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TOPOLOGICAL ALGORITHMS TO SOLVE INVERSE PROBLEM IN ELECTRICAL TOMOGRAPHY

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Abstract. *In this paper, there were investigated topological algorithms to solve the inverse problem in electrical tomography. The level set method, material derivative, shape derivative and topological derivative are based on shape and topology optimization approach to electrical impedance tomography problems with piecewise constant conductivities. The cost of the numerical algorithm is enough good, because the shape is captured on a fixed grid. The proposed solution is initialized by using topological sensitivity analysis. Shape derivative and material derivative (or topological derivative) have been incorporated with level set methods to investigate shape optimization problems.*

Keywords: topological methods, inverse problem, finite element method, electrical impedance tomography

ALGORYTMY TOPOLOGICZNE DO ROZWIĄZYWANIA ZAGADNIENIA ODWROTNEGO W TOMOGRAFII ELEKTRYCZNEJ

Streszczenie*. W artykule przedstawiono algorytmy topologiczne do rozwiązania problemu odwrotnego w tomografii elektrycznej. Metoda zbiorów poziomicowych, pochodna materialna, pochodna kształtu i pochodna topologiczna zostały oparte na topologii optymalizacji do rozwiązania odwrotnego w elektrycznej tomografii impedancyjnej. Koszt algorytmu numerycznego jest wystarczająco dobry, ponieważ kształt jest osadzony na stałej siatce. Proponowany algorytm inicjowano za pomocą topologicznej analizy wrażliwościowej. Pochodna kształtu, pochodna materialna (lub pochodna topologiczna) zostały połączone z metodą zbiorów poziomicowych do badania problemów optymalizacji kształtu.*

Słowa kluczowe: metody topologiczne, zagadnienie odwrotne, metoda elementów skończonych, tomografia impedancyjna

Introduction

In this paper there was investigated the application of the topological algorithm for the topology optimization based on shape derivative, material derivative or topological derivative. Numerical methods of the shape and the topology optimization were based on the level set representation and there were made possible topology changes during the optimization process [1, 3, 14, 15, 16]. Level set methods have been applied very successfully in many areas of the scientific modelling, for example in propagating fronts and interfaces [3, 4, 6–8, 18]. Therefore, they are used to study shape optimization problems. Instead of using the physically driven velocity, the level set method typically moves the surfaces by the gradient flow of an energy functional. These approaches based on shape sensitivity include the boundary design of elastic. There are two features that make these methods suitable for the topology optimization. The structure is represented by an implicit function such that its zero level set defines the boundary of the object. This function is often discretized on a regular grid that conveniently coincides with the finite or boundary element mesh used for structural analysis. The next valid feature is the simple update of the implicit function using the Hamilton-Jacobi equation, where the velocity function is determined by the shape sensitivity of the structure. These properties enable natural topology changes. The discussed technique can be applied to the solution of inverse problems in electrical impedance tomography [5, 9–13, 17].

1. Numerical Methods

1.1. Level Set Method

Level set methods have been applied very successfully in many areas of the scientific modelling, for example in propagating fronts and interfaces (Fig. 1). Therefore, they are used to study shape optimization problems. Instead of using the physically driven velocity, the level set method typically moves the surfaces by the gradient flow of an energy functional. These approaches based on shape sensitivity include the boundary design of elastic. There are two features that make these methods suitable for the topology optimization. The structure is represented by an implicit function such that its zero level set defines the boundary of the object.

Figure 2 presents six algorithms of the level set methods: the level set method, the parametric level set method, the Mumford-Shah model, the variational level set methods, the Mumford-Shah variational level set methods, the Levenberg–Marquardt level set method.

