Non-Linear Vibrations of a Non-Uniform Beam with SymmetricallyLocated Piezoelectric Patches

Krzysztof KULIŃSKI
Częstochowa University of Technology, Institute of Mechanics and Machine Design Foundations, Dąbrowskiego 71,42 200 Częstochowa, krzysztku@gmail.com

Jacek PRZYBYLSKI
jacek.pr@imipkm.pcz.pl

Abstract

In this paper the non-linear vibration behaviour and its modification due to the piezoelectric actuation of a beam with varying cross section and resting on an elastic foundation has been discussed. Due to assumed end conditions the stretching force emerges during the system vibrations. That force can be modified by an axial residual force to enhance or reduce the value of vibrations frequency of the beam. The system is divided onto three segments with the central segment consisted of the core beam and two colocally and perfectly bonded piezo patches. In order to obtain the approximate solutions of the non-linear frequency of the systems the Lindstedt-Poincare method has been utilized. Vast number of numerical results shows that not only the structural parameters of the system have significant effect on its non-linear vibration behaviour at a given amplitude but also the residual force and the elastic foundation modulus.

Keywords: non-linear vibrations, piezoactuators, amplitude-frequency relation, Winkler foundation

1. Introduction

The non-linear lateral vibrations of beam structures have been the subject of interest of many researchers. From the engineering point of view the beam-type structures are very interesting due to their wide application in civil and mechanical engineering, automotive, aviation, aeronautics industry, medical systems and equipment and many more. It is well known that any mechanical structure or its part should be protected from exposure to long time periods of resonance. Piezoelectric materials which are also called "smart materials" allow to modify the vibration frequency and buckling load of a given structure due to the inverse piezoelectric effect. That effect result in dimension changes of piezoelement which depend on the applied electric field vector. It should be noted that direct piezoelectric effect is also widely utilized in many areas of life such as sound processing, pacemakers, airbags, lighters etc.

As the research precursor of non-linear frequency studies shall be deemed to Wojnowsky-Krieger [1] whose thesis concerned the effect of the axial force on the non-linear frequency of simply supported beams. In the subsequent years there were vast number of literature positions published and experimental studies performed concerning the problem of the non-linear vibrations. Azrar et al. [2] presented mathematical approach concerning the second order approximation to obtain the non-linear vibration frequency for pinned-pinned and clamped-clamped beams which are close to the exact solution in a large amplitude frequency range. Moreover authors presented a very thorough discussion about increasing the accuracy of the obtained amplitude-frequency solutions. Benamar et al. [3] proposed a general model of the non-linear vibrations at large amplitudes for standardly supported beams to describe the influence of amplitude on both the mode shapes and the natural frequency. It was observed that near the clamps there were a great increase in beam curvatures which caused increased deflection resulting in highly non-linear increase of bending strain. A vast literature overview concerning the active, passive, semi-active and hybrid vibration control of the systems was presented by Song et al [4]. It was stated that piezoelectric materials despite some limitations have many advantages such as low-cost, low weight and ease of implementation. On the basis of Faria [5] as well as Zehetner and Irschik [6] considerations it can be stated that only for the beams which ends are mounted to prevent their axial displacement, both the stability and vibration frequency can be modified by piezoelectric actuation. Oguamanan et al. [7] investigated the influence of piezoelectric material in plane stress on beams mechanical performance. Authors showed that in systems where piezoelectric material was bonded both to the upper and bottom surface of the beam, especially the first frequency can be significantly modified. It was observed that depending on the applied electric field vector direction, vibration frequency can be enhanced or reduced. Moreover authors demonstrated that piezoactuators localized near the beam supports, give slightly more effective control of the system vibrations. The influence of piezoactuators length, its localization and the piezoelectric force on the amplitude-nonlinear frequency relationship in a slender pinned-pinned beam has been studied by Przybylski [8]. It was proved that stretching piezoelectric force result in an increase of the natural frequency and decrease of non-linear frequency, whereas compressive piezo-force resulted in opposite system behaviour. A broader literature overview with wider area of study of slender systems with bonded piezoelectric materials can be found in [9].

