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SURFACES WITH CONSTANT SLOPE AND THEIR GENERALISATION

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Abstract: Surfaces with a constant slope with respect to the given surface π are defined in the first part of the paper, which may not be developable in relation to the surfaces of a constant slope. It is shown that rotational conical surface and one-sheet rotational hyperboloid are the only two rotational surfaces with a constant slope. The condition is derived for the surface with a constant slope to be a torsal surface, and a link to the surface of tangents to the space curve is also given. Generalized surfaces with a constant slope are defined in the second part of the paper. Their generating lines are determined by points on a space curve and they have a constant slope with respect to a specific system of planes. Mathematical description of these surfaces enables the creation of various surfaces with a constant slope and their modelling on computer.

Keywords: surface with a constant slope, rotational surface with a constant slope, torsal surface with a constant slope, surface of tangents to a space curve, generalized surfaces with a constant slope

1 Surfaces with a constant slope

Surface of a constant slope with respect to the given plane π is the term used for the torsal surface whose generating lines have the same deviation $\gamma \in \left]0, \frac{\pi}{2}\right[$ from the plane π. Number $\sigma = \operatorname{tg} \gamma$, $\sigma \in]0,+\infty[$ is called the slope of the surface with respect to the plane π.

Let us consider ruled surfaces in the Euclidean space $\mathbf{E_3}$ whose generating lines have the same slope σ with respect to the given plane π but these surfaces need not be developable in general. They will be called surfaces with a constant slope.

Furthermore, let the generating lines of the surface κ with the constant slope σ with respect to the plane π be given by points on the curve $\mathcal{K} \subset \pi$ and by direction vectors in such way, that they have the slope σ with respect to the plane π.

1.1 Mathematical description of surfaces with a constant slope

Let us determine the Cartesian coordinate system $[O, x, y, z]$ in the space $\mathbf{E_3}$, respectively in its vector space $V(\mathbf{E_3})$. The plane π is the plane xy. The curve $\mathcal{K} \subset xy$ is given by the vector function

$$\mathbf{r}(s) = (x(s), y(s), 0), \quad s \in \boldsymbol{I},$$

s is an arc of the curve $\mathcal{K}$. Let

$$\mathbf{t} = \mathbf{t}(s), \quad \mathbf{n} = \mathbf{n}(s), \quad s \in \boldsymbol{I},$$

are vector functions of the Frenet-Serret trihedron of the curve $\mathcal{K}$.

Direction vectors of generating lines of the surface κ in the points on the curve $\mathcal{K}$ are given by the vector function

$$\mathbf{u}(s) = \sin\omega(s)\ \mathbf{t}(s) + \cos\omega(s)\ \mathbf{n}(s) + \varepsilon\ \sigma\ \mathbf{e}_3, \quad s \in \boldsymbol{I}, \tag{1}$$

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where ω is an arbitrary real function that is at least $C^{(1)}$ continuous on interval $\boldsymbol{I}$, $\varepsilon = \pm 1$ and the vector $\mathbf{e}_3 = (0, 0, 1)$.

The surface with a constant slope is parameterized by the vector function

$$\mathbf{x}(s,u) = \mathbf{r}(s) + u\,(\sin\omega(s)\,\mathbf{t}(s) + \cos\omega(s)\,\mathbf{n}(s) + \varepsilon\,\sigma\,\mathbf{e}_3)\,,\quad s \in \boldsymbol{I}\,,\quad u \in \boldsymbol{R} \qquad (2)$$

Choosing $\varepsilon = 1$ or $\varepsilon = -1$ we receive, in general, two different surfaces κ_1 a κ_2 that are symmetric with respect to the plane π.

The surface κ parameterized by the vector function (2) is determined by the curve $\mathcal{K}$, function ω, and by the slope σ. The curve $\mathcal{K}$ will be called the generatrix of the surface κ.

1.2 Examples of surfaces with a constant slope

Example 1. The generatrix $\mathcal{K}$ is a segment of the evolvent to the circle $\mathcal{L}$ with a centre in the origin and the radius r. $\mathcal{K}$ is parameterized by the vector function.

$$\mathbf{r}(s) = \left(r\left(\cos\sqrt{\frac{2s}{r}} + \sqrt{\frac{2s}{r}}\sin\sqrt{\frac{2s}{r}}\right), \mathrm{r}\left(\sin\sqrt{\frac{2s}{\mathrm{r}}} - \sqrt{\frac{2s}{r}}\cos\sqrt{\frac{2s}{r}}\right), 0 \right),\quad s \in [0, d] \qquad (3)$$

Vector functions of the Frenet-Serret trihedron elements of the evolvent $\mathcal{K}$ are

$$\mathbf{t}(s) = \left(\cos\sqrt{\frac{2s}{r}}, \sin\sqrt{\frac{2s}{r}}, 0\right),\quad \mathbf{n}(s) = \left(-\sin\sqrt{\frac{2s}{r}}, \cos\sqrt{\frac{2s}{r}}, 0\right),\quad s \in [0, d].$$

