Opuscula Math. 38, no. 1 (2018), 95–115
https://doi.org/10.7494/OpMath.2018.38.1.95 Opuscula Mathematica

# ON THE STABILITY OF SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

N. Psarros, G. Papaschinopoulos, and C.J. Schinas

*Communicated by Stevo Stević*

**Abstract.** In this paper we prove the stability of the zero equilibria of two systems of difference equations of exponential type, which are some extensions of an one-dimensional biological model. The stability of these systems is investigated in the special case when one of the eigenvalues is equal to -1 and the other eigenvalue has absolute value less than 1, using centre manifold theory. In addition, we study the existence and uniqueness of positive equilibria, the attractivity and the global asymptotic stability of these equilibria of some related systems of difference equations.

**Keywords:** difference equations, asymptotic behaviour, global stability, centre manifold, biological dynamics.

**Mathematics Subject Classification:** 39A10.

## 1. INTRODUCTION

Difference equations and systems of difference equations containing exponential terms have numerous potential applications in biology. A large number of papers dealing with such or related equations have been published (see, e.g, [6, 16, 22–24]). In [34], the following model incorporating litter inhibition is discussed:

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \quad L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}},$$

where $B$ is the living biomass, $L$ the litter mass, $N$ the total soil nitrogen, $t$ the time (measured in years) and constants $a, b, c, d > 0$ and $0 < k < 1$. In this model, the living biomass ($B$) is reduced below its equilibrium, by litter. Litter decay is determined by $d$, while litter production is $k$ times the living biomass. The complexity of the grassland ecosystem makes its study interesting but complicated. In addition, in [18], the authors studied the boundedness and the persistence of the positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium, as well as the existence of periodic solutions of the following equation:

$$x_{n+1} = a\frac{x_n^2}{b + x_n} + c\frac{e^{k-dx_n}}{1 + e^{k-dx_n}},$$

where $a \in (0, 1)$, $a, b, c, d, k$ are positive constants and $x_0$ is a positive real number.

Motivated by this discrete time model and recent studies of symmetric and close to symmetric systems of difference equations (see, e.g, [9, 11, 19, 25, 26]), in this paper, we will study the stability of the zero equilibria of the following systems:

$$x_{n+1} = a_1\frac{y_n}{b_1 + y_n} + c_1\frac{x_n e^{k_1-d_1x_n}}{1 + e^{k_1-d_1x_n}}, \quad y_{n+1} = a_2\frac{x_n}{b_2 + x_n} + c_2\frac{y_n e^{k_2-d_2y_n}}{1 + e^{k_2-d_2y_n}} \tag{1.1}$$

and

$$x_{n+1} = a_1\frac{x_n}{b_1 + x_n} + c_1\frac{y_n e^{k_1-d_1y_n}}{1 + e^{k_1-d_1y_n}}, \quad y_{n+1} = a_2\frac{y_n}{b_2 + y_n} + c_2\frac{x_n e^{k_2-d_2x_n}}{1 + e^{k_2-d_2x_n}}, \tag{1.2}$$

where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$, $d_2$, $k_1$, $k_2$, are real constants and the initial values $x_0$ and $y_0$ are real numbers.

In addition, we will investigate the asymptotic behaviour of the positive solutions of the following systems of difference equations:

$$x_{n+1} = a_1\frac{{y_n}^2}{b_1 + y_n} + c_1\frac{e^{d_1-k_1x_n}}{1 + e^{d_1-k_1x_n}}, \quad y_{n+1} = a_2\frac{e^{b_2-k_2x_n}}{1 + e^{b_2-k_2x_n}} + c_2\frac{e^{d_2-k_3y_n}}{1 + e^{d_2-k_3y_n}}, \tag{1.3}$$

$$x_{n+1} = a_1\frac{{y_n}^2}{b_1 + y_n} + c_1\frac{e^{d_1-k_1x_n}}{1 + e^{d_1-k_1x_n}}, \quad y_{n+1} = a_2\frac{{x_n}^2}{b_2 + x_n} + c_2\frac{e^{d_2-k_2y_n}}{1 + e^{d_2-k_2y_n}}, \tag{1.4}$$

$$x_{n+1} = a_1\frac{{x_n}^2}{b_1 + x_n} + c_1\frac{e^{d_1-k_1y_n}}{1 + e^{d_1-k_1y_n}}, \quad y_{n+1} = a_2\frac{e^{b_2-k_2x_n}}{1 + e^{b_2-k_2x_n}} + c_2\frac{e^{d_2-k_3y_n}}{1 + e^{d_2-k_3y_n}}, \tag{1.5}$$

$$x_{n+1} = a_1\frac{{x_n}^2}{b_1 + x_n} + c_1\frac{e^{d_1-k_1y_n}}{1 + e^{d_1-k_1y_n}}, \quad y_{n+1} = a_2\frac{{y_n}^2}{b_2 + y_n} + c_2\frac{e^{d_2-k_2x_n}}{1 + e^{d_2-k_2x_n}}, \tag{1.6}$$

where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$, $d_2$, $k_1$, $k_2$, $k_3$ are positive constants and the initial values $x_0$ and $y_0$ are positive.

The results of this paper could be used to create more elaborate biological models to facilitate understanding the underlying ecological mechanisms. The results obtained for the systems (1.1) and (1.2) provide conditions for stability of the zero equilibria of those systems. Those equilibria correspond to the physical situation where both quantities ($x$ and $y$) vanish.

The asymptotic behaviour of positive solutions of scalar equations related to the previous systems is studied in [22]. For some related cyclic systems of difference equations see [9, 28, 31] and [32], as well as some three-dimensional systems (see, e.g., [19,27,33]). Finally we note that, since difference equations have several applications in applied sciences, there exists a rich bibliography concerning theory and applications of difference equations (see [1–34]).

## 2. STABILITY OF ZERO EQUILIBRIUM OF SYSTEM (1.1)

In the following, we prove the stability of the zero equilibrium of System (1.1), using Centre Manifold Theory.

**Proposition 2.1.** *Consider System* (1.1) *where* $a_1$, $b_1$, $b_2$, $c_2$, $k_1$, *are real positive constants and* $a_2$, $c_1$, $k_2$ *are real negative constants such that*

$$c_2 < 1 + e^{-k_2}, \quad -\frac{1+e^{k_1}}{e^{k_1}} < c_1 < -\frac{c_2 e^{k_2}(1+e^{k_1})}{e^{k_1}(1+e^{k_2})}, \tag{2.1}$$

$$\Big(1 + \frac{c_1 e^{k_1}}{1+e^{k_1}}\Big)\Big(1 + \frac{c_2 e^{k_2}}{1+e^{k_2}}\Big) = \frac{a_1 a_2}{b_1 b_2}, \tag{2.2}$$

$$d_2 > \max\Big\{0, A_1, A_2, A_3\Big\}, \tag{2.3}$$

*where*

$$A_1 = \frac{(1+e^{k_2})^2}{c_2 \Gamma^2 e^{k_2}} \left( \frac{c_1 d_1 \Gamma e^{k_1}}{(1+e^{k_1})^2} + \frac{a_1 \Gamma^3}{b_1^2} - \frac{a_2}{b_2^2} \right),$$
$$A_2 = \frac{(1+e^{k_2})^2}{c_2 \Gamma \Delta e^{k_2}} \left( \frac{c_1 d_1 \Delta e^{k_1}}{(1+e^{k_1})^2} + \frac{a_1 \Delta^2 \Gamma}{b_1^2} - \frac{a_2}{b_2^2} \right),$$
$$A_3 = \sqrt{\frac{2(1+e^{k_2}(1+c_2))(1+e^{k_2})^2}{b_1^2 c_2 e^{k_2}(1-e^{k_2})}}.$$