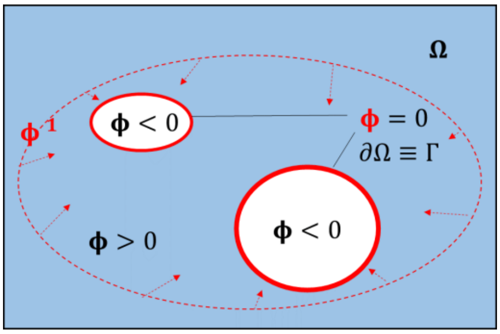

Fig. 1. The idea of the level set function

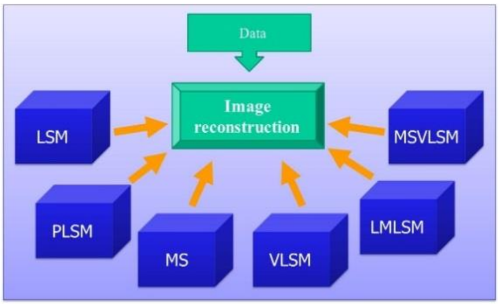

Fig. 2. The reconstruction methods

The representation of the level set method was shown in Figure 1. The level set function ϕ has the following properties:

$$\begin{aligned} &\phi(\vec{r},t) = 0 \text{ for } (x,y)\epsilon\partial\Omega(t) \equiv \Gamma(t) \\ &\phi(\vec{r},t) > 0 \; for \; (x,y)\epsilon\Omega(t) \\ &\phi(\vec{r},t) < 0 \; for \; (x,y) \notin \Omega(t) \end{aligned} \quad (1)$$

The motion is seen as the convection of values (levels) from the function ϕ with the velocity field $\vec{v}$ Such process is described by the Hamilton-Jacobi equation:

$$\frac{\partial\phi}{\partial t} + \vec{v}\cdot\nabla\phi = 0 \quad (2)$$

Here $\vec{v}$ is the desired velocity on the interface, and is arbitrary elsewhere. Actually, only the normal component of $\vec{v}$ is needed $v_n \equiv \vec{v}\cdot\vec{n} \equiv \vec{v}\cdot\nabla\phi/|\nabla\phi|$, so (2) becomes:

$$\frac{\partial\phi}{\partial t} + v_n\cdot|\nabla\phi| = 0 \quad (3)$$

There can update the level set function ϕ by solving discretized version of the Hamilton-Jacobi equation:

$$\frac{\phi^{k+1}-\phi^k}{\Delta t} + v_n^k \cdot |\nabla\phi^k| = 0 \quad (4)$$

Transforming above equation:

$$\phi^{k+1} = \phi^k - v_n^k \cdot |\nabla\phi^k|\Delta t \quad (5)$$

The gradient of the level set function in the k-th time step $(|\nabla\phi^k|)$ has been calculated by the essentially non-oscillatory (ENO) polynomial interpolation scheme. The stability of received solution is achieved by Courant-Friedreichs-Lewy condition (CFL condition):

$$\Delta t < \frac{min(\Delta x, \Delta y)}{max(|\vec{v}|)} \quad (6)$$

Inequality (6) is satisfied by choosing the CFL number α:

$$\Delta t \frac{max(|\vec{v}|)}{min(\Delta x, \Delta y)} = \alpha \quad (7)$$

where $0 < \alpha < 1$. The optimum value equals 0.9:

The calculated velocity must be extended off the interface to the whole domain. This process is called the extension of velocity and is based on the solution of the additional partial differential equation:

$$\frac{\partial v_n}{\partial t} + S(\phi)\frac{\nabla\phi}{|\nabla\phi|} \cdot \nabla v_n = 0 \quad (8)$$

where $S(\phi)$ is defined as following:

$$S(\phi) = \frac{\phi}{\sqrt{\phi^2+\varepsilon^2}} \quad (9)$$

In (9) $|\varepsilon| \ll 1$. Additionally, the velocity is extended to neighborhood of the interface, by defining velocity along normal direction.

Reinitialization is necessary when flat or steep regions complicate the determination of the zero contour. The level set function ϕ is signed distance function if at given time for every point (x,y):

$$|\nabla\phi| = 1 \quad (10)$$

Reinitialization is based on replacing ϕ by another function that has the same zero level set, but satisfies condition.