The surface κ has according to (2) the parametric expression for $\varepsilon = 1$ in the form

$$x = r\left(\cos\sqrt{\frac{2s}{r}} + \sqrt{\frac{2s}{r}}\sin\sqrt{\frac{2s}{r}}\right) + u\sin\left(\omega(s) - \sqrt{\frac{2s}{r}}\right),$$

$$y = r\left(\sin\sqrt{\frac{2s}{r}} - \sqrt{\frac{2s}{r}}\cos\sqrt{\frac{2s}{r}}\right) + u\cos\left(\omega(s) - \sqrt{\frac{2s}{r}}\right) \qquad (4)$$

$$z = u\,\sigma\,,\quad s \in [0, d]\,,\quad u \in \boldsymbol{R}\,.$$

Surface patch is visualised in the Figure 1 for $\omega(s) = 2s/\pi\,r$, $r = 2$, $\sigma = \sqrt{3}$, $u \in [0, 10]$, $d = 2\pi^2 r$.

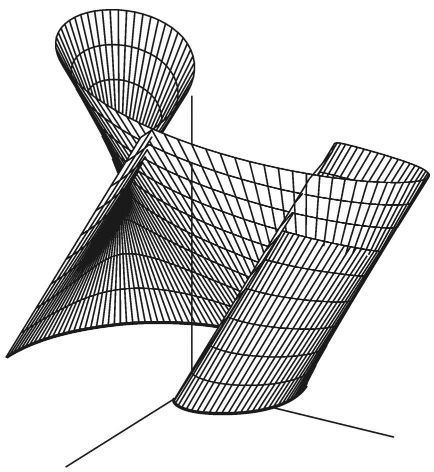

Fig. 1

1.3 Rotational surfaces with a constant slope

Rotational surfaces with a constant slope can be determined by the generatrix $\mathcal{K}$ in the form of a circle, while the function ω is constant on the interval $\boldsymbol{I}$.

The circle $\mathcal{K}$ is given by the vector function

$$\mathbf{r}(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0)\,, \quad s \in [0,\, 2\pi r]\ .$$

Vector functions of its Frenet-Serret trihedron are

$$\mathbf{t}(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}, 0\right), \quad \mathbf{n}(s) = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0\right), \quad s \in [0,\, 2\pi r]\ .$$

Let for $\forall s \in [0,\, 2\pi r]$ be $\omega(s) = c$, c is a constant from $\boldsymbol{R}$.

According to (2), the surface κ has the parametric representation

$$x = r\cos\frac{s}{r} - u\cos\left(\frac{s}{r} - c\right),\ y = r\sin\frac{s}{r} - u\sin\left(\frac{s}{r} - c\right),\ z = u\,\varepsilon\,\sigma\,, \tag{5}$$

$$s \in [0,\, 2\pi r]\,, \quad u \in \boldsymbol{R}\ .$$

Excluding parameters s and u from the equations (5) we receive the equation

$$\sigma^2\left(x^2 + y^2\right) - \left(z - \varepsilon\,\sigma\,r\cos c\right)^2 = \sigma^2 r^2 \sin^2 c\ . \tag{6}$$

Three cases might be considered:

a) c is a constant from $\boldsymbol{R}$, $c \neq k\dfrac{\pi}{2}$, $k \in \boldsymbol{Z}$.

Then (5), or (6) respectively, is the parametric representation, or equations of two rotational one-sheet hyperboloids κ_1 for $\varepsilon = 1$ and κ_2 for $\varepsilon = -1$, respectively. Surface patch is visualised in Figure 2 for $r = 10$, $\sigma = 2$, $c = 1/2$, $u \in [0, 13]$.

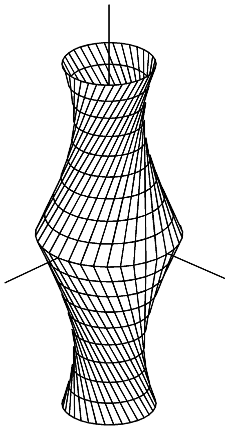

Fig. 2

b) $c = (2k+1)\dfrac{\pi}{2}$, $k \in \boldsymbol{Z}$.

The equation (6) has the form

$$\frac{x^2 + y^2}{r^2} - \frac{z^2}{\sigma^2 r^2} = 1$$

and this is the equation of the rotational one-sheet hyperboloid with the centre in the origin O, while the circle $\mathscr{K}$ is its neck circle. As there is no ε in the equation, therefore $\kappa_1 = \kappa_2$. The surface is symmetrical with respect to the plane π and it is created by any one of the two systems of lines with the slope σ with respect to the plane π (See Figure 3). This hyperboloid is the only surface with the property $\kappa_1 = \kappa_2$.

c) $c = k\pi$, $k \in \boldsymbol{Z}$.

The equation (6) for this c is

$$\sigma^2\left(x^2 + y^2\right) - \left(z - \varepsilon\,\sigma\,r\right)^2 = 0$$

and this is the equation of the conical surfaces κ_1 and κ_2 symmetrical with respect to the plane π. Patches of both surfaces are shown in Figure 4.