*Then the zero equilibrium of* (1.1) *is stable.*

*Proof.* The Jacobian matrix $J_0$ at the zero equilibrium for (1.1) is

$$J_0 = \begin{bmatrix} \frac{c_1 e^{k_1}}{1+e^{k_1}} & \frac{a_1}{b_1} \\ \frac{a_2}{b_2} & \frac{c_2 e^{k_2}}{1+e^{k_2}} \end{bmatrix}.$$

Calculating the eigenvalues of $J_0$, using (2.1) and (2.2) we obtain

$$\lambda_1 = -1, \quad \lambda_2 = 1 + \frac{c_1 e^{k_1}}{1+e^{k_1}} + \frac{c_2 e^{k_2}}{1+e^{k_2}} \quad \text{and so} \quad |\lambda_2| < 1.$$

Now, the initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \tag{2.4}$$

where

$$f(x,y) = \frac{a_1 y}{y+b_1} - \frac{a_1 y}{b_1} + \frac{c_1 x e^{k_1 - d_1 x}}{1+e^{k_1 - d_1 x}} - \frac{c_1 e^{k_1}}{1+e^{k_1}} x,$$
$$g(x,y) = \frac{a_2 x}{x+b_2} - \frac{a_2 x}{b_2} + \frac{c_2 y e^{k_2 - d_2 y}}{1+e^{k_2 - d_2 y}} - \frac{c_2 e^{k_2}}{1+e^{k_2}} y.$$

We let now

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where $T$ is the matrix that diagonalizes $J_0$ defined by

$$T = \begin{bmatrix} 1 & 1 \\ \Gamma & \Delta \end{bmatrix},$$

where

$$\Gamma = -\frac{b_1(1+e^{k_1}(1+c_1))}{a_1(1+e^{k_1})}, \quad \Delta = \frac{b_1(1+e^{k_2}(1+c_2))}{a_1(1+e^{k_2})}. \tag{2.5}$$

Then, (2.4) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \tag{2.6}$$

where

$$\begin{aligned}
&\hat{f}(u,v) \\
&= R\Bigg(\Delta\bigg(\frac{a_1(\Gamma u+\Delta v)}{b_1+\Gamma u+\Delta v} - \frac{a_1(\Gamma u+\Delta v)}{b_1} + \frac{c_1(u+v)e^{k_1-d_1(u+v)}}{1+e^{k_1-d_1(u+v)}} - \frac{c_1(u+v)e^{k_1}}{1+e^{k_1}}\bigg) \\
&- \bigg(\frac{a_2(u+v)}{b_2+u+v} - \frac{a_2(u+v)}{b_2} + \frac{c_2(\Gamma u+\Delta v)e^{k_2-d_2(\Gamma u+\Delta v)}}{1+e^{k_2-d_2(\Gamma u+\Delta v)}} - \frac{c_2(\Gamma u+\Delta v)e^{k_2}}{1+e^{k_2}}\bigg)\Bigg), \\
&\hat{g}(u,v) \\
&= R\Bigg(-\Gamma\bigg(\frac{a_1(\Gamma u+\Delta v)}{b_1+\Gamma u+\Delta v} - \frac{a_1(\Gamma u+\Delta v)}{b_1} + \frac{c_1(u+v)e^{k_1-d_1(u+v)}}{1+e^{k_1-d_1(u+v)}} - \frac{c_1(u+v)e^{k_1}}{1+e^{k_1}}\bigg) \\
&+ \frac{a_2(u+v)}{b_2+u+v} - \frac{a_2(u+v)}{b_2} + \frac{c_2(\Gamma u+\Delta v)e^{k_2-d_2(\Gamma u+\Delta v)}}{1+e^{k_2-d_2(\Gamma u+\Delta v)}} - \frac{c_2(\Gamma u+\Delta v)e^{k_2}}{1+e^{k_2}}\Bigg)
\end{aligned} \tag{2.7}$$

and

$$R = \frac{1}{\Delta - \Gamma}.$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$, $\psi(u) = \eta u^2 + \theta u^3$, $\eta, \theta \in \mathbb{R}$. The use of this approximation is justified by Theorem 7 of [1]. Consequently, according to Theorem 8 of [1] and using (2.6), the study of the stability of the zero equilibrium of System (1.1) reduces to the study of the stability of the zero equilibrium of the equation:

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \tag{2.8}$$

We now need to determine $\eta$ (the coefficient in the Taylor expansion). From (2.6), we conclude that map $h$ must satisfy the centre manifold equation (see [1, p. 34], [3, p. 243], [15], [16, p. 642] and [19]):

$$h(-u + \hat{f}(u, h(u))) - \lambda_2 h(u) - \hat{g}(u, h(u)) = 0.$$

Keeping the terms up to $u^3$ and using (2.7), we obtain

$$\eta = \frac{R}{1-\lambda_2}\left(\frac{c_1 d_1 \Gamma e^{k_1}}{(1+e^{k_1})^2} + \frac{a_1\Gamma^3}{b_1^2} - \frac{a_2}{b_2^2} - \frac{c_2 d_2 \Gamma^2 e^{k_2}}{(1+e^{k_2})^2}\right). \tag{2.9}$$

From (2.8) and (2.9) we obtain $G'(0) = -1$ and

$$\begin{aligned} G'''(0) = R\Bigg(\Delta\Bigg(&-\frac{6c_1 d_1^2 e^{2k_1}}{(1+e^{k_1})^3} + \frac{3c_1 d_1^2 e^{k_1}}{(1+e^{k_1})^2} - \frac{12 c_1 d_1 \eta e^{k_1}}{(1+e^{k_1})^2} - \frac{12 a_1 \Delta\Gamma\eta}{b_1^2} + \frac{6a_1\Gamma^3}{b_1^3}\Bigg) \\ &- \left(\frac{6a_2}{b_2^3} - \frac{12a_2\eta}{b_2^2} - \frac{12c_2 d_2 \Delta\Gamma e^{k_2}\eta}{(1+e^{k_2})^2} - \frac{6c_2 d_2^2 \Gamma^3 e^{2k_2}}{(1+e^{k_2})^3} + \frac{3c_2 d_2^2\Gamma^3 e^{k_2}}{(1+e^{k_2})^2}\right)\Bigg). \end{aligned} \tag{2.10}$$

From (2.10) and since $R > 0$, we deduce that if the following inequalities hold, then, $G'''(0) > 0$:

$$\eta\left(-\frac{c_1 d_1 \Delta e^{k_1}}{(1+e^{k_1})^2} - \frac{a_1\Delta^2\Gamma}{b_1^2} + \frac{a_2}{b_2^2} + \frac{c_2 d_2 \Delta\Gamma e^{k_2}}{(1+e^{k_2})^2}\right) > 0, \tag{2.11}$$

$$\Gamma^3\left(\frac{2a_1\Delta}{b_1^3} + \frac{c_2 d_2^2 e^{k_2}(e^{k_2}-1)}{(1+e^{k_2})^3}\right) > 0, \tag{2.12}$$

$$\frac{c_1 d_1^2 \Delta e^{k_1}(1-e^{k_1})}{(1+e^{k_1})^3} - \frac{2a_2}{b_2^3} > 0. \tag{2.13}$$

Now, from (2.3), we have that $d_2 > A_1$ and so from (2.9) we conclude that $\eta < 0$. Inequality (2.11) holds, since $\eta < 0$ and from (2.3) we obtain $d_2 > A_2$. Moreover, from (2.1), we obtain $c_1 > -\frac{1+e^{k_1}}{e^{k_1}}$ and therefore from (2.5) we get $\Gamma < 0$. Hence, (2.12) holds, since from (2.3) we have $d_2 > A_3$. Finally, (2.13) is always true, since $c_1, a_2 < 0$ and $\Delta, k_1 > 0$.