In the level set representation, the interface, which is the set of points (x,y) satisfying $\phi(x,t) = 0$ is not explicitly given. There is only information $\phi(x_i, t_i)$ at each grid point.

When flat or steep regions complicate the determination of the contour, reinitialization is necessary. The reinitialization procedure is based by replacing ϕ by another function that has the same zero level set but behaves better. This is based on following partial differential equation:

$$\frac{\partial}{\partial t}\phi + S(\phi)(|\nabla\phi| - 1) = 0 \quad (11)$$

where $S(\phi)$ is defined as:

$$S(\phi) = \begin{cases} -1 & \text{for} \quad \phi < 1 \\ 0 & \text{for} \quad \phi = 1 \\ 1 & \text{for} \quad \phi > 1 \end{cases} \quad (12)$$

1.2. Material derivative and shape derivative

The topological method is based on so-called conical differentiability of solutions to variational inequalities with respect to the coefficients of the governing differential operator. It is required that the metric projection in the energy space. Such property is sufficient to obtain the directional differentiability of solutions to the variational inequality with respect to the boundary variations with respect to the changes in the topology by the creation of a small object. A useful concept for calculating derivatives for cost functional is the so-called material and shape derivative of states u. In the application of inverse problems, these states typically are the solutions of partial differential equations which model the probing fields and which depend one way or another on the shape.

Let λ be the adjoint function satisfying:

$$-\Delta\lambda = u - u_m \quad (13)$$

The material derivative concept is applied to the formulation of an inverse obstacle problem. This is the derivative with respect to the geometry for a moving interface. The material derivative $\dot{u}(\vec{r})$ is given by:

$$\dot{u}(\vec{r}) \equiv \lim_{t\to 0}\frac{u_t(\vec{r}+t\vec{v}(\vec{r}))-u(\vec{r})}{t} \quad (14)$$

where $(x,y)\epsilon\Omega_t$.

The shape derivative concept is extended by material derivatives. This method is based on the fact that the definition of shape derivatives is presented to differentiate boundary and domain integrals with respect to a deformation of the contour. The shape derivative is defined as below:

$$u'(\vec{r}) \equiv \lim_{t\to 0}\frac{u_t(\vec{r})-u(\vec{r})}{t} = \dot{u}(\vec{r}) - \vec{v}(\vec{r}) \cdot \nabla u(\vec{r}) \quad (15)$$

The steepest descent direction $\vec{v}$ is given by:

$$\vec{v} = -(\nabla u \cdot \nabla\lambda)\vec{n} \quad (16)$$

The normal velocity is evaluated by using weighted least squares interpolation to get:

$$v_n^k = \nabla u^k \cdot \nabla\lambda^k + \varepsilon\kappa^k \quad (17)$$

In next step the level set function is updated:

$$\phi^{k+1} = \phi^k - (\nabla u^k \cdot \nabla\lambda^k + \varepsilon\kappa^k)|\nabla\phi^k|\Delta t \quad (18)$$

where Δt is obtained from CFL condition (11).

1.3. Topological derivative

Topological derivative (TD) is defined as the first term of the asymptotic expansion of a given shape functional (with expect to a small parameter that measures the size of singular domain perturbation). Topological derivative evaluates for a given shape functional defined in a geometrical domain and dependent on a classical solution to elliptic boundary value problem (Fig. 3).

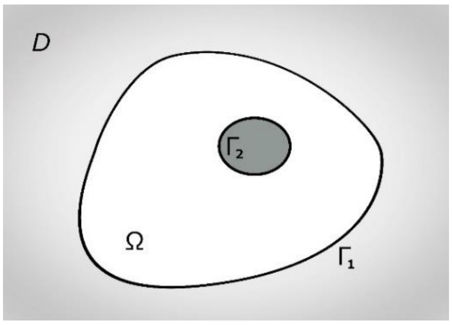

Fig. 3. The idea of topological derivative

Topological derivative represents the change of the shape (functional) when the domain is perturbed by holes, inclusions, defects or cracks. Shape derivative is defined as a minimization of a given shape functional (Fig. 4).