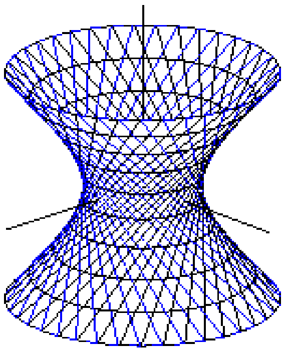

Fig. 3

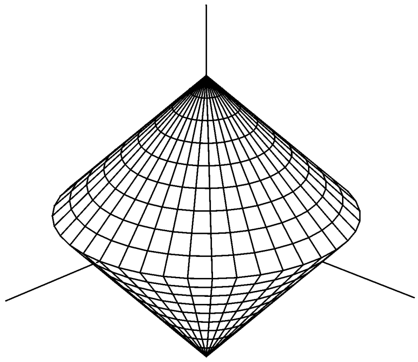

Fig. 4

The following proposition is valid:
The rotational surface with the constant slope σ, $\sigma \in \,]0,+\infty[$ *is either a rotational one-sheet hyperboloid or a rotational conical surface.*

1.4 Torsal generating line on a surface with a constant slope

Let us assume that the generatrix $\mathcal{K}$ is the regular curve. Vectors of the Frenet-Serret trihedron of the curve $\mathcal{K}$ determine an orthonormal basis, therefore their coordinates can be expressed as follows:

$$\mathbf{t}(s) = (\cos\alpha(s), \sin\alpha(s), 0), \quad \mathbf{n}(s) = (-\sin\alpha(s), \cos\alpha(s), 0), \quad s \in \boldsymbol{I}, \tag{7}$$

where α is a real function that is at least $C^{(1)}$ continuous on interval $\boldsymbol{I}$.

Derivatives of the vector functions (7) are vector functions

$$\mathbf{t}'(s) = \alpha'(s)\,\mathbf{n}(s), \quad \mathbf{n}'(s) = -\alpha'(s)\,\mathbf{t}(s) \tag{8}$$

The partial derivatives $\dfrac{\partial \mathbf{x}(s,u)}{\partial s}$ of the vector functions (2) can be adjusted using the formulas (8) as follows:

$$\frac{\partial \mathbf{x}(s,u)}{\partial s} = \big(1 + u(\omega'(s) - \alpha'(s))\cos\omega(s)\big)\,\mathbf{t}(s) - u(\omega'(s) - \alpha'(s))\sin\omega(s)\,\mathbf{n}(s). \tag{9}$$

The vector function (9) describes direction vectors of tangents to the parametric curves for the constant u. For $u = 0$ there is

$$\frac{\partial \mathbf{x}(s,u)}{\partial s} = \mathbf{t}(s), \tag{10}$$

and for $u = 1$ it is

$$\frac{\partial \mathbf{x}(s,u)}{\partial s} = \big(1 + (\omega'(s) - \alpha'(s))\cos\omega(s)\big)\,\mathbf{t}(s) - (\omega'(s) - \alpha'(s))\sin\omega(s)\,\mathbf{n}(s). \tag{11}$$

Vectors (10) and (11) must be linearly dependent in order to have the generating line to be a torsal line for some s. This is true if and only if

$$(\omega'(s) - \alpha'(s))\sin\omega(s) = 0,$$

thus

$$\text{i)}\;\; \omega'(s) - \alpha'(s) = 0 \;\Rightarrow\; \omega(s) = \alpha(s) + c,$$

$$\text{ii)}\;\; \sin\omega(s) = 0 \;\Rightarrow\; \omega(s) = k\pi, \quad k \in \boldsymbol{Z}.$$

If one of the equations i) or ii) is fulfilled for certain $s \in \boldsymbol{I}$, then the generating line is a torsal line.

Let us identify torsal generating lines on the surface from the example 1 with the parametric representation (4). For this surface there is

$$\omega(s) = \frac{2s}{\pi r}, \quad \alpha(s) = \sqrt{\frac{2s}{r}} \;\Rightarrow\; \omega'(s) = \frac{2}{\pi r}, \quad \alpha'(s) = \sqrt{\frac{1}{2rs}}\ , \quad s \neq 0\,.$$

The equation i) is satisfied for $s = \pi^2 r/8$ and a torsal generating line on the surface corresponds to this parameter.

Other torsal lines on the surface can be obtained from the equation ii), which appears for this surface in the form

$$\frac{2s}{\pi r} = k\pi \;\Rightarrow\; s = \frac{k\pi^2 r}{2}, \quad k = 0, 1, 2, 3, 4\ .$$

The surface has 6 torsal generating lines on the interval $[0, 2\pi^2 r]$.

From the rotational surfaces with a constant slope it is the rotational hyperboloid that is not a torsal surface, because

$$\omega(s) = c\,, \quad c \neq k\pi \;\text{ a }\; \alpha(s) = \frac{s}{r} \;\Rightarrow\; \omega'(s) = 0 \;\text{ a }\; \alpha'(s) = \frac{1}{r} \;\text{ for } \forall s \in [0,2\pi]\,.$$

Neither of the two equations i) and ii) is satisfied. The surface has no torsal generating lines. On the contrary, in the case of the rotational conical surface there is $\omega(s) = c$, $c = k\pi$ and according to ii) the surface is generated by torsal lines entirely, and it is therefore a torsal surface.