So, we have shown, that if the conditions in the proposition hold, then $G'''(0) > 0$. Hence, for the Schwarzian derivative (see [3], and [13]), we have $Sf(0) < 0$. Therefore, from Theorem 8 of [1], the zero equilibrium of (1.1) is stable. □

## 3. STABILITY OF ZERO EQUILIBRIUM OF SYSTEM (1.2)

In the following, we prove the stability of the zero equilibrium of System (1.2), using Centre Manifold Theory.

**Proposition 3.1.** *Consider System* (1.2) *where $b_1$, $b_2$, $c_1$, $c_2$, $k_1$, $k_2$ are real positive constants and $a_1$, $a_2$ are real negative constants such that*

$$-2 < \frac{a_1}{b_1} + \frac{a_2}{b_2} < 0, \quad -b_1 < a_1 < 0, \quad -b_2 < a_2 < 0. \tag{3.1}$$

*Let*

$$A_1 = \sqrt{\frac{2a_2(1+e^{k_1})^2(a_1+b_1)^4}{b_2^2(a_2+b_2)(1-e^{k_1})a_1^2e^{2k_1}}},$$
$$A_2 = \frac{a_2^2(a_1+b_1)^4(1+e^{k_1})^3(1+e^{k_2})(e^{k_2}-1)}{-2a_1b_1b_2^3e^{3k_1}e^{k_2}(a_2+b_2)},$$
$$A_3 = \left(1+\frac{a_1}{b_1}\right)\left(1+\frac{a_2}{b_2}\right)\frac{(1+e^{k_1})(1+e^{k_2})}{e^{k_1+k_2}},$$

$$K_1 = \frac{1+e^{k_1}}{b_2}\sqrt{\frac{2a_2}{(a_2+b_2)(1-e^{k_1})}},$$
$$K_2 = -\frac{a_1c_1e^{k_1}}{(a_1+b_1)^2},$$
$$K_3 = \sqrt{\frac{-2a_1(1+e^{k_2})^3(1+e^{k_1})(a_2+b_2)}{b_2c_1c_2b_1^3e^{k_1}e^{k_2}(e^{k_2}-1)}},$$
$$K_4 = -\frac{(a_1+b_1)^2a_2(1+e^{k_1})^2(1+e^{k_2})^2}{b_1^2b_2^2c_1^2c_2e^{2k_1}e^{k_2}}.$$

*Suppose also that the following relations hold:*

$$4(a_1+b_1)(a_2+b_2) < a_1a_2(e^{k_1}-1)(e^{k_2}-1), \quad A_1 < c_1 < \sqrt{\frac{A_2}{A_3}}, \quad c_2 \le \frac{A_3}{c_1} \tag{3.2}$$

*and*

$$K_1 < d_1 < K_2, \tag{3.3}$$
$$K_3 < d_2 < K_4. \tag{3.4}$$

*Then the zero equilibrium of* (1.2) *is stable.*

*Proof.* Firstly, we note that $4(a_1+b_1)(a_2+b_2) < a_1a_2(e^{k_1}-1)(e^{k_2}-1)$ implies that $A_1 < \sqrt{\frac{A_2}{A_3}}$. Next, we can easily see that $K_1 < K_2$ is true, since from (3.2), we have $c_1 > A_1$. Moreover, using $c_1c_2 < A_3$ and $c_1^2 < \frac{A_2}{A_3}$ from (3.2), we have $c_1^3c_2 < A_2$ and therefore $K_3 < K_4$ is true.

Now, the Jacobian matrix $J_0$ at the zero equilibrium for (1.2) is

$$J_0 = \begin{bmatrix} \frac{a_1}{b_1} & \frac{c_1e^{k_1}}{1+e^{k_1}} \\ \frac{c_2e^{k_2}}{1+e^{k_2}} & \frac{a_2}{b_2} \end{bmatrix}.$$

Calculating the eigenvalues of $J_0$, using (3.1) and (3.2) we obtain

$$\lambda_1 = -1, \quad \lambda_2 = 1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} \quad \text{and so} \quad |\lambda_2| < 1.$$

Now, the initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \tag{3.5}$$

where

$$\begin{aligned} f(x,y) &= \frac{a_1 x}{x+b_1} - \frac{a_1 x}{b_1} + \frac{c_1 y e^{k_1 - d_1 y}}{1+e^{k_1-d_1 y}} - \frac{c_1 e^{k_1}}{1+e^{k_1}} y, \\ g(x,y) &= \frac{a_2 y}{y+b_2} - \frac{a_2 y}{b_2} + \frac{c_2 x e^{k_2 - d_2 x}}{1+e^{k_2-d_2 x}} - \frac{c_2 e^{k_2}}{1+e^{k_2}} x. \end{aligned}$$

We let now

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where $T$ is the matrix that diagonalizes $J_0$ defined by

$$T = \begin{bmatrix} \Gamma & \Delta \\ 1 & 1 \end{bmatrix},$$

where

$$\Gamma = -\frac{b_1 c_1 e^{k_1}}{(b_1+a_1)(1+e^{k_1})}, \quad \Delta = \frac{b_2 c_1 e^{k_1}}{(a_2+b_2)(1+e^{k_1})}. \tag{3.6}$$

Then, (3.5) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \tag{3.7}$$

where

$$\begin{aligned} &\hat{f}(u,v) \\ &= R\Bigg(\frac{a_1(\Gamma u+\Delta v)}{b_1+\Gamma u+\Delta v} - \frac{a_1(\Gamma u+\Delta v)}{b_1} + \frac{c_1(u+v)e^{k_1-d_1(u+v)}}{1+e^{k_1-d_1(u+v)}} - \frac{c_1(u+v)e^{k_1}}{1+e^{k_1}} \\ &\quad - \Delta\bigg(\frac{a_2(u+v)}{b_2+u+v} - \frac{a_2(u+v)}{b_2} + \frac{c_2(\Gamma u+\Delta v)e^{k_2-d_2(\Gamma u+\Delta v)}}{1+e^{k_2-d_2(\Gamma u+\Delta v)}} - \frac{c_2(\Gamma u+\Delta v)e^{k_2}}{1+e^{k_2}}\bigg)\Bigg), \\ &\hat{g}(u,v) \\ &= -R\Bigg(\frac{a_1(\Gamma u+\Delta v)}{b_1+\Gamma u+\Delta v} - \frac{a_1(\Gamma u+\Delta v)}{b_1} + \frac{c_1(u+v)e^{k_1-d_1(u+v)}}{1+e^{k_1-d_1(u+v)}} - \frac{c_1(u+v)e^{k_1}}{1+e^{k_1}} \\ &\quad - \Gamma\bigg(\frac{a_2(u+v)}{b_2+u+v} - \frac{a_2(u+v)}{b_2} + \frac{c_2(\Gamma u+\Delta v)e^{k_2-d_2(\Gamma u+\Delta v)}}{1+e^{k_2-d_2(\Gamma u+\Delta v)}} - \frac{c_2(\Gamma u+\Delta v)e^{k_2}}{1+e^{k_2}}\bigg)\Bigg) \end{aligned} \tag{3.8}$$