In this paper the influence of vibrations amplitude, piezosegment length and Winkler elastic foundation modulus on the non-linear frequency for a pinned-pinned and clamped-clamped beams is investigated. Moreover the non-linear vibration adjustment due to piezoelectric actuation is examined. The object of study is a three segment system made of aluminium host beam with two symmetrically piezo patches bonded perfectly on the upper and bottom surface of the central segment. In order to obtain approximate solutions the Lindstedt-Poincare method has been utilized.

2. Problem formulation

The main objective of this work is to formulate and solve the problem of the non-linear vibrations of a stepped beam resting on the Winkler elastic foundation and to estimate the influence of both the structural parameters and the piezoelectric actuation on the non-linear frequency-amplitude relationship. Due to the moderately large amplitude of vibrations, the von Karman theory has been applied according to which during transversal vibrations, the axial inertia effect can be treated as insignificant.

The scheme of three-segmented system composed of a core beam with both ends clamped and two piezoelectric patches bonded along the central segment is shown in Fig. 1.

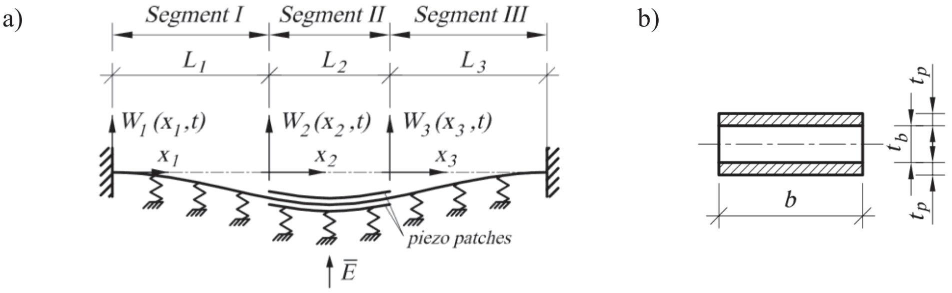

Figure 1. Scheme of clamped-clamped beam resting on elastic Winkler foundation with two piezoelectric patches colocally mounted along the central segment (a), piezosegment cross-section (b)

The applied voltage, symbolized by the electric field vector **E** in Fig. 1, is exactly the same for the upper and bottom piezo actuator which results in the axial stretching/compressive force being generated dependently on the electric field vector sense. A derivation of the residual force equation appearing along the stepped beams with *n*-pairs of piezoelectric actuators has been presented in [9]. According to these considerations for the three segmented system the residual force can be described as follows

$$F_r = F\left[1+\eta\left(\frac{L}{L_2}-1\right)\right]^{-1} \tag{1}$$

where: η denotes the relation of the piezosegment axial stiffness to that of the beam, $F = -2be_{31}V$ is the piezoelectric force induced by piezoceramic patches of width b, when piezo material is characterised by constant e_{31} and the applied voltage is equal to V, L is the length of the beam, L_2 is the length of piezosegment. According to von Karman theory and the actuality that algebraic sum of the axial displacement of three segments is equal to zero, the force which stretches the beam during its transverse vibration can be expressed as

$$S(t) = \frac{1}{2}\left[\sum_{i=1}^{3}\frac{L_i}{E_i A_i}\right]^{-1}\sum_{i=1}^{3}\int_0^{L_i}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2 \tag{2}$$