1.5 Ruled surfaces with a constant slope

As it was already stated, these surfaces are called surfaces of a constant slope.

The surface κ with a constant slope will be a torsal surface, if the equations i) or ii) will be valid for $\forall s \in \boldsymbol{I}$.

Using (7) we can rewrite the direction vectors (1) of generating lines of the surface κ as follows:

$$\mathbf{u}(s) = \big(\sin(\omega(s) - \alpha(s)), \cos(\omega(s) - \alpha(s)), \varepsilon\,\sigma\big), \quad s \in \boldsymbol{I}\ . \tag{12}$$

If the equation i) is the identity at the interval $\boldsymbol{I}$, then the vectors (12) are

$$\mathbf{u}(s) = (\sin c, \cos c, \varepsilon\,\sigma) \tag{13}$$

and the surface is a cylindrical surface. The curve $\mathcal{K}$ is its generatrix and (13) is the direction vector of its generating lines.

In Figure 5 we have depicted the cylindrical surface patch with the evolvent from the example 1 as its generatrix $\mathcal{K}$ parameterized by the vector function (3).

If the equation ii) is the identity on the intervale $\boldsymbol{I}$, the surface is parameterized by the vector function

$$\mathbf{x}(s,u) = \mathbf{r}(s) + u\big(\mathbf{n}(s) + \varepsilon\,\sigma\,\mathbf{e}_3\big), \quad s \in \boldsymbol{I}\,, \quad u \in \boldsymbol{R}\ .$$

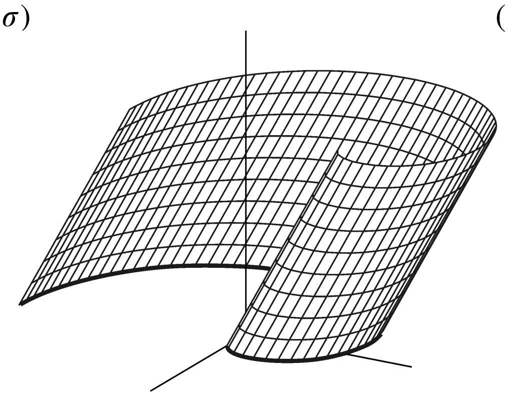

Fig. 5

Orthographic views of generating lines in the plane π are normals to the curve $\mathcal{K}$.

Therefore the preposition is valid:

Torsal surfaces with a constant slope determined by the generatrix $\mathcal{K} \subset \pi$ are

1. cylindrical surfaces,
2. surfaces, for which orthographic views of their generating lines in the plane π are normals to the generatrix $\mathcal{K}$. These include also a plane and a rotational conical surface.

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Example 2. Let the generatrix $\mathcal{K}$ be the ellipse given by the vector function

$$\mathbf{r}(t) = (a\cos t, b\sin t, 0), \quad t \in [0, 2\pi].$$

The unit direction vectors of normals to the ellipse are given by the vector function

$$\mathbf{n}(t) = \left(\frac{-b\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \frac{-a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, 0 \right), \quad t \in [0, 2\pi].$$

Direction vectors of the generating lines of the torsal surface with a constant slope can be determined by the vector function

$$\mathbf{u}(t) = \left(\frac{-b\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \frac{-a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \varepsilon\sigma \right), \quad t \in [0, 2\pi],\ \varepsilon \in \{-1, 1\}.$$

The parametric expression of surface κ is for $\varepsilon = 1$ the following

$$x = a\cos t - u\frac{b\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \quad y = b\sin t - u\frac{a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \quad z = u\sigma,$$
$$t \in [0, 2\pi], \quad u \in \boldsymbol{R}.$$

Intersection of the surface κ and the plane xz with the equation $y = 0$ is the curve $\mathcal{K}_1$ determined by the parametric representation

$$x = \frac{e^2}{a}\cos t, \quad y = 0, \quad z = \frac{b\sigma}{a}\sqrt{a^2\sin^2 t + b^2\cos^2 t}, \qquad t \in [0, 2\pi]. \tag{14}$$

Elimination of the parameter t from the equations (14) yields the equation

$$\frac{x^2}{e^2} + \frac{z^2}{b^2\sigma^2} = 1 \quad \wedge \quad y = 0,$$

which is the equation of the ellipse with vertices in the foci of the ellipse $\mathcal{K}$. The curve $\mathcal{K}_1$ is a segment of this ellipse (Figure 6b).

The intersection of the surface κ and the plane yz with the equation $x = 0$ is the curve $\mathcal{K}_2$ represented parametrically by the expression

$$x = 0, \quad y = \frac{-e^2}{b}\sin t, \quad z = \frac{a\sigma}{b}\sqrt{a^2\sin^2 t + b^2\cos^2 t}, \qquad t \in [0, 2\pi]. \tag{15}$$

Eliminating the parameter t from the equations (15) we receive the equation

$$-\frac{y^2}{e^2} + \frac{z^2}{a^2\sigma^2} = 1 \quad \wedge \quad x = 0$$

representing hyperbola. The curve $\mathcal{K}_2$ is a segment of this hyperbola (Figure 6b).

