exp[-\beta t] & \text{for} \quad t\ge 0,\ \beta>0,\end{cases}$$

and moreover

(i) $\lim_{n\to\infty} k_n=\infty$, $m_n/k_n=\lambda+o\left(\frac{1}{\sqrt{k_n}}\right)$, $0<\lambda<1$,

(ii) $a_n=\frac{\sqrt{1-\lambda}}{\beta l_n\sqrt{\lambda k_n}}$, $b_n=\frac{-\log\lambda}{\beta l_n}$,

then limit reliability functions of the system is

$$\tilde{\Re}_7^{(\lambda)}(t)=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-\frac{x^2}{2}}dx \quad \text{for } t\in(-\infty,\infty).$$

4. Conclusion

The paper proposes an approach to the solution of practically very important problem of determining the reliability functions of large scale systems by assuming that the number of system component tends to infinity and finding the system limit reliability function. This way, for sufficiently large systems their exact reliability functions may be approximated by their limit reliability functions. This approach gives practically important in everyday usage tool for reliability evaluation of large systems that can be met o instance in piping transportation systems considered in [8], where application of the proposed method is illustrated in the reliability evaluation of the port oil pipeline transportation system.

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