Introducing both residual piezo-force F_r and dynamic force $S(t)$ into the governing equation of motion for the *i*-th segment, that equation takes the following non-dimensional form

$$\frac{\partial^4 w_i(\xi_i,\tau)}{\partial \xi_i^4} \pm \varphi_i\left(f_r^2 + s^2(\tau)\right)\frac{\partial^2 w_i(\xi_i,\tau)}{\partial \xi_i^2} + \mu_i\omega^2\frac{\partial^2 w_i(\xi_i,\tau)}{\partial \tau^2} + \beta w_i(\xi_i,\tau) = 0, \quad \text{for } i = 1,2,3 \tag{3}$$

where the dimensionless parameters are defined as

$$w_i(\xi_i,\tau)=\frac{W_i(x_i,t)}{L},\ \ \xi_i=\frac{x_i}{L},\ \ l_i=\frac{L_i}{L},\ \ \varphi_i=(1+r_m)^{-\frac{1}{2}\left(j^{2i}+1\right)},\ \ r_m=\frac{E_pI_p}{E_bI_b},$$
$$j=\sqrt{-1},\ \ f_r^2=\frac{F_rL^2}{E_bI_b},\ \ s^2(\tau)=\frac{\lambda}{2}\left(\sum_{i=1}^{3}\eta^{-\frac{1}{2}\left(j^{2i}+1\right)}l_i\right)^{-1}\sum_{i=1}^{3}\int_0^{l_i}\left[\frac{\partial w_i(\xi_i,\tau)}{\partial\xi_i}\right]^2 d\xi_i\ ,$$
$$\lambda=A_bL^2/I_b,\ \ \mu_i=\left(\frac{\alpha_1+(\eta-1)\alpha_2}{\alpha_1(1+r_m)}\right)^{\frac{1}{2}\left(j^{2i}+1\right)},\ \ \alpha_1=\frac{E_p}{E_b},\ \ \alpha_2=\frac{\rho_p}{\rho_b},\ \ \eta=\frac{E_bA_b+E_pA_p}{E_bA_b},\qquad (4)$$
$$\omega^2=\Omega^2L^4\frac{\rho_bA_b}{E_bI_b},\ \ \tau=\Omega t,\ \ \beta=\frac{kL^4}{E_bI_b}$$

Following notation has been assumed: E_pI_p, E_bI_b - the bending stiffness of piezo patches and that of a beam, respectively, A_p, A_b - the cross section area of piezopatches and beam, respectively, ρ_p, ρ_b - the material densities of the actuators and beam, respectively, ω - the natural frequency of the system, t is time, k denotes the Winkler foundation modulus.

The non-dimensional boundary conditions for a pinned-pinned beam are:

$$w_1(\xi_1,\tau)\Big|_{\xi_1=0}=w_1^{II}(\xi_1,\tau)\Big|_{\xi_1=0}=w_3(\xi_3,\tau)\Big|_{\xi_3=l_3}=w_3^{II}(\xi_3,\tau)\Big|_{\xi_3=l_3}=0\qquad (5)$$

whereas for a clamped-clamped beam take the form:

$$w_1(\xi_1,\tau)\Big|_{\xi_1=0}=w_1^{I}(\xi_1,\tau)\Big|_{\xi_1=0}=w_3(\xi_3,\tau)\Big|_{\xi_3=l_3}=w_3^{I}(\xi_3,\tau)\Big|_{\xi_3=l_3}=0\qquad (6)$$

where: I and II are the Roman numerals denoting the order of the derivative with respect to the space variable ξ.

The continuity conditions are independent from the type of supports and describe the equality of the transversal force, moments, slopes and displacements between segments:

$$w_i(\xi_i,\tau)\Big|_{\xi_i=l_i}=w_{i+1}(\xi_{i+1},\tau)\Big|_{\xi_{i+1}=0},\ \ w_i^{I}(\xi_i,\tau)\Big|_{\xi_i=l_i}=w_{i+1}^{I}(\xi_{i+1},\tau)\Big|_{\xi_{i+1}=0},$$
$$(1+r_m)^{\frac{1}{2}\left[j^{2i}+1\right]}w_i^{Rn}(\xi_i,\tau)\Big|_{\zeta_i=l_i}=(1+r_m)^{\frac{1}{2}\left[j^{2(i+1)}+1\right]}w_{i+1}^{Rn}(\xi_{i+1},\tau)\Big|_{\xi_{i+1}=0},\ i=1,\,2,\ Rn=II,III\qquad (7)$$