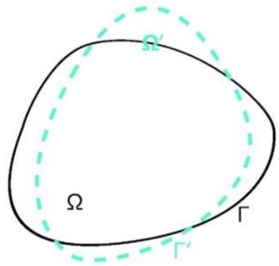

Fig. 4. The idea of shape derivative

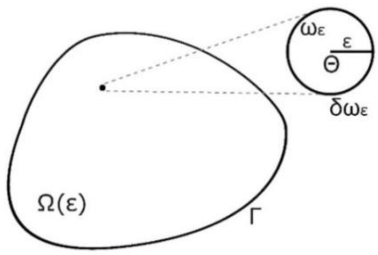

Fig. 5. Topological derivative – the small hole

Perforation of the domain by creating small holes inside Ω was presented in Figure 5. Figure 6 shows the domain with searched objects.

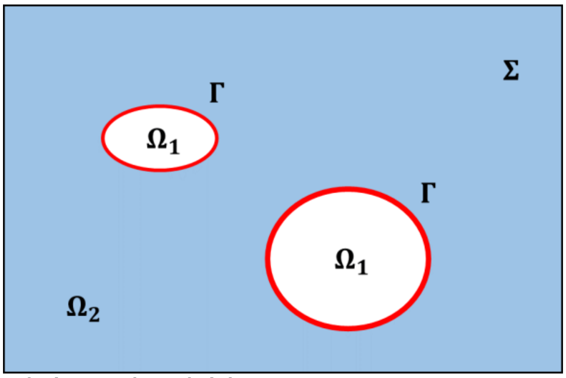

Fig. 6. The domain with searched objects

Topological derivative using the level set function to solve the inverse problem in electrical impedance tomography. Velocity in Hamilton-Jacobi equation is defined following:

$$v_n = -[(\gamma - \gamma_i)(\nabla p_i \cdot \nabla u) + \beta|\gamma - \gamma_i|\mathcal{H}] \text{ on } \Gamma_i \quad (19)$$

Laplace'a equaton is described as:

$$-\Delta u = 0 \text{ in } \Omega^*, q\partial_n u = f \text{ on } \Sigma, q\partial_n u = q_i \partial_n u_i \text{ on } \Gamma_i,$$
$$-\Delta u_i = 0 \text{ in } \Omega_i, u_i = u \text{ on } \Gamma_i \quad (20)$$

The adjoint equation is following:

$$-\Delta p = 0 \text{ in } \Omega^*, q\partial_n p = 2(u - u_m) \text{ on } \Sigma,$$
$$q\partial_n p = q_i \partial_n p_i \text{ on } \Gamma_i, -\Delta p_i = 0 \text{ in } \Omega_i, p_i = p \text{ on } \Gamma_i \quad (21)$$