and

$$R = \frac{1}{\Gamma - \Delta}.$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$, $\psi(u) = \eta u^2 + \theta u^3$, $\eta, \theta \in \mathbb{R}$. Using this approximation is justified by Theorem 7 of [1]. Consequently, according to Theorem 8 of [1] and using (3.7), the study of the stability of the zero equilibrium of System (1.2) reduces to the study of the stability of the zero equilibrium of the equation

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \tag{3.9}$$

We now need to determine $\eta$ (the coefficient in the Taylor expansion). From (3.7), we conclude that map $h$ must satisfy the centre manifold equation (see [1, p. 34], [3, p. 243], [15], [16, p. 642] and [19]):

$$h(-u + \hat{f}(u, h(u))) - \lambda_2 h(u) - \hat{g}(u, h(u)) = 0.$$

Keeping the terms up to $u^3$ and using (3.8), we obtain

$$\eta = \frac{R}{1-\lambda_2}\Big(\frac{c_1 d_1 e^{k_1}}{(1+e^{k_1})^2} + \frac{a_1 \Gamma^2}{b_1^2} - \Gamma\Big(\frac{a_2}{b_2^2} + \frac{c_2 d_2 \Gamma^2 e^{k_2}}{(1+e^{k_2})^2}\Big)\Big). \tag{3.10}$$

From (3.9) and (3.10) we obtain $G'(0) = -1$ and

$$\begin{aligned} G'''(0) = R\Big(&-\frac{6c_1 d_1^2 e^{2k_1}}{(1+e^{k_1})^3} + \frac{3c_1 d_1^2 e^{k_1}}{(1+e^{k_1})^2} - \frac{12 c_1 d_1 \eta e^{k_1}}{(1+e^{k_1})^2} - \frac{12 a_1 \Delta \Gamma \eta}{b_1^2} + \frac{6 a_1 \Gamma^3}{b_1^3} \\ &- \Delta\Big(\frac{6a_2}{b_2^3} - \frac{12 a_2 \eta}{b_2^2} - \frac{12 c_2 d_2 \Delta \Gamma e^{k_2} \eta}{(1+e^{k_2})^2} - \frac{6 c_2 d_2^2 \Gamma^3 e^{2k_2}}{(1+e^{k_2})^3} + \frac{3 c_2 d_2^2 \Gamma^3 e^{k_2}}{(1+e^{k_2})^2}\Big)\Big). \end{aligned} \tag{3.11}$$

From (3.11) and since $R < 0$, we deduce that if the following inequalities hold, then, $G'''(0) > 0$:

$$\eta\Big(-\frac{c_1 d_1 e^{k_1}}{(1+e^{k_1})^2} - \frac{a_1 \Delta \Gamma}{b_1^2} + \frac{a_2 \Delta}{b_2^2} + \frac{c_2 d_2 \Delta^2 \Gamma e^{k_2}}{(1+e^{k_2})^2}\Big) < 0, \tag{3.12}$$

$$\Gamma^3\Big(\frac{2a_1}{b_1^3} + \frac{2 c_2 d_2^2 \Delta e^{2k_2}}{(1+e^{k_2})^3} - \frac{c_2 d_2^2 \Delta e^{k_2}}{(1+e^{k_2})^2}\Big) < 0, \tag{3.13}$$

$$-\frac{2 c_1 d_1^2 e^{2k_1}}{(1+e^{k_1})^3} + \frac{c_1 d_1^2 e^{k_1}}{(1+e^{k_1})^2} - \frac{2\Delta a_2}{b_2^3} < 0. \tag{3.14}$$

Now, from the second inequalites of (3.3) and (3.4), we obtain from (3.10) that $\eta > 0$. In addition, from conditions (3.1) and equation (3.6) we obtain $\Gamma < 0$. Hence, (3.12) holds, since $\eta > 0$, $a_1, a_2, \Gamma < 0$ and $c_1, c_2, d_1, d_2 > 0$. Moreover, from the first inequality of (3.4) and since $\Gamma < 0$, we deduce that (3.13) is true. Finally, from the first inequality of (3.3), we obtain that (3.14) is also true.

So, we have shown, that if the conditions in the proposition hold, then $G'''(0) > 0$. Hence, for the Schwarzian derivative, we have $Sf(0) < 0$. Therefore, from Theorem 8 of [1], the zero equilibrium of (1.2) is stable. □

## 4. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.3)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.3). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

**Theorem 4.1.** *Let $f$, $g$, with $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let $a$, $A$, $b$, $B$ be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \to [a, A]$, $g : [a, A] \times [b, B] \to [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \tag{4.1}$$

*Suppose that $f(x, y)$ is a non-increasing function with respect to $x$ and a non-decreasing with respect to $y$. Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to $x$ and a non-increasing function with respect to $y$. Finally suppose that, if $m$, $M$, $r$, $R$ are real numbers such that if $M = f(m, R)$, $m = f(M, r)$, $R = g(m, r)$, $r = g(M, R)$, then $m = M$ and $r = R$. Then the system of difference equations* (4.1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of System* (4.1) *which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of* (4.1)*.*

We now prove the following result.

**Proposition 4.2.** *Consider System* (1.3)*. Suppose that the following relations hold true:*

$$k_1 c_1 < 1, \quad k_3 c_2 < 1 \tag{4.2}$$

*and*

$$a_1 a_2 k_2 < (1 - k_1 c_1)(1 - k_3 c_2). \tag{4.3}$$

*Then System* (1.3) *has a unique positive equilibrium and every positive solution of System* (1.3) *tends to the unique equilibrium of* (1.3) *as $n \to \infty$.*

*Proof.* System (1.3) can be written as $x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{y^2}{b_1 + y} + c_1 \frac{e^{d_1 - k_1 x}}{1 + e^{d_1 - k_1 x}} \tag{4.4}$$

and

$$g(x, y) = a_2 \frac{e^{b_2 - k_2 x}}{1 + e^{b_2 - k_2 x}} + c_2 \frac{e^{d_2 - k_3 y}}{1 + e^{d_2 - k_3 y}}. \tag{4.5}$$

We can now easily see that $f(x, y)$ is non-increasing in $x$ and non-decreasing in $y$, while $g(x, y)$ is non-increasing in $x$ and non-increasing in $y$.