3. Approximate solutions

In order to obtain approximate solutions of the non-linear boundary problem the Lindstedt-Poincare method has been utilized, according to which relevant quantities are expanded into exponential series with respect to the small amplitude parameter ε

$$w_i(\xi_i,\tau)=\sum_{n=1}^{N}\varepsilon^{2n-1}w_{i2n-1}(\xi_i,\tau)+O\left(\varepsilon^{2N+1}\right)\qquad (8)$$

$$s^2(\tau)=\sum_{n=1}^{N}\varepsilon^{2n}s_{2n}^2(\tau)+O\left(\varepsilon^{2(N+1)}\right)\qquad (9)$$

$$\omega^2 = \omega_0^2 + \sum_{n=1}^{N} \varepsilon^{2n} \omega_{2n}^2 + O\left(\varepsilon^{2(N+1)}\right) \tag{10}$$

where separation of space and time variable are described as:

$$w_{ij}(\xi_i,\tau) = \sum_{k=1}^{b} w_{ij}^{(2k-1)}(\xi_i)\cos(2k-1)\tau \text{ , for } b = \frac{j-1}{2}+1 \text{ and } j = 1,3,5,... \tag{11}$$

$$s_j^2(\tau) = \sum_{k=1}^{c} s_j^{2\,(2k-2))} \cos 2(k-1)\tau \text{ , for } c = \frac{j}{2}+1 \text{ and } j = 2,4,6,... \tag{12}$$

Introducing expansions from (8-10) into the equation of motion (3) and axial dynamic force $s^2(\tau)$ expressed in (4) and then equating the terms of respective ε exponents to zero, an infinite set of equations of motion and axial force is obtained.

By solving the first pair of equations from the infinite set of equations with use of boundary conditions (5-6) an infinite number of solutions for the natural frequency is obtained, whereas from the third equation after applying the orthogonally condition the second term of frequency ω_2 can be obtained. The relationship of non-linear frequency ω and amplitude ε are determined on the basis of equation (10), with a customary limit up to the second order.

4. Numerical results

In this chapter the numerical results concerning the non-linear frequency-amplitude relationship for clamped-clamped and pinned-pinned beams with piezosegment centrally localised are presented. All analysis can be performed by using the non-dimensional quantities, but to show its usefulness for engineering applications it has been assumed that the host beam thickness $t_b = 3.0$ [mm] and piezo patches $t_p = 0.5$ [mm] each, whereas both the beam and piezo patches width $b = 20$ [mm]. The influence of adhesive layer thickness has been taken as negligibly small. The beam was made of a homogeneous elastic isotropic aluminium, while piezoceramic actuators were made of a homogeneous elastic transversely isotropic P41 material (Annon Piezo Technology Ltd. Co.). Electromechanical properties of the adopted materials for the numerical analysis are shown in Tab. 1.

Table 1. Material properties of beam and piezo patches

Property	**Unit**	**Beam**	**Piezoceramic**
E	GPa	70.00	83.33
ρ	kg/m^3	2720	7450
d_{31}	C/N	-	$1.00\cdot10^{-10}$
U_{max}	V/mm	-	2000

The first group of plots presented in Fig. 2 shows the influence of structural parameters of the beam and the elastic foundation stiffness modulus on the mentioned relationship, whereas in Fig. 3 the role of piezoelectric actuation in modification the obtained curves for the system with piezosegment of length $l_2 = 0.80$.

