Pointwise completeness and pointwise degeneracy of fractional descriptor continuous-time linear systems with regular pencils

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Abstract. Pointwise completeness and pointwise degeneracy of the fractional descriptor continuous-time linear systems with regular pencils are addressed. Conditions for the pointwise completeness and pointwise degeneracy of the systems are established and illustrated by an example.

Key words: pointwise completeness, pointwise degeneracy, descriptor, fractional, continuous-time, linear, system.

1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–14]. The eigenvalues and invariants assignment by feedbacks have been investigated in [9, 10] and the minimum energy control of descriptor linear systems in [11]. The stability of positive descriptor systems has been investigated in [14]. The positive linear systems with different fractional orders have been addressed in [15]. Descriptor standard positive linear systems by the use of Drazin inverse have been addressed in [1, 2, 8, 10, 16]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [7]. The reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [12]. A new class of descriptor fractional linear discrete-time system has been introduced in [15]. The pointwise completeness and pointwise degeneracy for standard linear systems have been investigated in [17–27] and for fractional linear systems in [17, 20, 22]. The Drazin inverse of matrices has been applied to find the solutions of the state equations of the fractional descriptor continuous-time linear systems with regular pencils in [8].

In this paper the pointwise completeness and pointwise degeneracy of the fractional descriptor continuous-time linear systems with regular pencil is addressed.

The paper is organized as follows. In Sec. 2 some definitions, lemmas and theorems concerning the fractional descriptor continuous-time linear systems are recalled. The main result of the paper is presented in Sec. 3, where the conditions for the pointwise completeness and pointwise degeneracy of the fractional descriptor continuous-time linear systems with regular pencils are established and illustrated by an example. Concluding remarks are given in Sec. 4.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices and $\Re^n = \Re^{n \times 1}$, Z_+ – the set of $n \times n$ nonnegative matrices, I_n – the $n \times n$ identity matrix, ker A (im A) – the kernel (image) of the matrix A.

2. Preliminaries

Consider the autonomous fractional descriptor continuous-time linear system

$$E \, _0D_t^\alpha x(t) = Ax(t), \quad 0 < \alpha < 1, \tag{1}$$

where α is fractional order, $x(t) \in \Re^n$ is the state vector, E, $A \in \Re^{n \times n}$ and

$$_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{dx(\tau)}{d\tau} d\tau \tag{2}$$

is the Caputo definition of $\alpha \in \Re$ order derivative of $x(t)$ and

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{3}$$

is the Euler gamma function.

It is assumed that det $E = 0$ but the pencil (E, A) of (1) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \tag{4}$$
(the field of complex numbers).

Assuming that for some chosen $c \in \mathbf{C}$, $\det[Ec - A] \neq 0$ and premultiplying (1) by $[Ec - A]^{-1}$ we obtain

$$\overline{E} \, _0D_t^\alpha x(t) = \overline{A}x(t), \tag{5a}$$

where

$$\overline{E} = [Ec - A]^{-1}E, \quad \overline{A} = [Ec - A]^{-1}A. \tag{5b}$$

Note that the Eqs. (1) and (5a) have the same solution $x(t)$.

Definition 1. [3, 10]. The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n \times n}$ if

$$\text{rank} \, \overline{E}^q = \text{rank} \, \overline{E}^{q+1}. \tag{6}$$

Definition 2. [3, 8, 10]. A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n \times n}$ if it satisfies the conditions

$$\overline{E} \, \overline{E}^D = \overline{E}^D \overline{E}, \tag{7a}$$

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$$\overline{E}^D \overline{E} \overline{E}^D = \overline{E}^D, \tag{7b}$$

$$\overline{E}^D \overline{E}^{q+1} = \overline{E}^q, \tag{7c}$$

where q is the index of $\overline{E}$ defined by (6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [3, 10]. If $\det \overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [10].

Lemma 1. [3, 8, 10]. The matrices $\overline{E}$ and $\overline{A}$ defined by (5b) satisfy the following equalities
1.

$$\overline{AE} = \overline{EA} \quad \text{and} \quad \overline{A}^D \overline{E} = \overline{E} \overline{A}^D,$$
$$\overline{E}^D \overline{A} = \overline{A} \overline{E}^D, \quad \overline{A}^D \overline{E}^D = \overline{E}^D \overline{A}^D, \tag{8a}$$

2.

$$\ker \overline{A} \cap \ker \overline{E} = \{0\}, \tag{8b}$$

3.

$$\overline{E} = T \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1},$$
$$\overline{E}^D = T \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \tag{8c}$$

$\det T \neq 0$, $J \in \Re^{n_1 \times n_1}$, is nonsingular, $N \in \Re^{n_2 \times n_2}$ is nilpotent, $n_1 + n_2 = n$,
4.

$$(I_n - \overline{E} \overline{E}^D) \overline{A} \overline{A}^D = I_n - \overline{E} \overline{E}^D$$
$$\text{and} \quad (I_n - \overline{E} \overline{E}^D)(\overline{E} \overline{A}^D)^q = 0. \tag{8d}$$

Theorem 1. [8] The solution to the Eq. (1) is given by

$$x(t) = \Phi_0(t) \overline{E} \overline{E}^D w, \tag{9a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{(\overline{E}^D \overline{A})^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{9b}$$

and the vector $w \in \Re^n$ is arbitrary.

From (9a) we have $x(0) = x_0 = \overline{E} \overline{E}^D v$ and $x_0 \in \operatorname{im}(\overline{E} \overline{E}^D)$ where im denotes the image of $\overline{E} \overline{E}^D$.