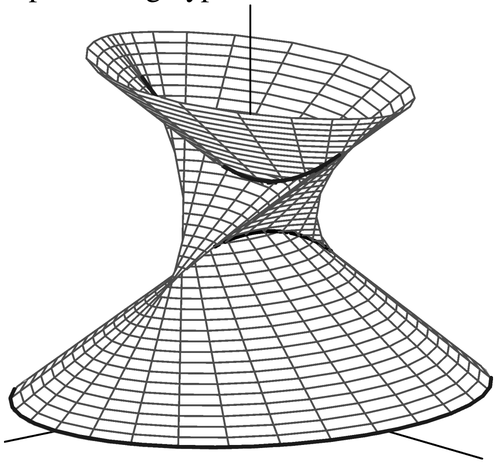

Fig. 6a

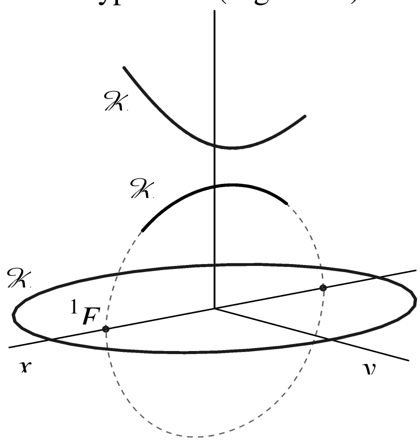

Fig. 6b

In Figure 6a, we have viewed the respective patch of the surface choosing $a = 4$, $b = 3$, $\sigma = 3/2$. When $\sigma = \dfrac{e^2}{b^2}$, $\mathscr{K}_1$ is the circular arc.

Let the generatrix $\mathscr{K}$ be the regular curve with non-constant first curvature in all points and let the orthographic views of generating lines of the surface κ with a constant slope be normals to the curve $\mathscr{K}$. Then κ must be a torsal surface, but with respect to the stated assumptions, it cannot be a rotational cylindrical surface or a rotation conical surface. The surface κ must be the surface of tangents to the space curve, let us denote it as $\mathscr{M}$. It follows from the preceding considerations that the curve $\mathscr{M}$ is located on the cylindrical surface determined by the evolute to the curve $\mathscr{K}$ and with generating lines perpendicular to the plane π. The curve $\mathscr{M}$ is parameterized by the vector function

$$\mathbf{z}(t) = \mathbf{x}(t) + R(t)\,\mathbf{n}(t) + \sigma\, R(t)\,\mathbf{e}_3\,, \quad t \in \boldsymbol{J}\,, \tag{16}$$

where the real function $R(t)$, $t \in \boldsymbol{J}$ is the function of radii of the curve $\mathscr{K}$ osculating circles.

The evolute to the ellipse $\mathscr{K}$ from the example 2 has the parametric representation

$$x = \frac{e^2}{a}\cos^3 t\,, \quad y = \frac{-e^2}{b}\sin^3 t\,, \quad z = 0\,, \quad t \in [0, 2\pi]\,.$$

The radii of the osculating circles of the ellipse $\mathscr{K}$ are values of the function

$$R(t) = \frac{\sqrt{(a^2\sin^2 t + b^2\cos^2 t)^3}}{a\,b}\,, \quad t \in [0, 2\pi]\,.$$

According to (16) the parametric expression of the curve $\mathscr{M}$ is

$$x = \frac{e^2}{a}\cos^3 t\,, \quad y = \frac{-e^2}{b}\sin^3 t\,, \quad z = \frac{\sigma\sqrt{(a^2\sin^2 t + b^2\cos^2 t)^3}}{a\,b}\,, \quad t \in [0, 2\pi]\,.$$

In Figure 7a, we can see the ellipse $\mathscr{K}$, its evolute $\mathscr{E}$ and the curve $\mathscr{M}$. In Figure 7b we have mapped the patch of the surface of tangents to the curve $\mathscr{K}$, see also Figure 6a, while its ground view is in Figure 7c.

Let the generatrix $\mathscr{K}$ be a segment of an evolvent to the circle $\mathscr{L}$ from the Example 1. It is well-known that an evolute to the curve $\mathscr{K}$ is the circle $\mathscr{L}$ and the curve $\mathscr{M}$ is a segment of a cylindrical helix. The torsal surface with a constant slope is then the surface of tangents to the helix $\mathscr{M}$.