We now show that $f : [a, A] \times [b, B] \to [a, A]$ and $g : [a, A] \times [b, B] \to [b, B]$, where

$$B = a_2 + c_2, \quad A = a_1 \frac{B^2}{b_1 + B} + c_1,$$

$$b = a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} \quad \text{and} \quad a = a_1 \frac{b^2}{b_1 + b} + c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}}.$$

Indeed, from (4.5), we can easily obtain that $g(x, y) \leq a_2 + c_2 = B$. Now, for $y \leq B$, from (4.4), we get $f(x, y) \leq a_1 \frac{B^2}{b_1 + B} + c_1 = A$. Moreover, for $x \leq A$ and $y \leq B$ from (4.5), we obtain

$$g(x, y) \geq a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} = b.$$

Finally, taking $x \leq A$, $y \geq b$ from (4.4), we conclude that

$$f(x, y) \geq a_1 \frac{b^2}{b_1 + b} + c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}} = a.$$

Now, let $m, M, r, R$ be positive real numbers such that $M = f(m, R)$, $m = f(M, r)$, $R = g(m, r)$ and $r = g(M, R)$. Therefore, we obtain $M - m = f(m, R) - f(M, r)$ and $R - r = g(m, r) - g(M, R)$. So, we can write

$$M - m = f(m, R) - f(M, R) + f(M, R) - f(M, r)$$

and

$$R - r = g(m, r) - g(M, r) + g(M, r) - g(M, R).$$

Using the Mean Value Theorem, we obtain

$$M - m = f_x(\xi_1, R)(m - M) + f_y(M, \xi_2)(R - r)$$

and

$$R - r = g_x(\xi_3, r)(m - M) + g_y(M, \xi_4)(r - R)$$

for some $\xi_1, \xi_3 \in (m, M)$ and $\xi_2, \xi_4 \in (r, R)$. Therefore, we can write

$$|M - m| \leq |f_x(\xi_1, R)||m - M| + |f_y(M, \xi_2)||R - r| \tag{4.6}$$

and

$$|R - r| \leq |g_x(\xi_3, r)||m - M| + |g_y(M, \xi_4)||r - R|. \tag{4.7}$$

However, we have

$$f_x(\xi_1, R) = \frac{-c_1 k_1 e^{d_1 - k_1 \xi_1}}{(1 + e^{d_1 - k_1 \xi_1})^2}, \quad f_y(M, \xi_2) = a_1 \frac{2 b_1 \xi_2 + \xi_2^2}{(b_1 + \xi_2)^2}, \tag{4.8}$$

$$g_x(\xi_3, r) = \frac{-a_2 k_2 e^{b_2 - k_2 \xi_3}}{(1 + e^{b_2 - k_2 \xi_3})^2}, \quad g_y(M, \xi_4) = \frac{-c_2 k_3 e^{d_2 - k_3 \xi_4}}{(1 + e^{d_2 - k_3 \xi_4})^2}. \tag{4.9}$$

From (4.6), (4.7), (4.8), (4.9), we obtain

$$|M - m|(1 - k_1 c_1) \leq a_1 |R - r|, \quad |R - r|(1 - k_3 c_2) \leq a_2 k_2 |M - m|. \tag{4.10}$$

Therefore, from (4.10), we conclude that

$$|M-m| \le \frac{a_1 a_2 k_2}{(1-k_1c_1)(1-k_3c_2)}|M-m|. \tag{4.11}$$

Hence, from (4.3), (4.10) and (4.11), we obtain

$$M=m, \quad R=r.$$

Let now $(x_n, y_n)$ be an arbitrary solution of (1.3). From the discussion above, it is obvious that $y_1 \le B$. Then we can see that $x_2 \le A$. In addition, we also have $y_2 \le B$ and so we can get $y_3 \ge b$. Finally, since we also have $x_3 \le A$, we obtain $x_4 \ge a$. Hence, we have shown that $x_n \in [a, A]$, $y_n \in [b, B]$, for all $n \ge 4$.

Therefore, from Theorem 4.1, System (1.3) has a unique positive equilibrium $(\tilde{x}, \tilde{y})$ and every positive solution of System (1.3) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

**Proposition 4.3.** *Consider System* (1.3), *where the conditions in Proposition 4.2 hold. In addition, suppose that the following relation holds true:*

$$k_1k_3c_1c_2 + k_2a_1a_2 < 1. \tag{4.12}$$

*Then the unique positive equilibrium* $(\tilde{x}, \tilde{y})$ *of System* (1.3) *is globally asymptotically stable.*

*Proof.* First, we will prove that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. The linearised system of (1.3) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

where

$$\alpha = -\frac{c_1k_1e^{d_1-k_1\tilde{x}}}{(1+e^{d_1-k_1\tilde{x}})^2}, \quad \beta = \frac{a_1(\tilde{y}^2+2b_1\tilde{y})}{(b_1+\tilde{y})^2},$$
$$\gamma = -\frac{a_2k_2e^{b_2-k_2\tilde{x}}}{(1+e^{b_2-k_2\tilde{x}})^2}, \quad \delta = -\frac{c_2k_3e^{d_2-k_3\tilde{y}}}{(1+e^{d_2-k_3\tilde{y}})^2}.$$

The characteristic equation of $A$ is $\lambda^2-(\alpha+\delta)\lambda+\alpha\delta-\beta\gamma=0$. From relation (4.12), we obtain $\alpha\delta-\beta\gamma<1$. Moreover, form (4.2), we obtain $|\alpha+\delta|<1+\alpha\delta-\beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. Using Proposition 4.2, we conclude that $(\tilde{x}, \tilde{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

## 5. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.4)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.4). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

**Theorem 5.1.** *Let $f$, $g$, with $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let $a$, $A$, $b$, $B$ be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \to [a, A]$, $g : [a, A] \times [b, B] \to [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \tag{5.1}$$

*Suppose that $f(x, y)$ is a non-increasing function with respect to $x$ and a non-decreasing with respect to $y$. Moreover, suppose that $g(x, y)$ is a non-decreasing function with respect to $x$ and a non-increasing function with respect to $y$. Finally suppose that, if $m$, $M$, $r$, $R$ are real numbers such that if $m = f(M, r)$, $M = f(m, R)$, $r = g(m, R)$, $R = g(M, r)$, then $m = M$ and $r = R$. Then the system of difference equations* (5.1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system* (5.1) *which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of* (5.1)*.*

We now prove the following result.

**Proposition 5.2.** *Consider system* (1.4)*. Suppose that the following relations hold true:*

$$k_1 c_1 < 1, \quad k_2 c_2 < 1, \quad a_1, a_2 < 1 \tag{5.2}$$

*and*

$$a_1 a_2 < (1 - k_1 c_1)(1 - k_2 c_2). \tag{5.3}$$

*Then System* (1.4) *has a unique positive equilibrium and every positive solution of the System* (1.4) *tends to the unique equilibrium of* (1.4) *as $n \to \infty$.*

*Proof.* System (1.4) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{y^2}{b_1 + y} + c_1 \frac{e^{d_1 - k_1 x}}{1 + e^{d_1 - k_1 x}} \tag{5.4}$$

and

$$g(x, y) = a_2 \frac{x^2}{b_2 + x} + c_2 \frac{e^{d_2 - k_2 y}}{1 + e^{d_2 - k_2 y}}. \tag{5.5}$$

We can now easily see that $f(x, y)$ is non-increasing in $x$ and non-decreasing in $y$, while $g(x, y)$ is non-decreasing in $x$ and non-increasing in $y$.