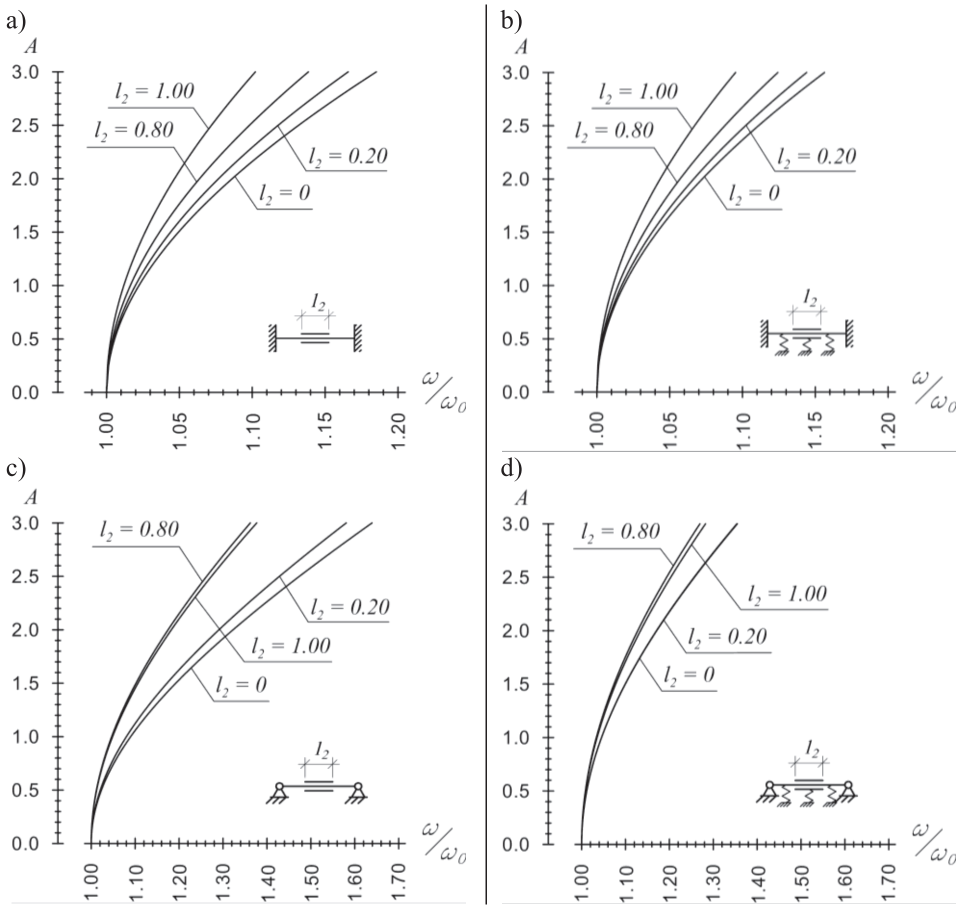

Figure 2. The influence of piezosegment length on amplitude – non-linear frequency relationship in clamped-clamped (a, b) and pinned-pinned (c, d) beams; remaining parameters: Winkler elastic foundation modulus $\beta = 0$ (a, c), $\beta = 100$ (b, d)

Comparing the curve courses for the clamped-clamped support (Fig. 2a, b) it can be stated than the longer the piezosegment length the smaller the amplitude influence on the non-linear frequency. For the pinned-pinned beam (Fig. 2c,d) at the whole range of the amplitude, the non-linear frequency is lower for the piezosegment of length $l_2 = 0.80$ than for the piezosegment mounted at the entire beam ($l_2 = 1.00$), whereas for the lengths $l_2 = 0.0$ and $l_2 = 0.20$ the vibrations aims to be the same with increased elastic foundation modulus. In both clamped-clamped and pinned-pinned system together with an increase of the elastic foundation modulus, the non-linear frequency decreases at the whole range of amplitudes and for any value of the piezo patches length.

In order to examine the piezoelectric actuation influence on the non-linear frequency – amplitude relationship two values of piezoelectric force has been chosen $f^2 = \pm\pi^2$. It should be noted that the range of non-dimensional residual force resulting from the applied electric field is far below the depoling field for the piezoceramic material.