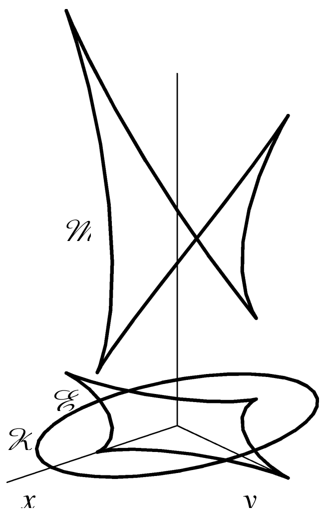

Fig. 7a

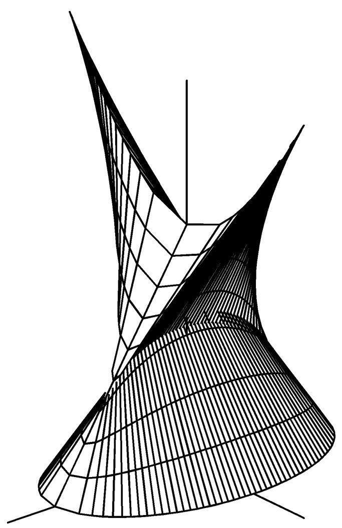

Fig. 7b

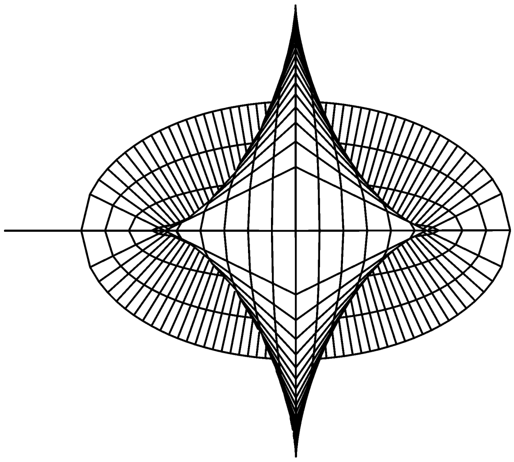

Fig. 7c

2 Generalized surfaces with a constant slope

2.1 Generalized surfaces with a constant slope with respect to the given surface

Let us replace the plane π by a general regular surface π and let a regular curve $\mathscr{K}$ be located on the surface π. Let us create the ruled surface κ, whose generating lines are given by points on the curve $\mathscr{K}$, while in all this points they have the constant slope with respect to the relevant tangent planes to the surface π. This ruled surface will be called generalized surface with a constant slope with respect to the surface π. It is evident that surfaces with a constant slope form a special class of generalized surfaces with a constant slope.

2.2 Mathematical description of generalized surfaces with a constant slope

Let the surface π be parametrized by the vector function

$\mathbf{x} = \mathbf{x}(u,v)$ on the definition domain $\boldsymbol{G}$

and the curve $\mathscr{K} \subset \pi$ be defined by functions

$$u = u(s) \ , \quad v = v(s) \ , \quad s \in \boldsymbol{I} \ ,$$

s is the arc of the curve $\mathscr{K}$ parametrized by the vector function

$$\mathbf{r}(s) = \mathbf{x}(u(s), v(s)) \ , \quad s \in \boldsymbol{I} \ .$$

Vectors

$$\mathbf{t} = \mathbf{t}(s) \ , \quad \mathbf{e} = \mathbf{e}(s) \ , \quad \mathbf{n} = \mathbf{n}(s)$$

generate the orthonormal basis at every point on the curve $\mathscr{K}$. The vector $\mathbf{t}$ is the direction vector of a tangent to the curve $\mathscr{K}$, $\mathbf{n}$ is the direction vector of a normal to the surface and $\mathbf{e} = \mathbf{n} \times \mathbf{t}$ is the direction vector of the intersection line of a tangent plane to the surface π and the normal plane to the curve $\mathscr{K}$ at the respective point.

Direction vectors of generating lines to the surface κ are given by the vector function

$$\mathbf{u}(s) = \sin\omega(s)\ \mathbf{t}(s) + \cos\omega(s)\ \mathbf{e}(s) + \varepsilon\ \sigma\ \mathbf{n}(s) \ , \quad s \in \boldsymbol{I} \ , \quad \varepsilon = \pm 1$$

and the generalized surface κ with the constant slope σ with respect to the surface π is parametrized by the vector function

$$\mathbf{x}(s,t) = \mathbf{r}(s) + t\ \mathbf{u}(s) \ , \quad s \in \boldsymbol{I} \ , \quad t \in \boldsymbol{R} \ .$$

Example 3. Let the surface π be the sphere defined by the vector function

$$\mathbf{x}(u,v) = (r\cos u \cos v, r\cos u \sin v, r\sin u), \quad u \in [-\pi/2, \pi/2], \quad v \in [0, 2\pi] \ .$$

The curve $\mathscr{K}$ is the v-parametric curve for $u = \pi/4$, so this circle is parametrized by the vector function

$$\mathbf{r}(s)=\left(\frac{r}{\sqrt{2}}\cos\frac{s\sqrt{2}}{r},\frac{r}{\sqrt{2}}\sin\frac{s\sqrt{2}}{r},\frac{r}{\sqrt{2}}\right),\quad s\in[0,\pi\, r\sqrt{2}]\ .$$

Vector functions **t**, **e**, and **n** are

$$\mathbf{t}(s)=\left(-\sin\frac{s\sqrt{2}}{r},\cos\frac{s\sqrt{2}}{r},0\right),$$

$$\mathbf{e}(s)=\frac{1}{\sqrt{2}}\left(-\cos\frac{s\sqrt{2}}{r},-\sin\frac{s\sqrt{2}}{r},1\right),$$

$$\mathbf{n}(s)=\frac{1}{\sqrt{2}}\left(\cos\frac{s\sqrt{2}}{r},\sin\frac{s\sqrt{2}}{r},1\right).$$