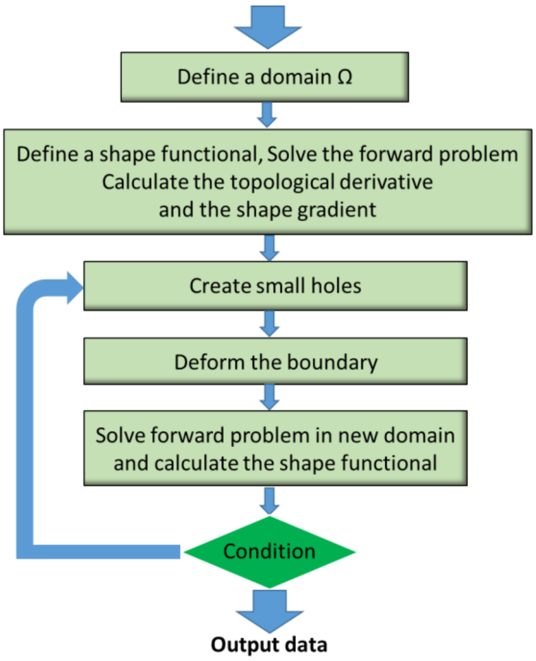

Fig. 7. The iterative algorithm – the inverse problem

For some $\varepsilon > 0$, the topological derivative $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon = \left(\varepsilon|S_\varepsilon^{N-1}|\right)^{-1} \sum_{k=1}^{4} \mathcal{T}_{k,\varepsilon} \quad (22)$$

The algorithm is initialized by $\Omega_2 := \Omega$ and

$$\Omega_1 := \{x \in \Omega \mid \mathcal{T}_\varepsilon(x) < \xi \min_{y \in \Omega} \mathcal{T}_\varepsilon(y)\} \quad (23)$$

where $0 < \xi < 1$ is a given threshold.

The proposed algorithm (Fig. 7):

- define a domain Ω for the shape optimization,
- define a shape functional,
- solve the forward problem,
- calculate the topological derivative and the shape gradient,
- create small holes,
- deform the boundary,
- solve forward problem in new domain and calculate the shape functional,
- minimization $\mathcal{T}(\Omega; u)$ (go to the step create small holes or finish the process).

2. Electrical Impedance Tomography

There were proposed algorithms based on level set function to solve the inverse problem in electrical impedance tomography. The conductivity values in different regions are determined by the finite element method. Numerical iterative algorithm is a combination of the level set methods for following the evolving step edges and the finite element method for computing the velocity. The objective function is defined as the difference between the potential due to the applied current and the measured potential.

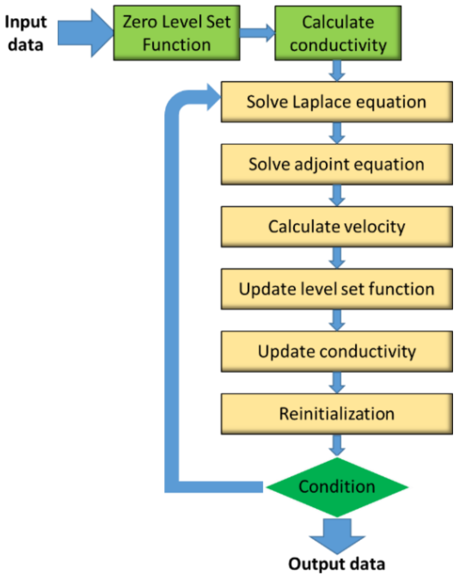

Fig. 8. The iterative algorithm – EIT

For the minimization problem, the level set method and the topological gradient method has been proposed. Both methods are gradient-type algorithms, and the coupled approach can be cast into the framework of alternate directions descent algorithms. The level set method relies on shape derivative, while the topological gradient method is based on the topological derivative or material derivative.

The proposed algorithm is iterative method, structured as follows (Fig.8):

- from the level set function at initial time, find necessary interface information;
- use the Finite/Boundary Element Method to solve the Laplace's equation and next compute the difference of the obtained solution with the observed data;
- solve the Poisson's equation (adjoint equation);
- find velocity in the normal direction;
- update the level set function;
- reinitialize the level set function.

3. Results

In examples reported below, several numerical models are presented. Additionally, there was present different geometries of the conductivity distributions. The conductivity of searched objects is known. The representation of the boundary shape and its evolution during an iterative reconstruction process is achieved by the level set method and the gradient method coupled together. In forward problem, which is given by Laplace's equation, the finite element method has been used. Additionally, different zero level set functions have been selected.