We now show that $f : [a, A] \times [b, B] \to [a, A]$ and $g : [a, A] \times [b, B] \to [b, B]$, where

$$A = \frac{a_1 c_2 + c_1}{1 - a_1 a_2} + \theta, \quad B = \frac{a_2 c_1 + c_2}{1 - a_1 a_2} + \theta,$$

$$a = c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}} \quad \text{and} \quad b = c_2 \frac{e^{d_2 - k_2 B}}{1 + e^{d_2 - k_2 B}}$$

for some $\theta \in (0, \infty)$. If $y \le B$, from (5.4), we have

$$f(x, y) \le a_1 y + c_1 \le a_1 \Big( \frac{a_2 c_1 + c_2}{1 - a_1 a_2} + \theta \Big) + c_1 \le A.$$

Next, if $x \le A$, from (5.5), we get

$$g(x,y) \le a_2\Big(\frac{a_1c_2+c_1}{1-a_1a_2}+\theta\Big)+c_2 \le B.$$

Moreover, if $x \le A$, from (5.4), we obtain

$$f(x,y) \ge c_1\frac{e^{d_1-k_1A}}{1+e^{d_1-k_1A}} = a.$$

Finally, for $y \le B$, from (5.5) we obtain

$$g(x,y) \ge c_2\frac{e^{d_2-k_2B}}{1+e^{d_2-k_2B}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers $m$, $M$, $r$, $R$ such that if $m = f(M,r)$, $M = f(m,R)$, $r = g(m,R)$, $R = g(M,r)$, then $m = M$ and $r = R$.

Moreover, from (1.4), we can see that $x_{n+1} \le a_1y_n + c_1$, $y_{n+1} \le a_2x_n + c_2$. So, we get $x_{n+2} \le a_1a_2x_n + a_1c_2 + c_1$, $y_{n+2} \le a_1a_2y_n + a_2c_1 + c_2$. We now let $w_{n+2} = a_1a_2w_n + a_1c_2 + c_1$ and $z_{n+2} = a_1a_2z_n + a_2c_1 + c_2$ with $x_0 = w_0$, $y_0 = z_0$, $x_1 = w_1$ and $y_1 = z_1$. These two equations are essentially linear difference equations of first order, so solvable ones (many recent equations and systems are solved by using the equation such as [25–27, 29, 33]). Solving these equations, we obtain

$$w_n = A_1(a_1a_2)^{\frac{n}{2}} + A_2(-1)^n(a_1a_2)^{\frac{n}{2}} + \frac{a_1c_2+c_1}{1-a_1a_2}$$

and

$$z_n = B_1(a_1a_2)^{\frac{n}{2}} + B_2(-1)^n(a_1a_2)^{\frac{n}{2}} + \frac{a_2c_1+c_2}{1-a_1a_2},$$

respectively, where $A_1$, $A_2$, $B_1$, $B_2$ depend on the initial conditions. Now, since $x_0 = w_0$, $y_0 = z_0$, $x_1 = w_1$, $y_1 = z_1$, working inductively, we obtain that $x_n \le w_n$, $y_n \le z_n$, $n \in \mathbb{N}$, and therefore we conclude that for some $n_0 \in \mathbb{N}^+$ and $\theta \in (0,\infty)$, we have $x_n \le \frac{a_1c_2+c_1}{1-a_1a_2} + \theta = A$ and $y_n \le \frac{a_2c_1+c_2}{1-a_1a_2} + \theta = B$, $n \ge n_0$. Hence, we can easily see that $x_n \ge a$, $n > n_0$ and $y_n \ge b$, $n > n_0$. So, we have obtained that $x_n \in [a,A]$ and $y_n \in [b,B]$, $n > n_0$.

Therefore, using Theorem 5.1, we deduce that System (1.4) has a unique positive equilibrium $(\tilde{x},\tilde{y})$, and every positive solution of System (1.4) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

**Proposition 5.3.** *Consider System* (1.4)*. If the conditions in Proposition 5.2 hold, then the unique positive equilibrium* $(\tilde{x},\tilde{y})$ *of System* (1.4) *is globally asymptotically stable.*

*Proof.* Firstly, we will prove that $(\tilde{x},\tilde{y})$ is locally asymptotically stable. The linearised system of (1.4) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = -\frac{c_1 k_1 e^{d_1 - k_1 \tilde{x}}}{(1 + e^{d_1 - k_1 \tilde{x}})^2}, \quad \beta = \frac{a_1(\tilde{y}^2 + 2b_1\tilde{y})}{(b_1 + \tilde{y})^2},$$

$$\gamma = \frac{a_2(\tilde{x}^2 + 2b_2\tilde{x})}{(b_2 + \tilde{x})^2}, \quad \delta = -\frac{c_2 k_2 e^{d_2 - k_2 \tilde{y}}}{(1 + e^{d_2 - k_2 \tilde{y}})^2}.$$

The characteristic equation of $A$ is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. Therefore, from relation (5.2), we obtain $\alpha\delta - \beta\gamma < 1$ and from (5.3), we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. Using Proposition 5.2, we conclude that $(\tilde{x}, \tilde{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

## 6. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.5)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.5). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

**Theorem 6.1.** *Let $f$, $g$, with $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let $a$, $A$, $b$, $B$ be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \to [a, A]$, $g : [a, A] \times [b, B] \to [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \ldots \tag{6.1}$$

*Suppose that $f(x, y)$ is a non-decreasing function with respect to $x$ and a non-increasing with respect to $y$. Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to $x$ and a non-increasing function with respect to $y$. Finally suppose that, if $m$, $M$, $r$, $R$ are real numbers such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, r)$, $r = g(M, R)$, then $m = M$ and $r = R$. Then the system of difference equations* (6.1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system* (6.1) *which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of* (6.1).

We now prove the following result.

**Proposition 6.2.** *Consider System* (1.5)*. Suppose that the following relations hold true:*

$$k_3 c_2 < 1, \quad a_1 < 1 \tag{6.2}$$

*and*

$$k_1 c_1 a_2 k_2 < (1 - k_3 c_2)(1 - a_1). \tag{6.3}$$

*Then System* (1.5) *has a unique positive equilibrium and every positive solution of System* (1.5) *tends to the unique equilibrium of* (1.5) *as $n \to \infty$.*

*Proof.* System (1.5) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x,y) = a_1 \frac{x^2}{b_1 + x} + c_1 \frac{e^{d_1 - k_1 y}}{1 + e^{d_1 - k_1 y}} \tag{6.4}$$

and

$$g(x,y) = a_2 \frac{e^{b_2 - k_2 x}}{1 + e^{b_2 - k_2 x}} + c_2 \frac{e^{d_2 - k_3 y}}{1 + e^{d_2 - k_3 y}}. \tag{6.5}$$

We can now easily see that $f(x,y)$ is non-decreasing in $x$ and non-increasing in $y$, while $g(x,y)$ is non-increasing in $x$ and non-increasing in $y$.