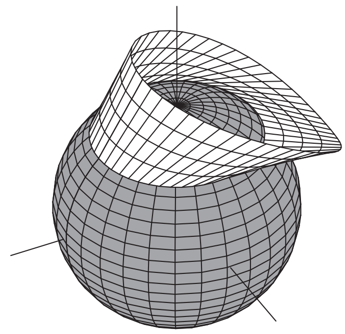

Fig. 8

The generalized surface κ with the constant slope σ with respect to the sphere has for $\varepsilon=1$ the parametric presentation

$$x=\frac{r}{\sqrt{2}}\cos\frac{s\sqrt{2}}{r}-t\left(\sin\frac{s\sqrt{2}}{r}\sin\omega(s)+\frac{1}{\sqrt{2}}\cos\frac{s\sqrt{2}}{r}\cos\omega(s)-\frac{\varepsilon\sigma}{\sqrt{2}}\cos\frac{s\sqrt{2}}{r}\right),$$

$$y=\frac{r}{\sqrt{2}}\sin\frac{s\sqrt{2}}{r}+t\left(\cos\frac{s\sqrt{2}}{r}\sin\omega(s)-\frac{1}{\sqrt{2}}\sin\frac{s\sqrt{2}}{r}\cos\omega(s)+\frac{\varepsilon\sigma}{\sqrt{2}}\sin\frac{s\sqrt{2}}{r}\right),$$

$$z=\frac{r}{\sqrt{2}}+\frac{t}{\sqrt{2}}\left(\cos\omega(s)+\varepsilon\,\sigma\right),\quad s\in\left[0,\pi\, r\sqrt{2}\right],\quad t\in R\ .$$

The patch of the surface received by choosing the function $\omega(s)=2s/r\sqrt{2}$, for $\sigma=1/4$, $t\in[0,3]$ and $r=4$ is shown in Figure 8.

2.3 Generalized surface with a constant slope with respect to osculating planes to its generating curve

Let $\mathcal{K}$ be a regular space curve which is parametrized by the vector function

$\mathbf{r}=\mathbf{r}(s)$, $s\in \boldsymbol{I}$, s is a natural parameter of the curve $\mathcal{K}$.

Vector functions of its Frenet-Serret trihedron are

$$\mathbf{t}=\mathbf{t}(s),\quad \mathbf{n}=\mathbf{n}(s),\quad \mathbf{b}=\mathbf{b}(s)\ .$$

Generating lines of the surface κ are given by points on the curve $\mathcal{K}$ and they have the constant slope σ with respect to the osculating planes to the curve at every point on the curve $\mathcal{K}$. The surface κ will be called the generalized surface with a constant slope with respect to osculating planes to a curve.

Direction vectors of generating line are given by the vector function

$$\mathbf{u}(s)=\sin\omega(s)\,\mathbf{t}(s)+\cos\omega(s)\,\mathbf{n}(s)+\varepsilon\,\sigma\,\mathbf{b}(s)\ ,\quad s=\pm1$$

and the surface κ is parametrized by the vector function

$$\mathbf{x}(s,t)=\mathbf{r}(s)+t\,\mathbf{u}(s)\,,\quad s\in\boldsymbol{I}\,,\quad t\in\boldsymbol{R}\,.$$

Surfaces with constant slope form again a special class, when $\mathcal{K}$ is a planar curve.

Example 4. Let the curve $\mathcal{K}$ be a cylindrical helix parametrized by the vector function

$$\mathbf{r}(s)=\left(r\cos\frac{s}{d},r\sin\frac{s}{d},\frac{v_0\,s}{d}\right),\quad s\in[0,c],$$

where r is the radius of the helix, v_0 is the helical movement pitch, $d=\sqrt{r^2+v_0^2}$ and c is a real positive constant.

The Frenet-Serret trihedron is given by the vector functions

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$$\mathbf{t}(s) = \left(-\frac{r}{d}\sin\frac{s}{d}, \frac{r}{d}\cos\frac{s}{d}, \frac{v_0}{d} \right),$$

$$\mathbf{n}(s) = \left(-\cos\frac{s}{d}, -\sin\frac{s}{d}, 0 \right),$$

$$\mathbf{b}(s) = \left(\frac{v_0}{d}\sin\frac{s}{d}, -\frac{v_0}{d}\cos\frac{s}{d}, \frac{r}{d} \right).$$

Direction vectors of generating lines of the surface κ are given by the vector function

$$\mathbf{u}(s) = \left(-\frac{r}{d}\sin\omega(s)\sin\frac{s}{d} - \cos\omega(s)\cos\frac{s}{d} + \frac{v_0\,\varepsilon\,\sigma}{d}\sin\frac{s}{d}, \right.$$

$$\frac{r}{d}\sin\omega(s)\cos\frac{s}{d} - \cos\omega(s)\sin\frac{s}{d} - \frac{v_0\,\varepsilon\,\sigma}{d}\cos\frac{s}{d},$$

$$\left. \frac{v_0}{d}\sin\omega(s) + \frac{r\,\varepsilon\,\sigma}{d} \right).$$