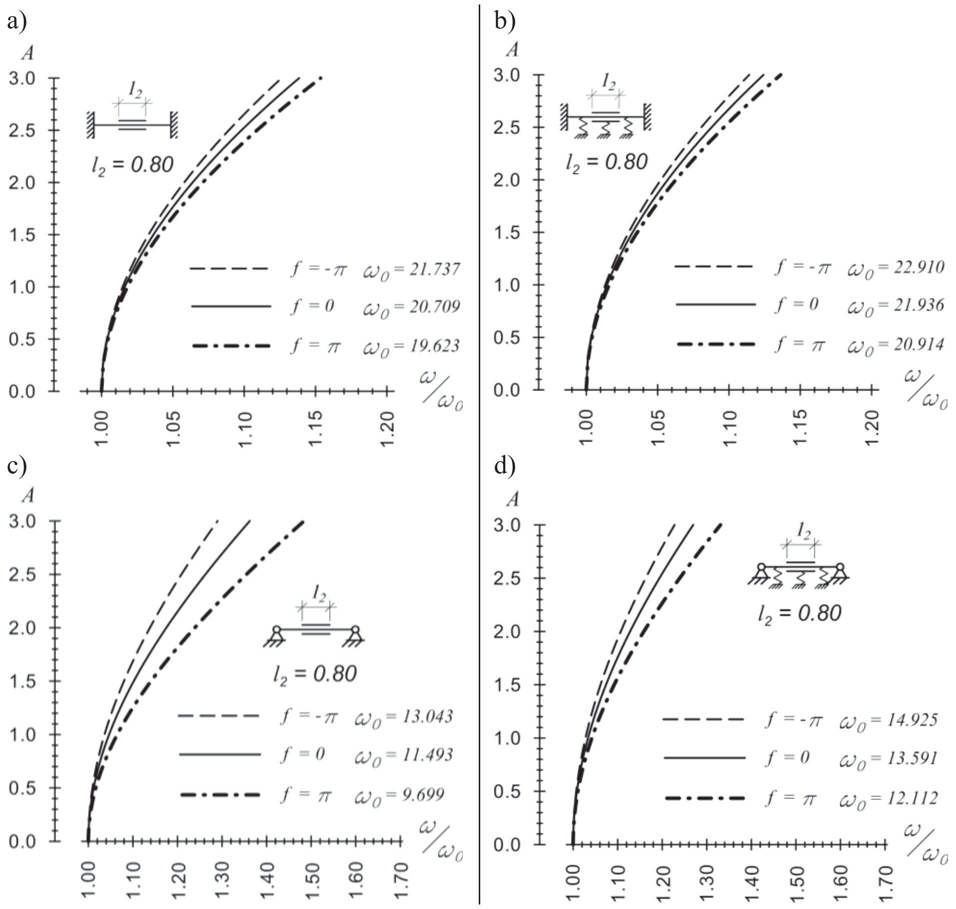

Figure 3. The influence of piezoelectric actuation on the amplitude-non-linear frequency relationship for clamped-clamped (a, b) and pinned-pinned (c, d) beams; remaining parameters: Winkler elastic foundation modulus $\beta = 0$ (a, c), $\beta = 100$ (b, d)

As it is presented in Fig. 3 in both cases (clamped-clamped and pinned-pinned beam) at any given amplitude and elastic foundation modulus the tensile piezoelectric force reduce the non-linear frequency, while the natural frequency is increased comparing to the beam without piezoactuation, whereas compressive piezo-force acts in an opposite way. Moreover the higher stiffness of Winkler elastic foundation the lower value of non-linear frequency at the whole range of amplitude. It should also be noted that more significant affection of Winkler elastic foundation on the non-linear frequency – amplitude relationship for systems with lower external support stiffness.

5. Conclusions

In this study the problem of non-linear vibrations for the non-uniform Euler-Bernoulli beams has been discussed. Moreover the enhancement and reduction of non-linear vibrations due to the piezoelectric actuation has been examined. It should be noted that performed studies can be useful in the manufacture of elements which are responsible of controlling static and dynamic response of structures.

It was also shown in this paper that regardless of system external support, the higher value of Winkler foundation modulus parameter results in decreasing of the non-linear frequency. There was also proved that piezoelectric actuation can enhance the non-linear vibration frequency via compressive force induced, while the natural frequency is increased and the opposite system behaviour is obtained for the tensile piezo-force.

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