We now show that $f : [a,A] \times [b,B] \to [a,A]$ and $g : [a,A] \times [b,B] \to [b,B]$, where

$$A = \frac{c_1}{1 - a_1} + \theta, \quad B = a_2 + c_2,$$

$$a = c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} \quad \text{and} \quad b = a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}}$$

for some $\theta \in (0, \infty)$. If $x \le A$, then from (6.4) we get

$$f(x,y) \le a_1 x + c_1 \le a_1 \Big( \frac{c_1}{1 - a_1} + \theta \Big) + c_1 \le A.$$

Now, from (6.5), we can easily see that $g(x,y) \le a_2 + c_2 = B$. So, if $y \le B$, from (6.4), we obtain

$$f(x,y) \ge c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} = a.$$

Finally, if $x \le A$ and $y \le B$, from (6.5) we obtain

$$g(x,y) \ge a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers $m$, $M$, $r$, $R$ such that if $M = f(M,r)$, $m = f(m,R)$, $R = g(m,r)$, $r = g(M,R)$, then $m = M$ and $r = R$.

Moreover, we have $x_{n+1} \le a_1 x_n + c_1$, $n \in \mathbb{N}$. We consider the difference equation $z_{n+1} = a_1 z_n + c_1$, $n \in \mathbb{N}$ and we let $z_n$ be the solution such that $z_0 = x_0$. Then, we have $z_n = (x_0 - \frac{c_1}{1-a_1}) a_1^n + \frac{c_1}{1-a_1}$, $n \in \mathbb{N}$. Now, since $z_0 = x_0$, and working inductively, we can show that $x_n \le z_n$, $n \in \mathbb{N}$. Therefore, since $a_1 \in (0,1)$, we get that for some $\theta \in (0,\infty)$ and $n_0 \in \mathbb{N}^+$, we have $x_n \le \frac{c_1}{1-a_1} + \theta = A$, $n \ge n_0$. Finally, we can also easily see that $x_n \ge a$, $n \ge 2$, and $y_n \le B$, $n \ge 1$, and $y_n \ge b$, $n > n_0$. Hence, we have obtained that $x_n \in [a,A]$ and $y_n \in [b,B]$, $n > n_0$.

Therefore, using Theorem 6.1, System (1.5) has a unique positive equilibrium $(\tilde{x}, \tilde{y})$, and every positive solution of System (1.5) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

**Proposition 6.3.** *Consider System* (1.5). *If the conditions in Proposition 6.2 hold, then the unique positive equilibrium* $(\tilde{x}, \tilde{y})$ *of System* (1.5) *is globally asymptotically stable.*

*Proof.* Firstly, we will prove that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. The linearised system of (1.5) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = \frac{a_1(\tilde{x}^2 + 2b_1\tilde{x})}{(b_1 + \tilde{x})^2}, \quad \beta = -\frac{c_1 k_1 e^{d_1 - k_1\tilde{y}}}{(1 + e^{d_1 - k_1\tilde{y}})^2},$$
$$\gamma = -\frac{a_2 k_2 e^{b_2 - k_2\tilde{x}}}{(1 + e^{b_2 - k_2\tilde{x}})^2}, \quad \delta = -\frac{c_2 k_3 e^{d_2 - k_3\tilde{y}}}{(1 + e^{d_2 - k_3\tilde{y}})^2}.$$

The characteristic equation of $A$ is $\lambda^2 - (\alpha+\delta)\lambda + \alpha\delta - \beta\gamma = 0$. Now, $\alpha\delta - \beta\gamma < 0 < 1$. Moreover, from relation (6.3) we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. Using Proposition 6.2, we conclude that $(\tilde{x}, \tilde{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

## 7. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.6)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.6). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

**Theorem 7.1.** *Let $f$, $g$, with $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let $a$, $A$, $b$, $B$ be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \to [a, A]$, $g : [a, A] \times [b, B] \to [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \ldots \tag{7.1}$$

*Suppose that $f(x, y)$ is a non-decreasing function with respect to $x$ and a non-increasing with respect to $y$. Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to $x$ and a non-decreasing function with respect to $y$. Finally suppose that, if $m$, $M$, $r$, $R$ are real numbers such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, R)$, $r = g(M, r)$, then $m = M$ and $r = R$. Then the system of difference equations* (7.1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of System* (7.1) *which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of* (7.1).

We now prove the following result.

**Proposition 7.2.** *Consider System* (1.6)*. Suppose that the following relations hold true:*

$$a_1, a_2 < 1 \tag{7.2}$$

*and*

$$c_1 c_2 k_1 k_2 < (1 - a_1)(1 - a_2). \tag{7.3}$$

*Then System* (1.6) *has a unique positive equilibrium and every positive solution of System* (1.6) *tends to the unique equilibrium of* (1.6) *as* $n \to \infty$.

*Proof.* System (1.6) can be written as $x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n)$, where

$$f(x,y) = a_1 \frac{x^2}{b_1 + x} + c_1 \frac{e^{d_1 - k_1 y}}{1 + e^{d_1 - k_1 y}} \tag{7.4}$$

and

$$g(x,y) = a_2 \frac{y^2}{b_2 + y} + c_2 \frac{e^{d_2 - k_2 x}}{1 + e^{d_2 - k_2 x}}. \tag{7.5}$$

We can now easily see that $f(x,y)$ is non-decreasing in $x$ and non-increasing in $y$, while $g(x,y)$ is non-increasing in $x$ and non-decreasing in $y$.

We now show that $f : [a, A] \times [b, B] \to [a, A]$ and $g : [a, A] \times [b, B] \to [b, B]$, where

$$A = \frac{c_1}{1 - a_1} + \theta, \quad B = \frac{c_2}{1 - a_2} + \theta, \quad a = c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} \quad \text{and} \quad b = c_2 \frac{e^{d_2 - k_2 A}}{1 + e^{d_2 - k_2 A}}$$

for some $\theta \in (0, \infty)$. If $x \le A$, from (7.4) we have

$$f(x,y) \le a_1 \Big( \frac{c_1}{1 - a_1} + \theta \Big) + c_1 \le A$$

and if $y \le B$, from (7.5) we obtain

$$g(x,y) \le a_2 \Big( \frac{c_2}{1 - a_2} + \theta \Big) + c_2 \le B.$$

Moreover, for $y \le B$, from (7.4) we get

$$f(x,y) \ge c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} = a$$

and for $x \le A$, from (7.5) we obtain

$$g(x,y) \ge c_2 \frac{e^{d_2 - k_2 A}}{1 + e^{d_2 - k_2 A}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers $m$, $M$, $r$, $R$ such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, R)$, $r = g(M, r)$, then $m = M$ and $r = R$.

Moreover, we have $x_{n+1} \le a_1 x_n + c_1$, $n \in \mathbb{N}$. We consider the difference equation $z_{n+1} = a_1 z_n + c_1$, $n \in \mathbb{N}$ and we let $z_n$ be the solution such that $z_0 = x_0$. Then, we have

$$z_n = \Big( x_0 - \frac{c_1}{1 - a_1} \Big) a_1^n + \frac{c_1}{1 - a_1}, \quad n \in \mathbb{N}.$$

Now, since $z_0 = x_0$, and working inductively, we can show that $x_n \le z_n$, $n \in \mathbb{N}$. Therefore, since $a_1 \in (0,1)$ we get that for some $\theta \in (0, \infty)$ and $n_0 \in \mathbb{N}^+$, we have $x_n \le A = \frac{c_1}{1 - a_1} + \theta$, $n \ge n_0$. In the same way, we can prove that for some $\theta \in (0, \infty)$

and $n_0 \in \mathbb{N}^+$, we have $y_n \leq B = \frac{c_2}{1-a_2} + \theta$, $n \geq n_0$. So, we can now easily see that $x_n \geq a$, $n > n_0$, and $y_n \geq b$, $n > n_0$. Hence, we have obtained that $x_n \in [a, A]$ and $y_n \in [b, B]$, $n > n_0$.