The surface κ has the parametric representation

$$x = r\cos\frac{s}{d} - t\left(\frac{r}{d}\sin\omega(s)\sin\frac{s}{d} + \cos\omega(s)\cos\frac{s}{d} - \frac{v_0\,\varepsilon\,\sigma}{d}\sin\frac{s}{d} \right),$$

$$y = r\sin\frac{s}{d} + t\left(\frac{r}{d}\sin\omega(s)\cos\frac{s}{d} - \cos\omega(s)\sin\frac{s}{d} - \frac{v_0\,\varepsilon\,\sigma}{d}\cos\frac{s}{d} \right),$$

$$z = \frac{v_0 s}{d} + t\left(\frac{v_0}{d}\sin\omega(s) + \frac{r\varepsilon\,\sigma}{d} \right), \quad s \in \langle 0, c \rangle, \quad t \in \boldsymbol{R}.$$

In Figures 9 to 11 examples of surface patches are displayed for selected function $\omega(s)$ and interval of parameter t. In all cases the same radius $r = 4$ of the helix, the helical movement pitch $v_0 = 3$, the slope $\sigma = 1/4$, $\varepsilon = 1$ and $c = 10\pi$ have been chosen.

The surface patch for $t \in [0, 8]$ and $\omega(s) = 0$ for all $s \in [0, 10\pi]$ is shown in Figure 9. Then the orthographic views of generating lines to the osculating planes form the main normals to the helix $\mathscr{K}$.

The surface patch for $t \in [0, 8]$ and $\omega(s) = \frac{4}{25}s$ is shown in Figure 10, in Figure 11

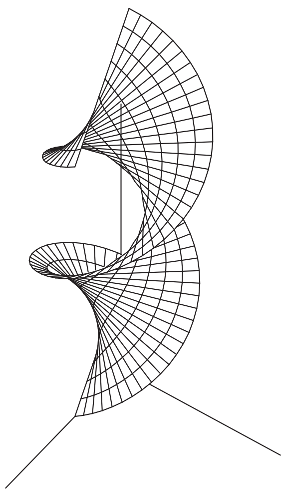

Fig. 9

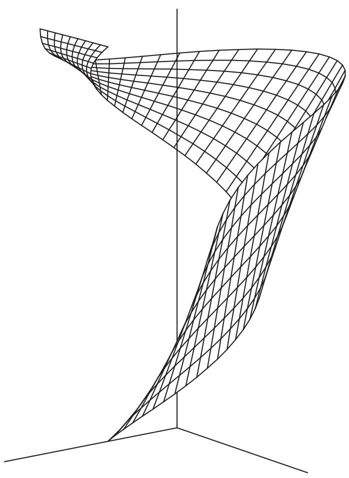

Fig. 10

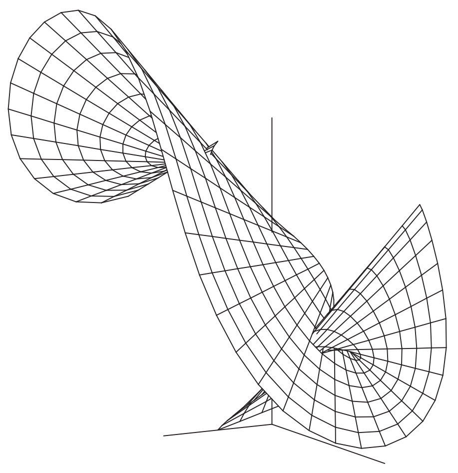

Fig. 11

example of choice $\omega(s) = \frac{s\sqrt{2}}{4}$ and $t \in [0, 16]$ is illustrated.

Note. Surfaces of revolution with a constant slope can be generated by revolution of a generating line about given axis. Composite revolution of a line about two, or more parallel axes determines a two-axial, or a multi-axial surface of revolution of cycloidal type, which is the surface with a constant slope to any plane perpendicular to both axes of revolution. Generalised surface with a constant slope can be generated e.g. by composite revolution of a line about two intersecting or skew axes, as the two-axial surfaces of revolution of spherical or Euler type. Two-axial surfaces of revolution are classified in details in [3].

Some surfaces with constant slope are not easily viewable without a computer. The mathematical description of these surfaces is almost essential for their display and modelling, or in search for their modifications.

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POWIERZCHNIE O STAŁYM NACHYLENIU I ICH UOGÓLNIENIA

Powierzchnie o stałym nachyleniu, omawiane w tej pracy, to powierzchnie, których tworzące są nachylone do pewnej płaszczyzny pod danym kątem. Warunek taki spełnia hiperboloida obrotowa jednopowłokowa. Stąd powierzchnie te nie muszą być powierzchniami rozwijalnymi. Okazuje się wtedy, że powierzchnia stożka obrotowego i hiperboloida jednopowłokowa obrotowa są jedynymi powierzchniami obrotowymi o stałym nachyleniu. Uogólnione powierzchnie o stałym nachyleniu mają tę własność, że ich tworzące są wyznaczone przez punkty pewnej krzywej przestrzennej i mają stałe nachylenie względem specjalnego układu płaszczyzn. Przedstawiony opis matematyczny uogólnionych powierzchni o stałym nachyleniu umożliwia tworzenie różnych takich powierzchni i ich modelowanie na komputerze.