Lemma 2. The matrix $\Phi_0(t)$ defined by (9b) is nonsingular for any matrix $A \in \Re^{n \times n}$ and time $t \geq 0$.

Proof is given in [8].

Theorem 2. Let

$$P = \overline{E} \overline{E}^D \quad \text{and} \quad Q = \overline{E}^D \overline{A}. \tag{10}$$

Then:
1) $P^k = P$ for $k = 2, 3, \ldots$, \hfill (11a)
2) $PQ = QP = Q$, \hfill (11b)
3) $P\Phi_0(t) = \Phi_0(t)$. \hfill (11c)

Proof is given in [8].

3. Pointwise completeness and pointwise degeneracy

Definition 3. The fractional descriptor system (1) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \Re^n$ there exists a vector of initial conditions $x_0 \in \operatorname{Im}(\overline{E} \overline{E}^D)$ such that $x(t_f) = x_f$.

Theorem 3. The fractional descriptor system (1) is pointwise complete for any $t = t_f$ and every final state $x_f \in \Re^n$ belonging to the set

$$x_f = \operatorname{im}[\Phi_0(t_f) x_0]. \tag{12}$$

Proof. Substituting in (9a) $t = t_f$ we obtain

$$x_f = x(t_f) = \Phi_0(t_f) x_0 \tag{13}$$

and

$$x_0 = [\Phi_0(t_f)]^{-1} x_f \tag{14}$$

since by Lemma 2 the matrix $\Phi_0(t)$ is nonsingular for any matrix $\overline{E}^D \overline{A}$ and time $t \geq 0$.

Definition 4. The fractional descriptor system (1) is called pointwise degenerated in the direction v for $t = t_f$ if there exists a nonzero vector $v \in \Re^n$ such that for all initial conditions $x_0 \in \operatorname{Im}(\overline{E} \overline{E}^D)$ the solution of (1) satisfy the condition

$$v^T x_f = 0 \tag{15}$$

where T denotes the transpose.

Theorem 4. The fractional descriptor system (1) is pointwise degenerated in the direction v defined by

$$v^T \overline{E} = 0 \tag{16}$$

for any $t_f \geq 0$ and all initial conditions $x_0 \in \operatorname{Im}(\overline{E} \overline{E}^D)$.

Proof. Postmultiplying (16) by $\overline{E}^D w$ and using $x_0 = \overline{E} \overline{E}^D w$ and (15) we obtain

$$v^T \overline{E} \overline{E}^D w = v^T x_0 = 0. \tag{17}$$

Taking into account (9b), (13) and (16) we obtain

$$v^T x_f = v^T \Phi_0(t_f) x_0 = \sum_{k=0}^{\infty} \frac{v^T (\overline{E}^D \overline{A})^k t_f^{k\alpha}}{\Gamma(k\alpha + 1)} x_0$$
$$= v^T x_0 + \sum_{k=1}^{\infty} \frac{v^T \overline{E} \overline{E}^D (\overline{E}^D \overline{A})^k t_f^{k\alpha}}{\Gamma(k\alpha + 1)} x_0 = 0 \tag{18}$$

since (7b) and (16) holds.

Example 1. Consider the system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad 0 < \alpha < 1. \tag{19}$$

The pencil of (19) is regular since

$$\det[Es - A] = \begin{vmatrix} s+1 & 0 \\ 0 & 2 \end{vmatrix} = 2(s+1) \neq 0. \tag{20}$$

and the assumption (4) is met.

Choosing $c = 1$ we obtain

$$\overline{E} = [Ec - A]^{-1}E$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\overline{A} = [Ec - A]^{-1}A$$

$$(21)$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\overline{E}^D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{A}^D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \quad (22)$$

In this case the admissible initial conditions are

$$x_0 = \text{im}(\overline{EE}^D) = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} \quad (23)$$

$$\text{and} \quad x_{10} \quad \text{is arbitrary.}$$

Using (9b) and (22) we obtain the nonsingular matrix

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{(\overline{E}^D\overline{A})^k t^{k\alpha}}{\Gamma(k\alpha + 1)}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}^k \quad (24a)$$

$$= \begin{bmatrix} 1 + \varphi(t) & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha + 1)}. \quad (24b)$$

By Theorem 3 the fractional descriptor system (1) with (19) is pointwise complete for $t = t_f$ and every $x_f \in \Re^2$ satisfying the condition (12). In this case from (14) and (24a), (24b) we have

$$x_0 = [\Phi_0(t_f)]^{-1}x_f = \begin{bmatrix} [1 + \varphi(t_f)]^{-1}x_{1f} \\ 0 \end{bmatrix}$$

$$\in \text{im}(\overline{EE}^D) = \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} \quad (25)$$

if and only if $x_f \in \text{Im}(\overline{EE}^D) = \begin{bmatrix} x_{1f} \\ 0 \end{bmatrix}$, where x_{1f} is arbitrary.

By Theorem 4 the system (1) with (19) is pointwise degenerated in the direction $v^T = \begin{bmatrix} 0 & v_2 \end{bmatrix}$ for any v_2 since

$$v^T\overline{E} = \begin{bmatrix} 0 & v_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \quad (26)$$

4. Concluding remarks

The pointwise completeness and the pointwise degeneracy of the fractional descriptor continuous-time linear systems with regular pencil have been addressed. The conditions for the pointwise completeness and the pointwise degeneracy of the systems have been established (Theorem 3 and 4). The considerations have been illustrated by an numerical example of fractional descriptor system with regular pencil. The considerations can be extended to the positive fractional descriptor linear systems.

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