Therefore, using Theorem 7.1, System (1.6) has a unique positive equilibrium $(\tilde{x}, \tilde{y})$, and every positive solution of System (1.6) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

**Proposition 7.3.** *Consider System* (1.6)*. If the conditions in Proposition 7.2 hold, then the unique positive equilibrium* $(\tilde{x}, \tilde{y})$ *of System* (1.6) *is globally asymptotically stable.*

*Proof.* First, we will prove that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. The linearised system of (1.6) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = \frac{a_1(\tilde{x}^2 + 2b_1\tilde{x})}{(b_1 + \tilde{x})^2}, \quad \beta = -\frac{c_1 k_1 e^{d_1 - k_1 \tilde{y}}}{(1 + e^{d_1 - k_1 \tilde{y}})^2},$$

$$\gamma = -\frac{c_2 k_2 e^{d_2 - k_2 \tilde{x}}}{(1 + e^{d_2 - k_2 \tilde{x}})^2}, \quad \delta = \frac{a_2(\tilde{y}^2 + 2b_2\tilde{y})}{(b_2 + \tilde{y})^2}.$$

The characteristic equation of $A$ is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. Now, from (7.2), we obtain that $\alpha\delta - \beta\gamma < \alpha\delta < 1$. Moreover, from relation (7.3) we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that $(\tilde{x}, \tilde{y})$ is locally asymptotically stable. Using Proposition 7.2, we conclude that $(\tilde{x}, \tilde{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

**Acknowledgements**
*The authors would like to thank the referees for their helpful suggestions.*

REFERENCES

[1] J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[2] D. Clark, M.R.S. Kulenovic, J.F. Selgrade, *Global asymptotic behavior of a two--dimensional difference equation modelling competition*, Nonlinear Anal. **52** (2003), 1765–1776.

[3] S. Elaydi, *Discrete Chaos*, 2nd ed., Chapman & Hall/CRC, Boca Raton, London, New York, 2008.

[4] E. El-Metwally, E.A. Grove, G. Ladas, R. Levins, M. Radin, *On the difference equation,* $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, Nonlinear Anal. **47** (2001), 4623–4634.

[5] E.A. Grove, G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC, 2005.

[6] E.A. Grove, G. Ladas, N.R. Prokup, R. Levis, *On the global behavior of solutions of a biological model*, Comm. Appl. Nonlinear Anal. **7** (2000), 33–46.

[7] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.

[8] L. Gutnik, S. Stević, *On the behaviour of the solutions of a second order difference equation*, Discrete Dyn. Nat. Soc. **14** (2007), Article ID 27562.

[9] B. Iričanin, S. Stević, *Some systems of nonlinear difference equations of higher order with periodic solutions*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **13** (3–4) (2006), 499–508.

[10] B. Iričanin, S. Stević, *Eventually constant solutions of a rational difference equation*, Appl. Math. Comput. **215** (2009), 854–856.

[11] B. Iričanin, S. Stević, *On two systems of difference equations*, Discrete Dyn. Nat. Soc. **4** (2010), Article ID 405121.

[12] C.M. Kent, *Converegence of solutions in a nonhyperbolic case*, Nonlinear Anal. **47** (2001), 4651–4665.

[13] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order With Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[14] M.R.S. Kulenovic, G. Ladas, *Dynamics of Second Order Rational Difference Equations*, Chapman & Hall/CRC, 2002.

[15] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Second Edition, Applied Mathematical Sciences, vol. 112, Spinger, New York, 1998.

[16] R. Luis, S. Elaydi, H. Oliveira, *Stability of a Ricker-type competion model and the competitive exclusion principle*, J. Biol. Dyn. **5** (2011) 6, 636–660.

[17] I. Ozturk, F. Bozkurt, S. Ozen, *On the difference equation* $y_{n+1} = \frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}$, Appl. Math. Comput. **181** (2006), 1387–1393.

[18] G. Papaschinopoulos, C.J. Schinas, G. Ellina, *On the dynamics of the solutions of a biological model*, J. Difference Equ. Appl. **20** (2014) 5–6, 694–705.

[19] N. Psarros, G. Papaschinopoulos, C.J. Schinas, *Study of the stability of a 3 × 3 system of difference equations using centre manifold theory*, Appl. Math. Lett., DOI: 10.1016/j.aml.2016.09.002 (2016).

[20] S. Stević, *Asymptotic behaviour of a sequence defined by iteration with applications*, Colloq. Math. **93** (2002) 1, 267–276.

[21] S. Stević, *On the recursive sequence* $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$, Taiwanese J. Math. **7** (2003) 2, 249–259.

[22] S. Stević, *Asymptotic behaviour of a nonlinear difference equation*, Indian J. Pure Appl. Math. **34** (2003) 12, 1681–1687.

[23] S. Stević, *A short proof of the Cushing-Henson conjecture*, Discrete Dyn. Nat. Soc. 2006 (2006), Article ID 37264.

[24] S. Stević, *On a discrete epidemic model*, Discrete Dyn. Nat. Soc. **10** (2007), Article ID 87519.

[25] S. Stević, *On a system of difference equations*, Appl. Math. Comput. **218** (2011), 3372–3378.

[26] S. Stević, *On a system of difference equations with period two coefficients*, Appl. Math. Comput. **218** (2011), 4317–4324.

[27] S. Stević, *On a third-order system of difference equations*, Appl. Math. Comput. **218** (2012), 7649–7654.

[28] S. Stević, *On some periodic systems of max-type difference equations*, Appl. Math. Comput. **218** (2012), 11483–11487.

[29] S. Stević, *On some solvable systems of difference equations*, Appl. Math. Comput. **218** (2012), 5010–5018.

[30] S. Stević, *On a solvable system of difference equations of $k$th order*, Appl. Math. Comput. **219** (2013), 7765–7771.

[31] S. Stević, *On a cyclic system of difference equations*, J. Difference Equ. Appl. **20** (2014) (5–6), 733–743.

[32] S. Stević, *Boundedness and peristence of some cyclic-type systems of difference equations*, Appl. Math. Lett. **56** (2016), 78–85.

[33] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda, *On a third-order system of difference equations with variable coefficients*, Abstr. Appl. Anal. **2012** (2012), Article ID 508523.

[34] D. Tilman, D. Wedin, *Oscillations and chaos in the dynamics of a perennial grass*, Nature **353** (1991), 653–655.

N. Psarros
nikosnpsarros@hotmail.com

Democritus University of Thrace
School of Engineering
Xanthi, 67100, Greece

G. Papaschinopoulos
gpapas@env.duth.gr

Democritus University of Thrace
School of Engineering
Xanthi, 67100, Greece

C.J. Schinas
cschinas@ee.duth.gr

Democritus University of Thrace
School of Engineering
Xanthi, 67100, Greece

*Received: March 30, 2017.*
*Revised: April 29, 2017.*
*Accepted: May 1, 2017.*