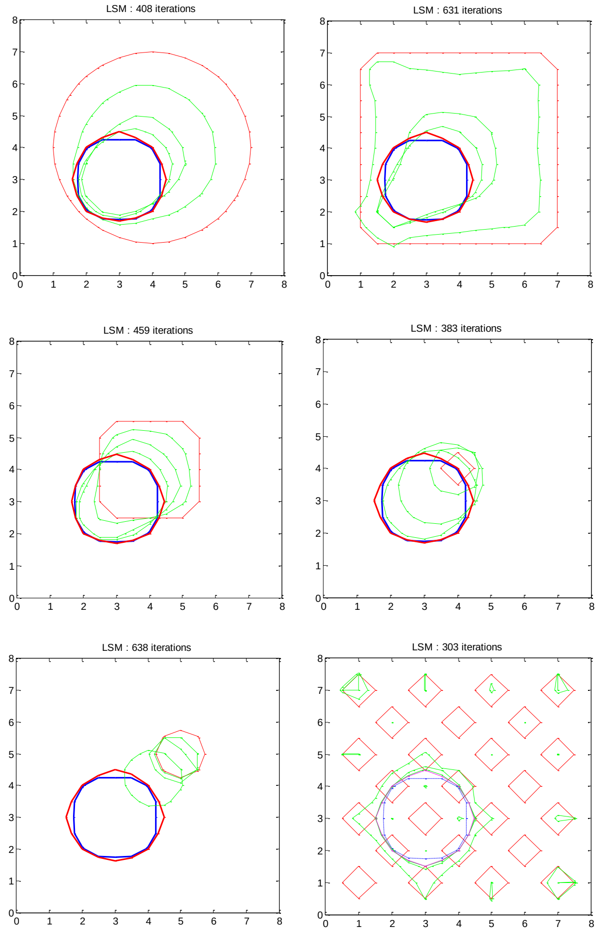

Fig. 9. The image reconstruction with the different zero level set function (red line – zero level set, bold red line – reconstructed object, blue line – original object, green line – steps of reconstruction)

Figure 9 presents the image reconstruction with the different zero level set function (red line – zero level set, bold red line – reconstructed object, blue line – original object, green line – steps of reconstruction). Figure 10 shows the two images: (a) the original objects and the zero contour from the level set function and (b) the process of the image reconstruction.

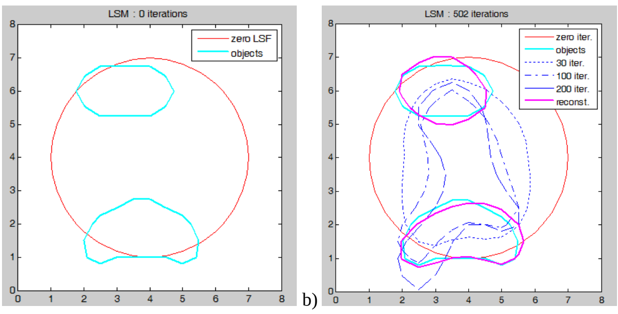

Fig. 10. Images reconstruction: a) the original objects and the zero contour from the level set function; b) the process of the image reconstruction

4. Conclusion

The algorithms based on material and shape derivative, topological derivative and the level set method have been proposed in this work. There are iterative algorithms where the shape boundary evolves smoothly and new small objects are detected. An efficient algorithm for solving the forward and inverse problems would also improve a lot of the numerical performances of the proposed methods. In model problem from EIT is required to identify unknown conductivities from near-boundary measurements of the potential. The number of iterations determine the position and shape of zero level set functions. In these algorithms, it can control the process of the image reconstruction. The level set function techniques have been shown to be successful to identify the unknown boundary shapes. The accuracy of the image reconstruction is better than gradient methods. The purpose of the presented method is obtaining the better image reconstruction than gradient methods and accelerates the iterative process by using different shapes of the zero level set functions. Applying the line measurement model is very effective to solve the inverse problem in the copper-mine ceiling and flood embankment. The right selection of the zero level set function gives the better results, reduces the time of the reconstruction process and improves the better quality of the image reconstruction.

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